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MAXIMAL NUMERICAL RANGE AND QUADRATIC ELEMENTS IN A C^*–ALGEBRA

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Dedicated to Professor A. Nokrane

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Abstract. In this paper, we give a description of the maximal numerical range of a hyponormal element and a characterization of a normaloid element in a C^*-algebra. We also give an explicit formula for the maximal numerical range of a quadratic operator acting on a complex Hilbert space. As a consequence, we determine the maximal numerical range of a rank-one operator.

1. Introduction

Let $\mathscr{A}$ be a complex C^*-algebra with unit e and let $\mathscr{A}'$ be its dual space. Define the state space of $\mathscr{A}$ by

$$\mathscr{S}(\mathscr{A}) = \{f \in \mathscr{A}' : f(e) = \|f\| = 1\}.$$

For $a \in \mathscr{A}$, the algebraic numerical range of a is given by

$$V(a) = \{f(a) : f \in \mathscr{S}(\mathscr{A})\}.$$

It is well-known that $V(a)$ $(a \in \mathscr{A})$ is a convex compact set and contains the convex hull of the spectrum $\sigma(a)$ of a; that is $co\big(\sigma(a)\big) \subseteq V(a)$, here co stands for the convex hull. This result follows at once from the corresponding properties of the set $\mathscr{S}(\mathscr{A})$. See, for more details, [16]. Let $w(a)$ denote the numerical radius of $a \in \mathscr{A}$; i.e., $w(a) = \sup\{|\lambda| : \lambda \in V(a)\}$. It is well-known that $w(\cdot)$ defines a norm on $\mathscr{A}$, which is equivalent to the C^*-norm $\|\cdot\|$. In fact, the following inequalities are well-known:

$$\frac{1}{2}\|a\| \leqslant w(a) \leqslant \|a\|,$$

for all $a \in \mathscr{A}$. An element $a \in \mathscr{A}$ is said to be normaloid if $w(a) = \|a\|$. Recall that an element $a \in \mathscr{A}$ is said to be positive and we write $a \geqslant 0$ if it is self-adjoint and if its spectrum contains only non-negative real numbers. Recall also that an element $a \in \mathscr{A}$

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is called normal (resp. hyponormal) if $a^*a = aa^*$ (resp. $a^*a - aa^* \geqslant 0$ or equivalently, $a^*a - aa^* = b^*b$ for some $b \in \mathscr{A}$). Here a^* is the adjoint of a. It is well-known that hyponormal, thus also normal, elements in $\mathscr{A}$ are normaloid.

Let $\mathscr{H}$ be a Hilbert space over the complex field $\mathbb{C}$ with inner product $\langle x, y\rangle$ and norm $\|x\| = \langle x, x\rangle^{1/2}$. Denote by $\mathscr{B}(\mathscr{H})$ the C^*-algebra of all bounded linear operators acting on $\mathscr{H}$. For $A \in \mathscr{B}(\mathscr{H})$, the numerical range of A is defined as the set

$$W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H},\ \|x\| = 1\}.$$

It is a celebrated result due to Toeplitz-Hausdorff that $W(A)$ is a convex set in the complex plane and it is well-known that $\overline{W(A)} = V(A)$, where $\overline{L}$ is the closure of a subset L of $\mathbb{C}$. The numerical range of an operator in $\mathscr{B}(\mathscr{H})$ is closed if $\dim(\mathscr{H}) < \infty$, but it is not always closed when $\dim(\mathscr{H}) = \infty$. For more details about the theory of numerical ranges, the reader is referred to [4, 5, 8, 9] and references therein.

The notion of the numerical range has been generalized in different directions. One such direction is the maximal numerical range. It is a relatively new concept in operator theory, having been introduced only in 1970 by Stampfli [17] and defined as follows.

