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LOCAL AND 2–LOCAL ISOMETRIES BETWEEN ABSOLUTELY CONTINUOUS FUNCTION SPACES

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Abstract. In this paper we give a complete description of local and 2-local isometries defined between spaces of scalar-valued absolutely continuous functions on arbitrary (not necessarily compact) subsets of the real line with at least two points.

1. Introduction

In the last years considerable work has been done on local maps, an area whose main problem of research is whether the local actions of some important classes of transformations (like derivations, automorphisms, isometries) on a given space determine the class under consideration completely. We refer the reader to [13] for more information.

In particular, given two Banach spaces A and B, the space of all bounded linear operators from A to B, $\mathscr{L}(A,B)$ and $\mathscr{S} \subseteq \mathscr{L}(A,B)$, a linear map $T : A \longrightarrow B$ is said to be *locally* in $\mathscr{S}$ if for each $a \in A$, there exists a map $T_a \in \mathscr{S}$ such that $Ta = T_a a$. Similarly, a map $T : A \longrightarrow B$ (which is not assumed to be linear) is said to be 2*-locally* in $\mathscr{S}$ if for any pair $a, a' \in A$, there is a map $T_{a,a'} \in \mathscr{S}$ such that $Ta = T_{a,a'}(a)$ and $Ta' = T_{a,a'}(a')$.

Local isometries are an active research area centered in the study of the algebraic reflexivity of certain function spaces. Let us recall that a Banach space A is algebraically reflexive if any linear map on A belonging locally to the group of all surjective linear isometries is surjective. For example, it is known (see [14] and [2]) that $C(X,\mathbb{C})$ (resp. $C_0(X,\mathbb{C})$) is algebraically reflexive provided X is a first countable compact space (resp. X is locally compact space whose one-point compactification is metrizable). Besides, $C(X,E)$ is algebraically reflexive if X is a first countable compact space and E is a finite-dimensional complex Banach space or E is a uniformly convex and algebraically reflexive Banach space (see [9]). One can also find recent results related to the algebraic reflexivity of Banach algebras of Lipschitz functions in

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[15]. It is worth mentioning that little is known concerning spaces of real-valued functions since the main tool for the above results, the Gleason-Kahane-Żelazko theorem, only applies to complex algebras.

On the other hand, 2-local maps were introduced by Šemrl in [16] when trying to drop the linearity assumption for certain local maps. Subsequently, many authors have worked on these maps. Namely Győry ([4]) proved that every 2-local isometry on $C_0(X,\mathbb{C})$ is a surjective linear isometry provided X is a first countable, σ-compact, locally compact space. Later, in [1], Al-Halees and Fleming generalized Győry's results to $C_0(X,E)$ for σ-compact metric spaces X under certain conditions on the Banach space E. Similar results have been achieved for other function spaces such as, for instance, (pointed) Lipschitz spaces ([10, 11, 12]), uniform algebras ([5, 12]) and spaces of functions of bounded variation ([7]).

In this paper we give a complete description of local and 2-local isometries defined between spaces of scalar-valued absolutely continuous functions on arbitrary (not necessarily compact) subsets of the real line with at least two points. In particular, we extend some previous results in [6, 7] to a noncompact framework. We would like to remark that, in most of the papers mentioned above, the compacity of the underlying spaces (or local compacity with functions vanishing at infinity) plays a crucial role.

2. Preliminaries

Let X be a subset of the real line $\mathbb{R}$ with at least two points. Let us recall that a scalar-valued function f on X has bounded variation if the total variation $\mathscr{V}(f)$ of f is finite, that is,

$$\mathscr{V}(f) := \sup\left\{\sum_{i=1}^{n} |f(x_i)-f(x_{i-1})| : n\in\mathbb{N}, x_0,x_1,\ldots,x_n\in X, x_0<x_1<\ldots<x_n\right\}<\infty.$$

Moreover, a scalar-valued function f on X is said to be *absolutely continuous* if given $\varepsilon>0$, there is a $\delta>0$ such that

$$\sum_{i=1}^{n} |f(b_i)-f(a_i)|<\varepsilon,$$

for each finite family of non-overlapping open intervals $\{(a_i,b_i): i=1,\cdots,n\}$ whose extreme points belong to X and $\sum_{i=1}^{n}(b_i-a_i)<\delta$. We denote by $AC_b(X)$ the space of all scalar-valued absolutely continuous functions of bounded variation on X, endowed with the norm $\|\cdot\| = \max\{\|\cdot\|_\infty, \mathscr{V}(\cdot)\}$, where $\|\cdot\|_\infty$ stands for the supremum norm of a function. Note that when X is bounded, each absolutely continuous function is automatically of bounded variation.

