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ON JORDAN BIDERIVATIONS OF TRIANGULAR RINGS

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Abstract. The aim of the paper is to give a description of Jordan biderivations for a certain class of triangular rings. It is shown that, under some mild conditions, every Jordan biderivation of the triangular rings is a biderivation. The result is then applied to some upper triangular matrix rings.

1. Introduction and preliminaries

Let R be an associative ring. An additive map δ from R into itself is said to be a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ holds for all a and b in R. We called δ a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for each a in R. Obviously, every derivation is a Jordan derivation. But the inverse is in general not true [4]. The standard problem is to find out whether a Jordan derivation is necessarily a derivation (for example, refer to [5, 7, 12, 13, 16] and the references therein).

More generally, a biadditive map $\varphi : R \times R \to R$ is called a biderivation if it is a derivation with respect to both components, meaning that

$$\varphi(ab,c) = \varphi(a,c)b + a\varphi(b,c) \quad \text{and} \quad \varphi(a,bc) = \varphi(a,b)c + b\varphi(a,c)$$

for all $a,b,c \in R$. If R is a noncommutative ring and $Z(R)$ is the centre of R, then the map

$$\varphi(x,y) = \lambda[x,y] \text{ for all } x,\ y \in R,$$

where $\lambda \in Z(R)$, is called an inner biderivation. φ is said to be an extremal biderivation if it is the form

$$\varphi(x,y) = [x,[y,a]] \text{ for all } x,\ y \in R,$$

where $a \in R$ and $a \notin Z(R)$ such that $[[R,R],a] = 0$ ([3], Remark 4.4). The inner biderivation and the extremal biderivation are two basic examples of biderivation. Papers [3, 6, 8, 10, 14, 15, 17] relatively studied biderivations on some rings and some operator algebras. We called φ a Jordan biderivation if

$$\varphi(a^2,b) = \varphi(a,b)a + a\varphi(a,b) \quad \text{and} \quad \varphi(a,b^2) = \varphi(a,b)b + b\varphi(a,b)$$

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for all $a,b \in R$. It is clear that every biderivation is a Jordan biderivation. Then the following question seems natural: whether a Jordan biderivation is necessarily a biderivation? But, so far, there are few papers on the study of Jordan biderivations. Abdioğlu and Lee in [1] proved that, for noncommutative prime ring R, every Jordan biderivation of R can be decomposed as $\zeta+\mu$, where ζ is an inner biderivation and μ is an biadditive map from $R\times R$ into the extended centroid satisfying $\mu(x^2,y)=0=\mu(x,y^2)$ for all $x,y\in R$. The Jordan biderivations of triangular matrix rings have been discussed in [2]. In the present note, we shall study the structure of Jordan biderivations of certain triangular rings. We show that, under some mild conditions, every Jordan biderivation of the triangular rings is a biderivation. As an application, we show that every Jordan biderivation of upper triangular matrix rings is a biderivation.

Throughout this note, R denotes a unital ring with a nontrivial idempotent e and with maximal left ring of quotients $Q_{ml}(R)$. The centre $C(R)$ of $Q_{ml}(R)$ is a field which is called the extended centroid of R (see [9] for details). For $A,\ B\subseteq Q_{ml}(R)$, set $C(A,\ B)=\{a\in A : ab=ba \text{ for all } b\in B\}$. R is called a triangular ring if $fRe=0$ and eRf is a faithful $(eRe, fRf)-bimodule$, where $f=1-e$. We close this section by listing here some well known properties of triangular ring R, which come from a good reference [11].

PROPERTIES.

(i) R is a subring of $Q_{ml}(R)$ with the same 1.

(ii) For any dense left ideal I of R and a left R-module homomorphism $h: I\to R$, there exists $q\in Q_{ml}(R)$ such that h is a right multiplication by q.

(iii) eR is a dense left ideal of R and for each $p\in Q_{ml}(R)$ the following hold: $eRfp=0$ implies $fp=0$ and $peRf=0$ implies $pe=0$.

(iv) $C(R)=\{z\in eQ_{ml}(R)e\oplus fQ_{ml}(R)f : zexf=exfz \text{ for all } x\in R\}=C(Q_{ml}(R),\ R)$.

(v) There exists a unique ring isomorphism $\tau : C(R)e\to C(R)f$ such that $\lambda exf=exf\tau(\lambda e)$ for all $x\in R,\ \lambda\in C(R)$.