DEFINITION 1.1. For $A \in \mathscr{B}(\mathscr{H})$, the maximal numerical range $W_0(A)$ of A is given by

$$W_0(A) = \{\lim_n \langle Ax_n, x_n\rangle :\ x_n \in \mathscr{H},\ \|x_n\| = 1,\ \lim_n \|Ax_n\| = \|A\|\}.$$

It was shown in [17] that $W_0(A)$ is nonempty, closed, convex and contained in the closure of the numerical range; $W_0(A) \subseteq \overline{W(A)}$. In the case of finite-dimensional spaces, the maximal numerical range is produced by maximal vectors for A (vectors $x \in \mathscr{H}$ such that $\|x\| = 1$ and $\|Ax\| = \|A\|$). Note that the notion of the maximal numerical range was introduced by Stampfli [17] (especially) for the purpose of calculating the norm of the inner derivation on $\mathscr{B}(\mathscr{H})$. Recall that the inner derivation δ_A associated with $A \in \mathscr{B}(\mathscr{H})$ is defined by

$$\delta_A : \mathscr{B}(\mathscr{H}) \longrightarrow \mathscr{B}(\mathscr{H}),\ X \longmapsto AX - XA.$$

Indeed, the author of [17] established the following. For any $A \in \mathscr{B}(\mathscr{H})$

$$\|\delta_A\| = 2\,\|A - c_A\|,$$

where c_A is the unique scalar c_A satisfying

$$\|A - c_A\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|.$$

The scalar c_A is called the center of mass of A. In the same paper [17], Stampfli proved that we always have $c_A \in \overline{W(A)}$. Furthermore, if A is a hyponormal operator, the center of mass c_A is exactly the center of the smallest disk containing the spectrum $\sigma(A)$.

Recently, considerable interests have been given to the maximal numerical range, see, for instance, [3, 6, 10, 12, 15]. For example, in [3], the authors gave the following description of the maximal numerical range $W_0(A)$ whenever A is hyponormal.

THEOREM 1.2. ([3]) *Let $A \in \mathscr{B}(\mathscr{H})$ be hyponormal. Then*

$$W_0(A) = co\big(\sigma_n(A)\big),$$

where $\sigma_n(A) := \{\lambda \in \sigma(A) : |\lambda| = \|A\|\}$.

In [6] the authors gave the following characterization of normaloid operator.

THEOREM 1.3. ([6]) *Let $A \in \mathscr{B}(\mathscr{H})$. Then A is normaloid if and only if $w(A) = w_0(A)$, where $w_0(a) := \sup\{|\lambda| : \lambda \in W_0(A)\}$, the maximal numerical radius of A.*

In [7], the author introduced the concept of the algebraic maximal numerical range of an element $a \in \mathscr{A}$ as follows.

DEFINITION 1.4. Let $a \in \mathscr{A}$. The algebraic maximal numerical range of a is the set

$$V_0(a) = \{f(a) : f \in \mathscr{S}_{max}(a)\},$$

where $\mathscr{S}_{max}(a)$ is the set of all maximal states for a defined by

$$\mathscr{S}_{max}(a) := \{f \in \mathscr{S}(\mathscr{A}) : f(a^*a) = \|a\|^2\}.$$

In the same paper [7], the author established the following.

THEOREM 1.5. ([7]) *Let $a \in \mathscr{A}$. Then $V_0(a)$ is a non-empty convex compact subset of $V(a)$. Moreover, if $\mathscr{A} = \mathscr{B}(\mathscr{H})$ then $V_0(a) = W_0(a)$.*

Recall that a bounded linear operator $A \in \mathscr{B}(\mathscr{H})$ is called quadratic if it satisfies some non-trivial quadratic equation $(A - \alpha I)(A - \beta I) = 0$, where I is the operator identity on $\mathscr{H}$ and $\alpha, \beta \in \mathbb{C}$. We have the following.

THEOREM 1.6. ([1, 14, 18]) *Let $A \in \mathscr{B}(\mathscr{H})$ be a quadratic operator satisfying $(A - \alpha I)(A - \beta I) = 0$ for some scalars α and β. Then*

(a) A is unitarily equivalent to an operator of the form

$$\alpha I_1 \oplus \beta I_2 \oplus \begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix} \text{ on } \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus (\mathscr{H}_3 \oplus \mathscr{H}_3),$$

where $\mathscr{H}_1, \mathscr{H}_2$ and $\mathscr{H}_3$ are complex Hilbert spaces with T being positive semi-definite on $\mathscr{H}_3$.