In the sequel, by $\overline{X}$ we denote the closure of X in $\mathbb{R}$. Also, for any $f\in AC_b(X)$, let $\overline{f}$ be the unique absolutely continuous extension of f to the closure $\overline{X}$ of X, which exists by [8, Lemma 3.1].

Given two Banach spaces A and B, we shall denote by $\mathrm{Iso}(A,B)$ the set of all surjective linear isometries from A onto B. When $A=B$, we shall write $\mathrm{Iso}(A)$ instead of $\mathrm{Iso}(A,A)$.

3. Local isometries of absolutely continuous function spaces

In the sequel, X and Y will be two arbitrary (not necessarily closed nor bounded) subsets of $\mathbb{R}$ with at least two points.

By $\mathrm{Iso}_1(AC_b(X),AC_b(Y))$ we denote the set of all surjective linear isometries $T : AC_b(X) \longrightarrow AC_b(Y)$ such that $T1$ is bounded away from zero, i.e., there exists $t > 0$ such that, for each $y \in Y$, we have $|T1(y)| \geqslant t$. Indeed, by [8, Theorem 3.13], we know that $\mathrm{Iso}_1(AC_b(X),AC_b(Y))$ is the set of all surjective linear isometries $T : AC_b(X) \longrightarrow AC_b(Y)$ of the form of a weighted composition operator. The following result, which is used several times in our proofs, describes the form of these isometries.

THEOREM 3.1. ([8, Theorem 3.13]) *If $T \in \mathrm{Iso}_1(AC_b(X),AC_b(Y))$, then there exist a unimodular scalar λ and a monotonic homeomorphism $\varphi : \overline{Y} \longrightarrow \overline{X}$ such that $\overline{Tf} = \lambda \overline{f} \circ \varphi$ for all $f \in AC_b(X)$.*

Moreover, it is worth pointing out that, according to Corollary 3.15 in [8], for the case X (and so Y) is connected we have $\mathrm{Iso}_1(AC_b(X),AC_b(Y)) = \mathrm{Iso}(AC_b(X),AC_b(Y))$.

Let us next adapt the concepts mentioned in the introduction to our context:

DEFINITION 3.2. A linear map $T : AC_b(X) \longrightarrow AC_b(Y)$ which is locally in $\mathrm{Iso}_1(AC_b(X),AC_b(Y))$ is called a local isometry. In fact, T is a local isometry if for each $f \in AC_b(X)$ there is a surjective linear isometry $T_f \in \mathrm{Iso}_1(AC_b(X),AC_b(Y))$ (depending on f) such that $Tf = T_f f$.

We can now provide a complete description of local isometries defined between spaces of absolutely continuous functions in a noncompact framework, which is a generalization of [6, Theorem 2.1].

THEOREM 3.3. *If $T : AC_b(X) \longrightarrow AC_b(Y)$ is a local isometry, then there exist a monotonic homeomorphism $\varphi : \overline{Y} \longrightarrow \overline{X}$, and a scalar λ with $|\lambda| = 1$ such that*

$$\overline{Tf}(y) = \lambda \overline{f}(\varphi(y)) \quad (f \in AC_b(X), y \in Y).$$

Proof. By [8, Lemma 3.1], each absolutely continuous function has a unique absolutely continuous function extension to the closure of the underlying space with the same total variation. This allows us to consider the map $S : AC_b(\overline{X}) \longrightarrow AC_b(\overline{Y})$ defined by $S(\overline{f}) = \overline{Tf}$ for all $f \in AC_b(X)$. It is easy to see that S is a local isometry. Therefore we can assume, without loss of generality, that X and Y are closed subsets of the real line. Moreover, since T is a local isometry, we infer from Theorem 3.1 that $T1$ is a unimodular constant function. Hence, by considering $\frac{T}{T1}$ instead of T, we can assume, without loss of generality, that $T1 = 1$. We now continue the proof through several steps.

Step 1. For each $f \in AC_b(X)$, $\|Tf\|_\infty = \|f\|_\infty$ and $\|Tf\| = \|f\|$.
Since T is a local isometry, it is an immediate consequence of Theorem 3.1.

REMARK. Before continuing with the proof, let us notice that if we had assumed compacity, we could have exploited the density of $AC_b(X)$ in $C(X)$ and would have obtained the representation of T from the famous Holsztyński Theorem (see, e.g., [3, Theorem 2.3.10]). However, in this noncompact framework, such density is not valid (see [8, Remark 3.4]) and this forces us to use another approach in order to describe T.

