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CANONICAL DECOMPOSITION FOR DISSIPATIVE LINEAR RELATIONS

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Abstract. This work presents two ways of decomposing dissipative linear relations based on the fundamental decompositions for contractions by Sz. Nagy-Foiaş-Langer and von Neumann-Wold. The invariant subspaces for contractions are treated using a minor variation of the Cayley transform. The obtained decompositions allow one to separate the selfadjoint and completely nonselfadjoint parts of a dissipative relation.

1. Introduction

This paper deals with the canonical decomposition of any closed dissipative linear relation into its selfadjoint and completely nonselfadjoint parts. In particular, this provides the corresponding decomposition for a symmetric relation even when deficiency indices are not necessarily equal (cf. [6, Sec. 3.4 and particularly Thm. 3.4.4]).

A linear relation in the Hilbert space $\mathscr{H}$ is a linear subset of $\mathscr{H} \oplus \mathscr{H}$. The concept of linear relation generalizes the notion of operator when it is identified with its graph. Indeed, a linear relation is an operator whenever its multivalued part (see Section 2) is the trivial subspace in $\mathscr{H}$. For this reason, some authors refer to relations as multivalued linear operators [13]. The fundamentals of linear relations are found in [1, 3, 13, 23, 25].

Linear relations are used extensively in extension theory for linear operators, particularly in the setting of boundary triplets and when extending nondensely defined operators. Relations are also relevant in singular perturbation theory, the theory of canonical systems, and some boundary value problems of partial differential equations. In all these applications, the class to which the linear relation belongs plays an important role; the multivalued generalization of dissipative, symmetric and selfadjoint operators are dissipative, symmetric and selfadjoint linear relations, respectively.

In this work we deal with dissipative relations (see Definition 6 and [5, 11, 15, 31, 32] for an account of the matter). Dissipative linear relations appear in problems of mathematical physics where one has dissipative systems, i.e., systems in which the energy is in general nonconstant and nonincreasing in time (see for example [14, 17, 19, 28]).

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R. S. Phillips introduced dissipative operators in his seminal work [29] motivated by the Cauchy problem for dissipative hyperbolic systems of partial differential equations. He showed that a maximal dissipative operator generates a strongly continuous semi-group of contraction operators (see also [38]). Other applications of dissipative operators can be found in [4, 20, 21, 39, 40].

The theory of dissipative operators is tightly connected with the theory of contractions, i.e., linear operators T such that $\|T\| \leqslant 1$ (see [36, 37] for an exhaustive exposition of the theory of contractions). The class of contractions has been amply studied and is a well-understood class of operators (some generalizations of the concept of contraction can be found in [12, 16]). We would point out that a motivation for studying contractions stems from the invariant subspace problem [22, 30, 37]. By means of a Cayley transform [37, Chap. 4, Sec. 4], any dissipative operator is transformed into a contraction. Thus, one class of operators can be studied through the other class. Although this bridge can be extended even in the case of dissipative relations, we use in this work a slight variation of the Cayley transform, namely the Z transform (cf. [18]). The use of this transform leads to some simplifications that will be evident below.

Contractions admit decompositions which turn out to be crucial for this work. We focus our attention on two ways of decomposing contractions, namely the Sz. Nagy-Foiaş-Langer and the von Neumann-Wold decompositions [27, 37] (see [35] for a more general setting). The main goal is to give a decomposition of any closed dissipative linear relation, in which one isolates its selfadjoint and completely nonselfadjoint parts (see Theorem 5). This decomposition is carried out for arbitrary maximal dissipative operators in [10, Thm. 7.6], where additionally it is shown how the decomposition is related to the corresponding dilations. Theorem 7 shows in particular that any symmetric relation, which does not admit dissipative proper extensions, is separated into its selfadjoint part and its elementary-maximal part, which is a relation whose Z transform is a unilateral shift. These decompositions are made by means of transforming invariant subspaces for contractions. It is worth mentioning that the decompositions given in this work have applications in several problems where dissipative relations naturally arise, in particular, in the theory of boundary and quasi-boundary triples (see [7, 8, 24, 33]) and the functional models for dissipative operators [10, 26, 34].