(b)

$$\|A\| = \left\| \begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\| = \frac{1}{\sqrt{2}} \sqrt{u + \sqrt{u^2 - v}},$$

where $u = |\alpha|^2 + |\beta|^2 + \|T\|^2$ and $v = 4|\alpha|^2|\beta|^2$.

PROPOSITION 1.7. ([1]) *Let $A \in \mathscr{B}(\mathscr{H})$ be a quadratic operator satisfying $(A-\alpha I)(A-\beta I)=0$ for some scalars α and β. Then, the center of mass of A is*

$$c_A = \frac{\alpha+\beta}{2}.$$

THEOREM 1.8. ([10]) *Let $A = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix}$, where $\alpha,\beta,\gamma \in \mathbb{C}$. Then*

$$\begin{cases} W_0(A) = \left\{ \dfrac{\|A\|^2(\alpha+\beta)-\alpha\beta(\overline{\alpha}+\overline{\beta})}{2\|A\|^2-|\alpha|^2-|\beta|^2-|\gamma|^2} \right\}, \ \textit{if } \gamma \neq 0 \textit{ or } |\alpha| \neq |\beta|; \\ W_0(A) = [\alpha,\beta], \ \textit{otherwise.} \end{cases}$$

In Section 2, we establish some results regarding hyponormal elements and normaloid elements in a complex C^*-algebra that generalize Theorem 1.2 and Theorem 1.3. We point out a gap in the proof of [7, Proposition 5.2] and give a correct proof of it. In Section 3, we provide an explicit formula for the maximal numerical range of a quadratic operator using the fact that a quadratic operator is unitarily equivalent to a direct sum of operators relatively well-known. As a corollary, we determine the maximal numerical range and the center of mass of a rank-one operator.

2. The algebraic maximal numerical range of a hyponormal element in a C^*-algebra

We give a description of the algebraic maximal numerical range $V_0(a)$ when $a \in \mathscr{A}$ is hyponormal which will be a generalization of Theorem 1.2. We also give a generalization of Theorem 1.3. For this purpose, we need the following results. The first one is known as the Gelfand-Naimark theorem.

THEOREM 2.1. ([2]) *Let $\mathscr{A}$ be a C^*-algebra with unit e. Then, there exist a complex Hilbert space $\mathscr{H}$ and an isometric $*$-morphism T from $\mathscr{A}$ onto a closed self-adjoint subalgebra $\mathfrak{B}$ of $\mathscr{B}(\mathscr{H})$.*

In the sequel, we shall denote $T(a)$ by T_a for all $a \in \mathscr{A}$. Therefore, we have $\|T_a\| = \|a\|$, $T_{ab} = T_aT_b$, $T_e = I$ (where I is the operator identity on $\mathscr{H}$) and $T_{a^*} = (T_a)^*$ for all $a,b \in \mathscr{A}$. Moreover, $a \in \mathscr{A}$ is invertible if and only if T_a is invertible. In that case, $(T_a)^{-1} = T_{a^{-1}}$. In particular, $\sigma(a) = \sigma(T_a)$. As a consequence of these properties, we have the following. Let $a \in \mathscr{A}$, then there is a unique scalar c_a (also called the center of mass of a) such that

$$\|a - c_a\| = \inf_{\lambda \in \mathbb{C}} \|a - \lambda\|.$$

Moreover, $c_a = c_{T_a}$.

LEMMA 2.2. *Let $a \in \mathscr{A}$. Then the following hold*

1. $V(a) = V(T_a)$;
2. $V_0(a) = V_0(T_a)$.