Before stating the next step, we need to fix some notation. For each $x \in X$, we set

$$\mathscr{F}_x := \{f \in AC_b(X) : \|f\|_\infty = 1 = f(x)\}$$

which is clearly non-empty. Moreover, we also define

$$\mathscr{I}_x := \bigcap \{M_{Tf} : f \in \mathscr{F}_x\},$$

where $M_{Tf} := \{y \in Y : |Tf(y)| = 1 = \|Tf\|_\infty\}$.

Step 2. Given $x \in X$, $\mathscr{I}_x$ is a non-empty subset of Y.

Let us define

$$\widetilde{\mathscr{I}_x} := \bigcap \{M_{\widetilde{Tf}} : f \in \mathscr{F}_x\},$$

where $\widetilde{Tf}$ is the unique extension of Tf to the Stone-Čech compactification, βY, of Y. And, as expected, $M_{\widetilde{Tf}} = \{y \in \beta Y : |\widetilde{Tf}(y)| = 1 = \|\widetilde{Tf}\|_\infty\}$, which is obviously a non-empty compact subset of βY.

We now prove, by means of a standard technique in a compact framework (see, e.g., [8, Lemma 3.8]), that $\widetilde{\mathscr{I}_x}$ is a nonempty subset of βY. To this end, let $f_1, \dots, f_n \in \mathscr{F}_x$. Define $f = \sum_{i=1}^n \frac{f_i}{n}$. Clearly, $f \in AC_b(X)$ with $\|\widetilde{Tf}\|_\infty = \|Tf\|_\infty = \|f\|_\infty = 1$, by Step 1. Hence there is a point $y \in \beta Y$ such that $|\widetilde{Tf}(y)| = 1$, and so

$$1 = |\widetilde{Tf}(y)| = \left| \sum_{i=1}^n \frac{\widetilde{Tf_i}(y)}{n} \right| \leqslant \sum_{i=1}^n \frac{|\widetilde{Tf_i}(y)|}{n} \leqslant \sum_{i=1}^n \frac{\|\widetilde{Tf_i}\|_\infty}{n} = \sum_{i=1}^n \frac{\|f_i\|_\infty}{n} = 1,$$

which implies that $|\widetilde{Tf_i}(y)| = 1$ for every $i \in \{1, \dots, n\}$. Thus $y \in \bigcap_{i=1}^n M_{\widetilde{Tf_i}}$. Therefore, the family $\{M_{\widetilde{Tf}} : f \in \mathscr{F}_x\}$ has the finite intersection property and then $\widetilde{\mathscr{I}_x} \neq \emptyset$, as desired.

Finally, let us check that $\mathscr{I}_x = \widetilde{\mathscr{I}_x}$. Take $f \in \mathscr{F}_x$ with compact support and $\{t \in X : |f(t)| = 1\} = \{x\}$. From the representation given by Theorem 3.1, it follows that Tf has compact support in Y. As a consequence, $\widetilde{\mathscr{I}_x} \subseteq Y$ because it is clear that $\widetilde{\mathscr{I}_x} \subseteq \mathrm{Supp}(Tf)$. Therefore, $\mathscr{I}_x = \widetilde{\mathscr{I}_x}$ is a non-empty subset of Y, as required.

Step 3. If $x \in X$ and $f \in AC_b(X)$ with $f(x) = 0$, then $Tf(y) = 0$ for all $y \in \mathscr{I}_x$.

This step is verified by a argument similar to the proof of Lemma 3.10 in [8]. Contrary to what we claim, suppose that $x \in X$ and $f \in AC_b(X)$ with $f(x) = 0$, but $Tf(y) \neq 0$ for some $y \in \mathscr{I}_x$. Let $r > \|f\|_\infty$ and choose $h \in \mathscr{F}_x$ such that $0 \leqslant h \leqslant 1$

and $\||f|+rh\|_\infty = \|f \pm rh\|_\infty = r$, by [8, Lemma 3.6 (1)]. Since $y \in \mathscr{I}_x$, $|Th(y)| = 1$, and so we have

$$\begin{aligned} r = \|f \pm rh\|_\infty &= \|T(f \pm rh)\|_\infty \\ &\geqslant \max\{|Tf(y)+rTh(y)|, |Tf(y)-rTh(y)|\} > r, \end{aligned}$$

which is impossible. This completes the proof of Step 3.