2. Main results and proofs

In this section, we shall discuss the structure of Jordan biderivations of certain triangular rings. We ought perhaps to mention that our approach is simple but efficient. Our main result reads as follows.

THEOREM 2.1. *Let R be a triangular ring and e the nontrivial idempotent of it. Assume that, $\delta:\ R\times R\to R$ is a Jordan biderivation and $C(fQ_{ml}(R)f, fRf)=C(R)f$ and either $eRC(R)e$ or $fRC(R)f$ does not contain nonzero central ideals.*

(i) If $\delta(e,e)\neq 0$, then it has the form $\delta=\phi+\sigma$, where $\phi:\ R\times R\to R$ is an inner biderivation and $\sigma:\ R\times R\to R$ is an extremal biderivation.

(ii) If $\delta(e,e)=0$, then δ is an inner biderivation.

In other words, every Jordan biderivation of triangular rings is a biderivation.

To prove Theorem 2.1, we need some lemmas. The following lemma will be used repeatedly, whose proof is similar to ([13], Lemma 3.1-3.2) and will be skipped.

LEMMA 2.2. *Let R be a triangular ring and $\delta : R\times R \to R$ be a Jordan biderivation. Then for any $x,y,z,w \in R$, the following hold:*

(i) $\delta(xy+yx,z) = \delta(x,z)y + x\delta(y,z) + \delta(y,z)x + y\delta(x,z)$;

(ii) $\delta(z,xy+yx) = \delta(z,x)y + x\delta(z,y) + \delta(z,y)x + y\delta(z,x)$;

(iii) $\delta(xyx,z) = \delta(x,z)yx + x\delta(y,z)x + xy\delta(x,z)$;

(iv) $\delta(z,xyx) = \delta(z,x)yx + x\delta(z,y)x + xy\delta(z,x)$;

(v) $\delta(xyz+zyx,w) = \delta(x,w)yz + x\delta(y,w)z + xy\delta(z,w) + \delta(z,w)yx + z\delta(y,w)x + zy\delta(x,w)$;

(vi) $\delta(w,xyz+zyx) = \delta(w,x)yz + x\delta(w,y)z + xy\delta(w,z) + \delta(w,z)yx + z\delta(w,y)x + zy\delta(w,x)$.

LEMMA 2.3. *Let R be a triangular ring and $\delta : R\times R \to R$ be a Jordan biderivation. Then $\delta(e,e) = -\delta(e,f) = -\delta(f,e) = \delta(f,f)$.*

Proof. It follows from Lemma 2.2 (iii)-(iv) that $\delta(x,1) = \delta(1,x) = 0$ for all $x \in R$. Then $\delta(e,e) = \delta(e,1-f) = -\delta(e,f)$. Similarly, we get $\delta(e,f) = \delta(1-f,1-e) = \delta(f,e)$ and $\delta(f,f) = -\delta(f,e)$. Hence $\delta(e,e) = -\delta(e,f) = -\delta(f,e) = \delta(f,f)$. □

LEMMA 2.4. *Let R be a triangular ring and $\delta : R\times R \to R$ be a Jordan biderivation. Then*

$$[\delta(x,z),[w,y]] + [\delta(x,w),[z,y]] = [\delta(y,w),[x,z]] + [\delta(y,z),[x,w]]$$

for all $x,y,z,w \in R$.

Proof. For any $x,y,z,w \in R$, by Lemma 2.2 (i)-(ii), on the one hand, we have

$$\begin{aligned}
&\delta(xy+yx,zw+wz)\\
=\ &\delta(x,zw+wz)y + x\delta(y,zw+wz) + \delta(y,zw+wz)x + y\delta(x,zw+wz)\\
=\ &(\delta(x,z)w + z\delta(x,w) + \delta(x,w)z + w\delta(x,z))y\\
&+x(\delta(y,z)w + z\delta(y,w) + \delta(y,w)z + w\delta(y,z))\\
&+(\delta(y,z)w + z\delta(y,w) + \delta(y,w)z + w\delta(y,z))x\\
&+y(\delta(x,z)w + z\delta(x,w) + \delta(x,w)z + w\delta(x,z)),
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&\delta(xy+yx,zw+wz)\\
=\ &\delta(xy+yx,z)w + z\delta(xy+yx,w) + \delta(xy+yx,w)z + w\delta(xy+yx,z)\\
=\ &(\delta(x,z)y + x\delta(y,z) + \delta(y,z)x + y\delta(x,z))w\\
&+z(\delta(x,w)y + x\delta(y,w) + \delta(y,w)x + y\delta(x,w))\\
&+(\delta(x,w)y + x\delta(y,w) + \delta(y,w)x + y\delta(x,w))z\\
&+w(\delta(x,z)y + x\delta(y,z) + \delta(y,z)x + y\delta(x,z)).
\end{aligned}$$