Proof. We give a proof of the second assertion, the proof of the first one is similar. Let $\lambda \in V_0(a)$. Then, there is $f \in \mathscr{S}_{max}(a)$ such that $f(a) = \lambda$. Define g on $\mathfrak{B}$ by $g(T_x) := f(x)$ for all $x \in \mathscr{A}$. It is clear that $g \in \mathscr{S}(\mathfrak{B})$. By the Hahn-Banach theorem, we may extend g to $\tilde{g} \in \mathscr{S}(\mathscr{B}(\mathscr{H}))$. Moreover,

$$\tilde{g}(T_a^* T_a) = g(T_a^* T_a) = g(T_{a^*a}) = f(a^*a) = \|a\|^2 = \|T_a\|^2.$$

Thus, $\tilde{g} \in \mathscr{S}_{max}(T_a)$ and since $\tilde{g}(T_a) = g(T_a) = f(a) = \lambda$, then $\lambda \in V_0(T_a)$. Consequently, $V_0(a) \subseteq V_0(T_a)$. A similar argument gives the other inclusion. We then obtain the desired result. $\square$

PROPOSITION 2.3. *Let $a \in \mathscr{A}$ be hyponormal. Then*

$$V_0(a) = co\big(\sigma_n(a)\big),$$

where $\sigma_n(a) := \{\lambda \in \sigma(a) : |\lambda| = \|a\|\}$.

Proof. Let $a \in \mathscr{A}$ be hyponormal. It is easy to show that the operator T_a is hyponormal and $\sigma_n(T_a) = \sigma_n(a)$. Using Lemma 2.2 and Theorem 1.2, we get

$$V_0(a) = V_0(T_a) = co\big(\sigma_n(T_a)\big) = co\big(\sigma_n(a)\big)$$

as required. $\square$

REMARK 2.4. Let $a \in \mathscr{A}$ and define the maximal numerical radius of a as follows

$$w_0(a) := \sup\{|\lambda| : \lambda \in V_0(a)\}.$$

From Lemma 2.2, we derive that $w(a) = w(T_a)$ and $w_0(a) = w_0(T_a)$. So, since $\|a\| = \|T_a\|$, it follows that a is normaloid if and only if T_a is normaloid. According to Theorem 1.3, we have the following.

PROPOSITION 2.5. *Let $a \in \mathscr{A}$. Then a is normaloid if and only if $w(a) = w_0(a)$.*

In the proof of [7, Proposition 5.2], Fong used the following statement. If $a \in \mathscr{A}$, then $\mathscr{S}_{max}(a) = \mathscr{S}_{max}(a^*)$. But, this statement is not true in general. Indeed, let $\mathscr{H} = \ell_2$ be the complex Hilbert space of square summable sequences and let S be the right shift operator on $\mathscr{H}$ defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. We show that $\mathscr{S}_{max}(S) \neq \mathscr{S}_{max}(S^*)$. It is known that $\|S\| = 1$ and $S^*S = I$. Then, for any $f \in \mathscr{S}(\mathscr{B}(\mathscr{H}))$ we have $f(S^*S) = f(I) = 1 = \|S\|^2$. It results that $\mathscr{S}_{max}(S) = \mathscr{S}(\mathscr{B}(\mathscr{H}))$. But, $\langle SS^*(1,0,0,\ldots), (1,0,0,\ldots)\rangle = 0$, so $0 \in V(SS^*)$ and hence $0 = g(SS^*)$ for some $g \in \mathscr{S}(\mathscr{B}(\mathscr{H}))$. Since $\|S^*\| = 1$, $g \notin \mathscr{S}_{max}(S^*)$.

PROPOSITION 2.6. [7, Proposition 5.2] *Let $a \in \mathscr{A}$. Then*

$$V_0(a^*) = V_0(a)^*,$$

where for a subset Λ of $\mathbb{C}$, $\Lambda^* := \{\overline{\lambda} : \lambda \in \Lambda\}$.

We now give a correct proof of this proposition.

Proof. Let $a \in \mathscr{A}$. According to [11, Proposition 2], $V_0(T_a^*) = V_0(T_a)^*$ and by Lemma 2.2, $V_0(a^*) = V_0(T_a^*) = V_0(T_a)^* = V_0(a)^*$. $\square$

3. Maximal numerical range of a quadratic operator

In this section, we calculate the maximal numerical range of a quadratic operator on a complex Hilbert space. Let $A \in \mathscr{B}(\mathscr{H})$ be a quadratic operator satisfying the following quadratic equation $(A-\alpha I)(A-\beta I) = 0$, where $\alpha, \beta \in \mathbb{C}$. From Theorem 1.6, there exist complex Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ and $\mathscr{H}_3$ such that A is unitarily equivalent to an operator of the form

$$\alpha I_1 \oplus \beta I_2 \oplus \begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix} \text{ on } \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus (\mathscr{H}_3 \oplus \mathscr{H}_3),$$

with T being positive semi-definite on $\mathscr{H}_3$. According to [11, Lemma 2], $W_0(A) = W_0\left(\begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix}\right)$. Therefore, we can assume that $A = \begin{bmatrix} \alpha I & T \\ 0 & \beta I \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, with T is positive. The following theorem is a generalization of Theorem 1.8.