Step 4. For any two distinct points x and x' in X, $\mathscr{I}_x \bigcap \mathscr{I}_{x'} = \emptyset$.

Assume, on the contrary, that there exists a point y in $\mathscr{I}_x \bigcap \mathscr{I}_{x'}$. Choose a function $f \in \mathscr{F}_x$ with $f(x') = 0$. By Step 3, $Tf(y) = 0$ because $y \in \mathscr{I}_{x'}$. On the other hand, $|Tf(y)| = 1$ since $f \in \mathscr{F}_x$ and $y \in \mathscr{I}_x$, which is impossible. This argument yields $\mathscr{I}_x \bigcap \mathscr{I}_{x'} = \emptyset$.

We can now introduce the following nonempty subset of Y:

$$Y_0 := \{y \in Y : y \in \mathscr{I}_x \text{ for some } x \in X\}.$$

Besides, we can define a mapping $\varphi : Y_0 \longrightarrow X$ such that $\varphi(y) = x$ if $y \in \mathscr{I}_x$. According to the preceding step, φ is well-defined. Meantime, it is clear that φ is surjective.

Step 5. For each $f \in AC_b(X)$ and $y \in Y_0$, we have $Tf(y) = f(\varphi(y))$.

Let $f \in AC_b(X)$ and $y \in Y_0$. Since $(f - f(\varphi(y)))(\varphi(y)) = 0$, we have $T(f - f(\varphi(y)))(y) = 0$ by Step 3. Hence $Tf(y) = T(f(\varphi(y)))(y) = f(\varphi(y))$, as desired.

Step 6. $Y_0 = Y$.

Let us first define the function $h : X \longrightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} 2 + \frac{1}{x-1} & \text{if } x \in (-\infty, 0) \cap X, \\ \frac{1}{x+1} & \text{if } x \in [0, +\infty) \cap X. \end{cases}$$

Clearly, h is an injective function which belongs to $AC_b(X)$ with $\mathscr{V}(h) \leqslant 2$. Since T is a local isometry, there exist a monotonic homeomorphism $\varphi_h : Y \longrightarrow X$, and a scalar λ_h with $|\lambda_h| = 1$ such that

$$Th = T_h h = \lambda_h (h \circ \varphi_h).$$

Hence, from Step 5, we get

$$h(\varphi(y)) = \lambda_h h(\varphi_h(y)) \quad (y \in Y_0),$$

which taking into account that $|\lambda_h| = 1$ and $h \geqslant 0$, implies that $h(\varphi(y)) = h(\varphi_h(y))$ $(y \in Y_0)$. Hence $\varphi(y) = \varphi_h(y)$ for all $y \in Y_0$ because h is injective.

Now we can prove that $Y_0 = Y$. Otherwise, there would exist a point $y_0 \in Y \setminus Y_0$. Set $x_0 = \varphi_h(y_0)$. Since φ is surjective, there is a point $y \in Y_0$ such that $x_0 = \varphi(y)$. Hence, from the above discussion, we conclude that $\varphi_h(y_0) = \varphi_h(y)$, which contradicts the injectivity of φ_h. Therefore, $Y_0 = Y$.

Then, as observed above, $\varphi = \varphi_h$ is a monotonic homeomorphism from Y onto X, and also we have

$$Tf(y) = \lambda f(\varphi(y)) \quad (f \in AC_b(X), y \in Y),$$

which completes the proof of the theorem. $\square$

4. 2-Local isometries of absolutely continuous function spaces

In this section we present a complete description of 2-local isometries between absolutely continuous function spaces. First let us recall the definition of 2-local isometries in the following.

DEFINITION 4.1. A map $T : AC_b(X) \longrightarrow AC_b(Y)$ (no linearity nor surjectivity are assumed) is called a 2-local isometry if it belongs 2-locally to $\mathrm{Iso}_1(AC_b(X), AC_b(Y))$, i.e., for every $f, g \in AC_b(X)$ there exists a $T_{f,g} \in \mathrm{Iso}_1(AC_b(X), AC_b(Y))$ such that $Tf = T_{f,g}f$ and $Tg = T_{f,g}g$.