Comparing the above two equations, we see that

$$[\delta(x,z),[w,y]]+[\delta(x,w),[z,y]]=[\delta(y,w),[x,z]]+[\delta(y,z),[x,w]]$$

for all $x,y,z,w\in R$. $\square$

Proof of Theorem 2.1. In the following we will prove the statement (i) of theorem by checking several claims.

Claim 1. *The following statements hold:*
(i) $\delta(e,R)\subseteq eRf$*;*
(ii) $\delta(f,R)\subseteq eRf$*;*
(iii) $\delta(R,e)\subseteq eRf$*;*
(iv) $\delta(R,f)\subseteq eRf$*.*

We only prove statement (i). For other cases, the proofs are similar. Since $\delta(e,R)=\delta(e,R)e+e\delta(e,R)$, we have $e\delta(e,R)e=0$ and $f\delta(e,R)f=0$. It follows that $\delta(e,R)=e\delta(e,R)f\subseteq eRf$.

Since $\delta(e,e)\neq 0$, $\delta(e,e)=e\delta(e,e)f\notin Z(R)$. Taking $x=z=e$ in Lemma 2.4, by Claim 1, we get $[[w,y],\delta(e,e)]=0$ for all $w,y\in R$. Thus $\sigma(x,y)=[x,[y,\delta(e,e)]]$ is a extremal biderivation. Moreover, by Claim 1, we have

$$\sigma(e,e)=[e,[e,\delta(e,e)]]=\delta(e,e).$$

Let $\phi=\delta-\sigma$. Then ϕ is a Jordan biderivation satisfying $\phi(e,e)=0$.

Claim 2. *For any* $x,y\in R$*, the following is true:*
(i) $\phi(exf,y)\in eRf$*;*
(ii) $\phi(x,eyf)\in eRf$*.*

We only need to check statement (i), and the proof of statement (ii) is similar. In fact, for any $x,y\in R$, by Lemma 2.2 (v) and Claim 1, we have

$$\begin{aligned}\phi(exf,y)&=\phi(exf+fxe,y)\\&=\phi(e,y)xf+e\phi(x,y)f+ex\phi(f,y),\end{aligned}$$

which implies $e\phi(exf,y)e=0$ and $f\phi(exf,y)f=0$. So $\phi(exf,y)\in eRf$ for all $x,y\in R$.

Claim 3. *For any* $x,y\in R$*, the following statements hold:*
(i) $\phi(exe,fyf)=0$*;*
(ii) $\phi(fxf,eye)=0$*.*

For any $x,y\in R$, by Lemma 2.2 (iii) and Claim 1, we have

$$\begin{aligned}\phi(exe,fyf)&=\phi(e,fyf)exe+e\phi(exe,fyf)e+exe\phi(e,fyf)\\&=e\phi(exe,fyf)e+exe\varphi(e,fyf)\end{aligned}$$

It follows that

$$e\phi(exe,fyf)f=exe\phi(e,fyf)f \tag{2.1}$$

and

$$f\phi(exe, fyf)f = 0 \tag{2.2}$$

for all $x, y \in R$. Similarly, we obtain

$$\begin{aligned}\phi(exe, fyf) &= \varphi(exe, f)fyf + f\varphi(exe, fyf)f + fyf\varphi(exe, f)\\ &= \varphi(exe, f)fyf,\end{aligned}$$

which implies

$$e\phi(exe, fyf)f = e\phi(exe, f)fyf \tag{2.3}$$

and

$$e\phi(exe, fyf)e = 0 \tag{2.4}$$

for all $x, y \in R$. Comparing Eqs.(2.1) and (2.3) reduces to $exe\phi(e, fyf)f = e\phi(exe, f)fyf$ for all $x, y \in R$. Taking $x = e$ in the above equation, by Lemma 2.3 and the fact $\phi(e, e) = 0$, one can easily check that $e\phi(e, fyf)f = e\phi(e, f)fyf = 0$ for all $x, y \in R$. Combining this and Eqs.(2.1), (2.2) and (2.4), we get $\phi(exe, fyf) = 0$ for all $x, y \in R$.