THEOREM 3.1. *Let $A = \begin{bmatrix} \alpha I & T \\ 0 & \beta I \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. Then*

$$\begin{cases} W_0(A) = \left\{ \dfrac{\|A\|^2(\alpha+\beta) - \alpha\beta(\overline{\alpha}+\overline{\beta})}{2\|A\|^2 - |\alpha|^2 - |\beta|^2 - \|T\|^2} \right\}, & \text{if } T \neq 0 \text{ or } |\alpha| \neq |\beta|; \\ W_0(A) = [\alpha, \beta], & \text{otherwise.} \end{cases}$$

Proof. We show that $W_0(A) = W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$ and we then conclude by Theorem 1.8. If $T = 0$, the result is clear since A is normal and so we apply Theorem 1.2. If $T \neq 0$ and $\alpha = 0$ then by Theorem 1.8, $W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right) = \{\beta\}$. We also have $W_0(A) = \{\beta\}$. Indeed, let $\lambda \in W_0(A)$, then there is $x_n = y_n \oplus z_n \in \mathscr{H} \oplus \mathscr{H}$ with $\|y_n\|^2 + \|z_n\|^2 = 1$ such that $\lim_n \langle Ax_n, x_n\rangle = \lambda$ and $\lim_n \|Ax_n\|^2 = \|A\|^2 = \|T\|^2 + |\beta|^2$. Since $\|Ax_n\|^2 = \|Tz_n\|^2 + |\beta|^2\|z_n\|^2$, then $\lim_n \|z_n\| = 1$ and $\lim_n \|y_n\| = 0$. We derive that $\lim_n \langle Ax_n, x_n\rangle = \lim_n \left(\langle Tz_n, y_n\rangle + \beta |z_n|^2\right) = \beta$. Therefore, we may assume that $T \neq 0$ and $\alpha \neq 0$.

We show that $W_0(A) \subseteq W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$. Let $\lambda \in W_0(A)$, then there exists a unit vector sequence $\{x_n\}$ in $\mathscr{H} \oplus \mathscr{H}$ such that $\lim_n \|Ax_n\| = \|A\|$ and $\lim_n \langle Ax_n, x_n\rangle = \lambda$. We decompose x_n as $\alpha_n y_n \oplus \beta_n z_n$ where $|\alpha_n|^2 + |\beta_n|^2 = 1$ and $\|y_n\| = \|z_n\| = 1$. Note that we can assume that $\alpha\alpha_n\overline{\beta_n} \geqslant 0$. Therefore, we have

$$\begin{aligned}
\|Ax_n\|^2 &= |\alpha|^2|\alpha_n|^2 + 2\alpha\alpha_n\overline{\beta_n} Re\left(\langle Tz_n, y_n\rangle\right) + |\beta_n|^2\|Tz_n\|^2 + |\beta|^2|\beta_n|^2 \\
&\leqslant |\alpha|^2|\alpha_n|^2 + 2\alpha\alpha_n\overline{\beta_n}\|T\| + |\beta_n|^2\|T\|^2 + |\beta|^2|\beta_n|^2 \\
&= \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\|^2 \\
&\leqslant \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\|^2 \\
&= \|A\|^2 \quad \big(\text{by Theorem 1.6.(b)}\big).
\end{aligned}$$

Since $\lim_n \|Ax_n\| = \|A\|$, we derive that

$$\lim_n \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\|.$$

A simple computation shows that

$$\langle Ax_n, x_n\rangle = \alpha|\alpha_n|^2 + \beta_n\overline{\alpha_n}\langle Tz_n, y_n\rangle + \beta|\beta_n|^2$$

and

$$\left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\rangle = \alpha|\alpha_n|^2 + \beta_n\overline{\alpha_n}\|T\| + \beta|\beta_n|^2.$$