THEOREM 4.2. *For each 2-local isometry $T : AC_b(X) \longrightarrow AC_b(Y)$, there exist a monotonic homeomorphism $\varphi : \bar{Y} \longrightarrow \bar{X}$, and a scalar λ with $|\lambda| = 1$ such that $\overline{Tf}(y) = \lambda \overline{f} \circ \varphi$ for all $f \in AC_b(X)$.*

Proof. We will use a modification of the proof provided in [7, Theorem 2.5]. As in the beginning of the proof of Theorem 3.3, we can assume, without loss of generality, that X and Y are closed subsets of $\mathbb{R}$. Since $T1$ is a unimodular constant function, we can assume, without loss of generality that T is unital, i.e., $T1 = 1$. Then, following the proof of [7, Theorem 2.5] (see also [11, Theorem 2.1]), one can deduce that for each $x \in X$, the set $I_x := \bigcap_{f \in AC_b(X)} E_{x,f}$ is a singleton, where $E_{x,f} = \{z \in Y : Tf(z) = f(x)\}$. By $\psi(x)$ we denote the unique point in I_x. This allows us to define an injective map $\psi : X \longrightarrow Y$ such that $Tf(\psi(x)) = f(x)$ for all $x \in X$ and $f \in AC_b(X)$. Now, taking $Y_0 := \psi(X)$ and $\varphi := \psi^{-1}$ we will have the bijective map $\varphi : Y_0 \longrightarrow X$ such that

$$Tf(y) = f(\varphi(y)) \quad (f \in AC_b(X), y \in Y_0).$$

We now claim that $Y_0 = Y$. Let h be defined as in the proof of Theorem 3.2 (Step 6). Since T is a 2-local isometry, there exists $T_{h,1} \in \mathrm{Iso}_1(AC_b(X), AC_b(Y))$ such that $T_{1,h}(1) = 1$ and $Th = T_{1,h}(h)$. According to Theorem 3.1, there is a monotonic homeomorphism $\varphi_{1,h} : Y \longrightarrow X$ such that $T_{1,h}(h) = h \circ \varphi_{1,h}$. Combining the latter equations, we get

$$h(\varphi(y)) = h(\varphi_{1,h}(y)) \quad (y \in Y_0),$$

which easily implies that $\varphi = \varphi_{1,h}$ on Y_0 because of the injectivity of h.

In order to prove that $Y_0 = Y$, let y be an arbitrary point in Y. Since φ is surjective, there exists $y_0 \in Y_0$ with $\varphi(y_0) = \varphi_{1,h}(y)$. On the other hand, from the previous paragraph we have $\varphi(y_0) = \varphi_{1,h}(y_0)$, which implies that $\varphi_{1,h}(y_0) = \varphi_{1,h}(y)$. Consequently, $y = y_0$ because $\varphi_{1,h}$ is injective, which yields $y \in Y_0$. Therefore, we can conclude that $Y_0 = Y$.

Gathering all the information, we infer that $\varphi : Y \longrightarrow X$ is a monotonic homeomorphism ($\varphi = \varphi_{1,h}$) and also

$$Tf(y) = f(\varphi(y)) \quad (f \in AC_b(X), y \in Y),$$

as required. $\square$

REMARK 4.3. It should be noted that, for the complex case, one can exploit the spherical version of the Kowalski-Słodkowski theorem provided in [12] to obtain the description of 2-local isometries (see the remark at the end of [7]). More precisely, let $T : AC_b(X) \longrightarrow AC_b(Y)$ be a 2-local isometry. For each $y_0 \in Y$, define the map $T_{y_0} : (AC_b(X), \|\cdot\|_\Sigma) \longrightarrow \mathbb{C}$ by $T_{y_0}f = Tf(y_0)$ ($f \in AC_b(X)$), where $\|\cdot\|_\Sigma = \|\cdot\|_\infty + \mathscr{V}(\cdot)$. Clearly, since T is a 2-local isometry, T_{y_0} is 1-homogeneous, and according to Theorem 3.1, for each pair $f, g \in AC_b(X)$, there exist a scalar $\lambda_{f,g} \in \mathbb{T}$ and a monotonic homeomorphism $\varphi_{f,g} : Y \longrightarrow X$ such that

$$T_{y_0}f = \lambda_{f,g} f(\varphi_{f,g}(y_0)) \text{ and } T_{y_0}g = \lambda_{f,g}\, g(\varphi_{f,g}(y_0)).$$

Then

$$T_{y_0}f - T_{y_0}g = \lambda_{f,g}(f(\varphi_{f,g}(y_0)) - g(\varphi_{f,g}(y_0))) \in \mathbb{T}\,\sigma(f-g),$$

where $\sigma(f-g)$ is the spectrum of $f-g$. By Proposition 3.2 in [12] it follows that T_{y_0} is linear. Consequently, T is a linear map since y_0 was arbitrary, which especially implies that $T : AC_b(X) \longrightarrow AC_b(Y)$ is a local isometry. Thus the result follows from Theorem 3.2.

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