Using the same idea, we can prove statement (ii).

Claim 4. *For any $x, y \in R$, there exists $\lambda \in C(R)e$ such that*
(i) $\phi(exe, eyf) = -\phi(eyf, exe) = \lambda exeyf$;
(ii) $\phi(exf, fyf) = -\phi(fyf, exf) = \lambda exfyf$.

We only give the proof of statement (i). The proof of statement (ii) is similar. Define a map $h : eR \to R$ by

$$h(x) = \phi(e, exf)$$

for all $x \in eR$. Then, by Lemma 2.2 (ii) and Claim 1, we get

$$\begin{aligned}h(rx) &= \phi(e, erxf)\\ &= \phi(e, erxf + exfere)\\ &= \phi(e, ere)exf + ere\phi(e, exf)\\ &= ere\phi(e, exf)\\ &= rh(x)\end{aligned}$$

for all $x \in eR$, $r \in R$. This implies that h is a left R-module homomorphism. Property (ii) and (iii) show that there exists $q \in Q_{ml}(R)$ such that $h(x) = xq$ for all $x \in eR$. It is clear that $eq = h(e) = 0$. So $h(x) = xfqf$ for all $x \in eR$. Moreover, by Lemma 2.2 (ii) and Claim 1, we have

$$\begin{aligned}h(xfrf) &= \phi(e, exfrf)\\ &= \phi(e, exfrf + frfexf)\\ &= \phi(e, exf)frf + exf\phi(e, frf)\\ &= \phi(e, exf)frf\\ &= h(x)frf,\end{aligned}$$

which leads to $xfrfqf = xfqfrf$ for all $x \in eR$ and $r \in R$. So we get $eR(frfqf - fqfrf) = 0$ for all $r \in R$. Using Property (iii), we see that $frfqf = fqfrf$ for all $r \in R$. Then $fqf \in C(fQ_{ml}(R)f, fRf)$, and hence $fqf \in C(R)f$. Setting $\lambda = \tau^{-1}(fqf)$, by Property (v), we have $\lambda exf = xfqf$ for all $x \in eR$. So

$$\phi(e, exf) = \lambda exf \tag{2.5}$$

for all $x \in R$. Therefore, for any $x, y \in R$, by Lemma 2.2 (iii) and Claim 2, we have

$$\begin{aligned} \phi(exe, eyf) &= \phi(e, eyf)exe + e\phi(exe, eyf)e + exe\phi(e, eyf) \\ &= exe\varphi(e, eyf) \\ &= \lambda exeyf. \end{aligned}$$

With the similar argument, there exists $\mu \in C(R)e$ such that

$$\phi(exf, e) = \mu exf \tag{2.6}$$

for all $x \in R$. For any $x, y, z \in R$, by Lemma 2.4, we have

$$\begin{aligned} &[\phi(e, exf), [eye, eze]] + [\phi(e, eye), [exf, eze]] \\ = &[\phi(eze, eye), [e, exf]] + [\phi(eze, exf), [e, eye]], \end{aligned}$$

which yields that

$$-[eye, eze]\phi(e, exf) = [\phi(eze, eye), exf]. \tag{2.7}$$

On the other hand, we obtain

$$\begin{aligned} &[\phi(exf, e), [eye, eze]] + [\phi(exf, eye), [e, eze]] \\ = &[\phi(eze, eye), [exf, e]] + [\phi(eze, e), [exf, eye]], \end{aligned}$$

which means that

$$-[eye, eze]\phi(exf, e) = [\phi(eze, eye), -exf], \tag{2.8}$$

for all $x, y, z \in R$. By Eq.(2.5)-(2.8), we get $[eye, eze](\phi(e, exf) + \phi(exf, e)) = 0$, that is,

$$(\lambda + \mu)[eye, eze]exf = 0$$

for all $x, y, z \in R$. By Property (iii), we conclude that

$$(\lambda + \mu)[eRe, eRe] = 0.$$

This leads to

$$[(\lambda + \mu)eRC(R)e, eRC(R)e)] = 0.$$

Then $(\lambda + \mu)eRC(R)e$ is a central ideal of $eRC(R)e$. Assume without loss of generality that $eRC(R)e$ does not contain nonzero central ideals. Therefore, we arrive at $\mu = -\lambda$ and

$$\begin{aligned} \phi(eyf, exe) &= \phi(eyf, e)exe + e\phi(eyf, exe)e + exe\phi(eyf, e) \\ &= exe\varphi(eyf, e) \\ &= -\lambda exeyf, \end{aligned}$$

completing the proof of statement (i).