Note that the sequence $\{\beta_n\overline{\alpha_n}\}$ is bounded, so that we may assume, by passing to a subsequence if necessary, it is convergent. If $\lim_n \beta_n\overline{\alpha_n} = 0$, then $\lim_n \langle Ax_n, x_n\rangle = \lim_n \left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\rangle$, so $\lambda \in W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$. If $\lim_n \beta_n\overline{\alpha_n} \neq 0$, since $\lim_n \alpha_n\overline{\beta_n}\Big(Re\left(\langle Tz_n, y_n\rangle\right) - \|T\|\Big) = 0$, then $\lim_n Re\left(\langle Tz_n, y_n\rangle\right) = \|T\|$. This implies $\lim_n \langle Tz_n, y_n\rangle = \|T\|$ and, as above, we again have $\lambda \in W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$. Consequently, $W_0(A) \subseteq W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$.

We now show that $W_0\left(\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\right) \subseteq W_0(A)$. Let $\lambda \in W_0\left(\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\right)$, then there exist $a,b \in \mathbb{C}$ such that $|a|^2+|b|^2=1$,

$$\left\|\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}\right\| = \left\|\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\right\|$$

and

$$\left\langle\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}, \begin{bmatrix}a \\ b\end{bmatrix}\right\rangle = \lambda.$$

Let z_n be unit vectors in $\mathscr{H}$ such that $\lim_n \|Tz_n\| = \|T\|$. Set $y_n := Tz_n/\|Tz_n\|$ and $x_n := ay_n \oplus bz_n$. We have

$$\|Ax_n\|^2 = |\alpha|^2|a|^2 + 2Re\left(\alpha a\overline{b}\right)\|Tz_n\| + |b|^2\|Tz_n\|^2 + |\beta|^2|b|^2$$

and

$$\left\|\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}\right\|^2 = |\alpha|^2|a|^2 + 2Re\left(\alpha a\overline{b}\right)\|T\| + |b|^2\|T\|^2 + |\beta|^2|b|^2.$$

Hence

$$\lim_n \|Ax_n\| = \left\|\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}\right\| = \left\|\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\right\| = \|A\|.$$

On the other hand,

$$\langle Ax_n, x_n\rangle = \alpha|a|^2 + b\overline{a}\|Tz_n\| + \beta|b|^2$$

and

$$\left\langle\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}, \begin{bmatrix}a \\ b\end{bmatrix}\right\rangle = \alpha|a|^2 + b\overline{a}\|T\| + \beta|b|^2.$$

We derive that

$$\lim_n \langle Ax_n, x_n\rangle = \left\langle\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}, \begin{bmatrix}a \\ b\end{bmatrix}\right\rangle = \lambda.$$

It follows that $\lambda \in W_0(A)$. Thus, $W_0\left(\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\right) \subseteq W_0(A)$. In summary, $W_0(A) = W_0\left(\begin{bmatrix}\alpha & \|T\| \\ 0 & \beta\end{bmatrix}\right)$. This completes the proof. $\square$

REMARK 3.2. An element $a \in \mathscr{A}$ is called quadratic if there exist two scalars α, β such that $(a-\alpha e)(a-\beta e)=0$. It is clear that $a \in \mathscr{A}$ is quadratic if and only if T_a is quadratic. Then, from Lemma 2.2, Proposition 1.7 and Theorem 3.1, we have the following.

COROLLARY 3.3. *Let $a \in \mathscr{A}$ be a quadratic operator satisfying $(a-\alpha e)(a-\beta e)=0$ for some scalars α and β. The algebraic maximal numerical range of a is either a point or the line segment $[\alpha,\beta]$ connecting α and β. Moreover, the center of mass of a is $c_a = \dfrac{\alpha+\beta}{2}$.*

As a consequence, we give a result concerning rank-one operators. Every rank-one operator is quadratic. Indeed, there exist $f \in \mathscr{H}'$ and $u \in \mathscr{R}(T)$ (the range of T) such that $T(x) = f(x)u$ for all $x \in \mathscr{H}$. Then $T^2(x) - f(u)T(x) = 0$ for all $x \in \mathscr{H}$. That is $T^2 - f(u)T = 0$. Hence, T is quadratic $\big(\alpha = 0$ and $\beta = f(u)\big)$. Moreover, by the Riesz representation theorem, there exists $v \in \mathscr{H}$ such that $f(\cdot) = \langle \cdot, v\rangle$. Then $T = u \otimes v$. According to Theorem 3.1 and Proposition 1.7, we have the following result concerning the maximal numerical range and the center of mass of a rank-one operator on a complex Hilbert space.