Claim 5. *For any $x, y \in R$, the following is true:*
(i) $\phi(exe, eye) = \lambda[exe, eye]$*;*
(ii) $\phi(fxf, fyf) = \tau(\lambda)[fxf, fyf]$.

For any $x \in R$, by Lemma 2.2 (iv), we have $\phi(e, exe) = \phi(e,e)xe + e\phi(e,x)e + ex\phi(e,e)$. It follows from Claim 1 and the fact $\phi(e,e) = 0$ that $\phi(e, exe) = 0$ for all $x \in R$. Hence

$$\begin{aligned}\phi(exe, eye) &= \phi(e, eye)exe + e\phi(exe, eye)e + exe\phi(e, eye)\\ &= e\phi(exe, eye)e\end{aligned}$$

for all $x, y \in R$. Then, for any $x, y, z \in R$, we have

$$\begin{aligned}&[\phi(exe, eye), [e, ezf]] + [\phi(exe, e), [eye, ezf]]\\ &= [\phi(ezf, e), [exe, eye]] + [\phi(ezf, eye), [exe, e]],\end{aligned}$$

which implies that

$$\phi(exe, eye)ezf = -[exe, eye]\phi(ezf, e) = \lambda[exe, eye]ezf,$$

since Claim 4 (i). It follows that $(\phi(exe, eye) - \lambda[exe, eye])eRf = 0$ for all $x, y \in R$. By Property (iii), we obtain that $\phi(exe, eye) = \lambda[exe, eye]$ for all $x, y \in R$.

In an analogous manner, we have $\phi(fxf, fyf) = f\phi(fxf, fyf)f$ for all $x, y \in R$. So we get

$$\begin{aligned}&[\phi(fxf, fyf), [e, ezf]] + [\phi(fxf, e), [fyf, ezf]]\\ &= [\phi(ezf, e), [fxf, fyf]] + [\phi(ezf, fyf), [fxf, e]],\end{aligned}$$

that is,

$$ezf\phi(fxf, fyf) = \phi(e, ezf)[fxf, fyf]$$

for all $x, y, z \in R$. Therefore, by Claim 4 (i), we obtain

$$\begin{aligned}\phi(e, ezf)[fxf, fyf] &= \lambda ezf[fxf, fyf]\\ &= ezf\tau(\lambda)[fxf, fyf]\end{aligned}$$

for all $x, y, z \in R$. Then $eRf(\phi(fxf, fyf) - \tau(\lambda)[fxf, fyf]) = 0$ for all $x, y \in R$. It follows from Property (iii) that $\phi(fxf, fyf) = \tau(\lambda)[fxf, fyf]$ for all $x, y \in R$.

Claim 6. $\phi(exf, eyf) = 0$ *for all* $x, y \in R$.

For fixed $y \in R$, we define a map $h : eR \to R$ by

$$h_y(x) = \phi(exf, eyf)$$

for all $x \in eR$. Then, by Lemma 2.2 (i) and Claim 2, we get

$$\begin{aligned}h_y(rx) &= \phi(erxf, eyf)\\ &= \phi(erexf + exfere, eyf)\\ &= \phi(ere, eyf)exf + ere\phi(exf, eyf)\\ &= r\phi(exf, eyf)\\ &= rh_y(x)\end{aligned}$$

for all $x \in eR$, $r \in R$, and hence h_y is a left R-module homomorphism. Using Property (ii)-(iii), we see that there exists $q_y \in Q_{ml}(R)$ such that $h_y(x) = xq_y$ for all $x \in eR$. It is clear that $eq_y = h_y(e) = 0$. So $q_y = fq_yf$, which implies that $h_y(x) = xfq_yf$ for all $x \in eR$. Moreover, by Lemma 2.2 (i) and Claim 2, we have

$$\begin{aligned} h_y(xfrf) &= \phi(exfrf, eyf) \\ &= \phi(exfrf + frfexf, eyf) \\ &= \phi(exf, eyf)frf + exf\phi(frf, eyf) \\ &= \phi(exf, eyf)frf \\ &= h_y(x)frf \end{aligned}$$

and hence $xfrfq_yf = xfq_yfrf$ for all $x \in eR$ and $r \in R$. Then $eR(frfq_yf - fq_yfrf) = 0$ for all $r \in R$. In view of Property (iii), we get $frfq_yf = fq_yfrf$ for all $r \in R$. Consequently, by the assumption of Theorem 2.1, we have $fq_yf \in C(R)f$.