PROPOSITION 3.4. *Let $T \in \mathscr{B}(\mathscr{H})$ be a rank-one operator. Then $W_0(T) = \{\langle u,v\rangle\}$ and $c_T = \dfrac{\langle u,v\rangle}{2}$, where $u,v \in \mathscr{H}$ are such that $T = u\otimes v$.*

Note that we can obtain the previous result by observing that T is unitarily equivalent to $\begin{bmatrix} \langle u,v\rangle & \|v\|^2 \\ 0 & 0 \end{bmatrix} \oplus 0$ and using [11, Lemma 2] and Theorem 1.8.

REMARK 3.5. Note that for stating Theorem 3.1 we used [11, Lemma 2] which asserts the following result. Let $A_1 \in \mathscr{B}(\mathscr{H}_1)$ and $A_2 \in \mathscr{B}(\mathscr{H}_2)$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are complex Hilbert spaces. For A unitarily equivalent to $A_1 \oplus A_2$,

$$W_0(A) = co\Big(\bigcup_{\|A_j\|=\|A\|} W_0(A_j)\Big).$$

This result can be generalized by induction to the finite direct sum case. But, it is not true in the infinite direct sum case in general. Indeed, let $\{B_k\}$ for $k=1,2,\ldots,$ be the operators on the complex Hilbert space $\mathscr{H} = \mathbb{C}^2$ represented by

$$B_k = \begin{bmatrix} 1 & 0 \\ 0 & -1+\dfrac{1}{k} \end{bmatrix}, \quad k = 1,2,\ldots .$$

It is known that $\overline{\bigcup_k \sigma(B_k)} \subseteq \sigma\big(\oplus_k B_k\big)$. That is, $\overline{\bigcup_k \{-1+\frac{1}{k},1\}} \subseteq \sigma\big(\oplus_k B_k\big)$. It results that $\{-1,1\} \subset \sigma\big(\oplus_k B_k\big)$ and since $\|\oplus_k B_k\| = 1$, then $\{-1,1\} \subseteq \sigma_n\big(\oplus_k B_k\big)$. From [15, Lemma 1], $\sigma_n\big(\oplus_k B_k\big) \subseteq W_0(\oplus_k B_k)$. We derive that $\{-1,1\} \subseteq W_0(\oplus_k B_k)$. But, $W_0(B_k) = \{1\}$, for $k=1,2,\ldots$, then $\bigcup_k W_0(B_k) = \{1\}$. Consequently, $co\Big(\bigcup_k W_0(B_k)\Big) \subsetneq W_0(\oplus_k B_k)$. However, we have the following.

PROPOSITION 3.6. *Let $\{\mathscr{H}_n\}$ be a collection of complex Hilbert spaces, let $\{T_n\}$ be a collection of hyponormal operators with $T_n \in \mathscr{B}(\mathscr{H}_n)$. Assume that $\sup_n \|T_n\| < \infty$ and consider the direct sum $T = \oplus_n T_n \in \mathscr{B}\big(\oplus_n \mathscr{H}_n\big)$. Then*

$$W_0(T) = co\Big(\overline{\bigcup_k \sigma(T_k)} \cap C_T\Big),$$

where $C_T := \{\lambda : |\lambda| = \|T\|\}$.

Proof. Since $\{T_n\}$ is a collection of hyponormal operators, then T is hyponormal. By virtue of Theorem 1.2, $W_0(T) = co\big(\sigma_n(T)\big)$. According to [13, Proposition 2.F], we have $\sigma(T) = \overline{\bigcup_k \sigma(T_k)}$ and the result follows. $\square$

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