Now, for any $x, y, x', y' \in R$, by Lemma 2.4, we have

$$\begin{aligned} &[\phi(exe, eye), [ey'f, ex'f]] + [\phi(exe, ey'f), [eye, ex'f]] \\ &= [\phi(ex'f, ey'f), [exe, eye]] + [\phi(ex'f, eye), [exe, ey'f]], \end{aligned}$$

which implies that

$$\begin{aligned} 0 &= [exe, eye]\phi(ex'f, ey'f) \\ &= [exe, eye]ex'fq_{y'}f \\ &= \tau^{-1}(fq_{y'}f)[exe, eye]ex'f \end{aligned}$$

for all $x, y, x', y' \in R$. By Property (iii), we have

$$\tau^{-1}(fq_{y'}f)[eRe, eRe] = 0.$$

This leads to

$$[\tau^{-1}(fq_{y'}f)eRC(R)e, eRC(R)e] = 0.$$

It follows that $\tau^{-1}(fq_{y'}f)eRC(R)e$ is a central ideal of $eRC(R)e$. Assume without loss of generality that $eRC(R)e$ does not contain nonzero central ideals. Then $\tau^{-1}(fq_{y'}f) = 0$, which leads to $fq_{y'}f = 0$ for all $y' \in R$. So we conclude that $\phi(exf, eyf) = exfq_yf = 0$ for all $x, y \in R$, completing the proof of Claim 6.

Let $\nu = \lambda + \tau(\lambda) \in C(R)$. By Claim 3-6, one can easily check that $\phi(x, y) = \nu[x, y]$ for all $x, y \in R$. Hence $\delta = \phi + \sigma$, where ϕ is a inner biderivation and σ is a extremal biderivation. The proof of the statement (i) of theorem is completed.

For the statement (ii), by assumption $\delta(e, e) = 0$, the proof is similar to that of Claim 2-6 of the statement (i). We omit it here. Complete the proof of the theorem. $\square$

3. Application

In this section, we shall apply Theorem 2.1 to the upper triangular matrix rings.

Let S be a unital ring and $T_n(S)$ be the ring of all $n\times n$ upper triangular matrices over S. Then $T_n(S)$ is called the upper triangular matrix ring. It is clear that the upper triangular matrix ring $T_n(S)$ can be represented as a triangular ring. Indeed, let $e=e_{11}$ and $f=1-e_{11}$, where e_{11} is a standard matrix unit in $T_n(S)$ (that is, e_{11} is 1 in the $(1,1)$th entry and 0 elsewhere). Thus, e is a nontrivial idempotent such that $fT_n(S)e=0$ and $eT_n(S)f$ is a faithful $(eT_n(S)e, fT_n(S)f)$-bimodule. Hence, as an application of Theorem 2.1, we get the following theorem.

THEOREM 3.1. *Let $T_n(S)$ be a upper triangular matrix ring with $n>2$ and δ be a Jordan biderivation of $T_n(S)$.*

(i) If $\delta(e_{11},e_{11})\neq 0$, then $\delta=\phi+\sigma$, where ϕ is an inner biderivation of $T_n(S)$ and σ is an extremal biderivation of $T_n(S)$.

(ii) If $\delta(e_{11},e_{11})=0$, then δ is an inner biderivation of $T_n(S)$.

In other words, every Jordan biderivation of $T_n(S)$ is a biderivation.

Proof. Note that $Q_{ml}(T_n(S))=M_n(Q_{ml}(S))$ [3]. Then we have $C(T_n(S))=C(S)I_n$. It follows that

$$\begin{aligned} C(fQ_{ml}(T_n(S))f, fT_n(S)f) &= C(fM_n(Q_{ml}(S))f, fT_n(S)f)\\ &= C(Q_{ml}(T_{n-1}(S)), T_{n-1}(S))\\ &= C(T_{n-1}(S))\\ &= C(S)f. \end{aligned}$$

Moreover, it is easy to check that

$$fT_n(S)C(T_n(S))f = fT_n(S)C(S)f = fT_n(SC(S))f.$$

which means $fT_n(S)C(T_n(S))f$ does not contain nonzero central ideals, since $n>2$. So the conditions of Theorem 2.1 are satisfied, and hence, by Theorem 2.1, we obtain the results of Theorem 3.1. □

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