The paper is organized as follows. In Section 2 we briefly recall some standard definitions in the theory of linear relations. Also, this section tackles the problem of invariant and reducing subspaces for linear relations. It is shown here that for relations the adjoint of the decomposition given by a reducing subspace is the decomposition of the adjoint relations with respect to the same reducing subspace (see Theorem 1). Another result of this section is that any linear relation of the form $\mathscr{K} \oplus \mathscr{K}$, where $\mathscr{K}$ is a linear set, is invariant under the Z transform (see Remark 4). A consequence of this is that the Z transform preserves reducing subspaces for any linear relation (see Theorem 2). Section 3 deals with the general theory of contractions, in particular, the Sz. Nagy-Foiaş-Langer and the von Neumann-Wold decompositions. The Sz. Nagy-Foiaş-Langer decomposition is extended to any closed contraction (see Theorem 3). These results, together with the theory of reducing subspaces developed in the preceding section, are combined with the theory of the Z transform to obtain the required decomposition of any closed dissipative relation. Finally, as an illustration of the general results of this work, Section 4 presents a simple decomposition of a maximal symmetric relation with nontrivial multivalued part.

2. Invariant and reducing subspaces for linear relations

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space, with inner product antilinear in its left argument. We denote $\mathscr{H}\oplus\mathscr{H}$ as the orthogonal sum of two copies of the Hilbert space $\mathscr{H}$ (q.v. [9, Sec. 2.3]). Throughout this work, any linear set T in $\mathscr{H}\oplus\mathscr{H}$, is called a *linear relation* (or relation, for short). We use "linear set" instead of the usual "subspace" since this last term is reserved for closed linear sets. The following sets are associated with the linear relation T:

$$\operatorname{dom} T := \left\{ f\in\mathscr{H} : \binom{f}{g} \in T \right\} \qquad \operatorname{ran} T := \left\{ g\in\mathscr{H} : \binom{f}{g} \in T \right\}$$

$$\ker T := \left\{ f\in\mathscr{H} : \binom{f}{0} \in T \right\} \qquad \operatorname{mul} T := \left\{ g\in\mathscr{H} : \binom{0}{g} \in T \right\}$$

The concept of linear relation generalizes the notion of linear operator. Namely, a relation T is an operator (when it is identified by its graph) if and only $\operatorname{mul} T = \{0\}$.

For two relations T,S and $\zeta\in\mathbb{C}$, we denote the following linear relations:

$$T+S := \left\{ \binom{f}{g+h} : \binom{f}{g}\in T,\ \binom{f}{h}\in S \right\} \qquad \zeta T := \left\{ \binom{f}{\zeta g} : \binom{f}{g}\in T \right\}$$

$$ST := \left\{ \binom{f}{k} : \binom{f}{g}\in T,\ \binom{g}{k}\in S \right\} \qquad T^{-1} := \left\{ \binom{g}{f} : \binom{f}{g}\in T \right\}$$

The *adjoint* of a relation T is given by

$$T^* := \left\{ \binom{h}{k} \in \mathscr{H}\oplus\mathscr{H} : \langle k,f\rangle = \langle h,g\rangle,\ \forall \binom{f}{g}\in T \right\},$$

which turns out to be a closed relation with the following properties:

$$\begin{aligned} T^* &= (-T^{-1})^{\perp}, & S\subset T &\Rightarrow T^*\subset S^*, \\ T^{**} &= \overline{T}, & (\alpha T)^* &= \overline{\alpha}T^*, \text{ with } \alpha\neq 0, \\ (T^*)^{-1} &= (T^{-1})^*, & \ker T^* &= (\operatorname{ran} T)^{\perp}. \end{aligned} \tag{1}$$

The boundedness definition for relations is not unique (cf. [13]). Here, a relation T is *bounded* if there exists $C>0$ such that $\|g\|\leqslant C\|f\|$, for all $\binom{f}{g}\in T$. This condition implies that any bounded relation is an operator. In this fashion, the *regular* set of linear relation T is

$$\hat{\rho}(T) := \{\zeta\in\mathbb{C} : (T-\zeta I)^{-1} \text{ is bounded}\},$$

which is open. Besides, the *deficiency* space of T is given by

$$\mathbf{N}_\zeta(T) := \left\{ \binom{f}{\zeta f} \in T \right\}, \quad (\zeta\in\mathbb{C})$$

which is a bounded relation with $\operatorname{dom}\mathbf{N}_\zeta(T) = \ker(T-\zeta I)$. Moreover,

$$\dim \mathbf{N}_{\overline{\zeta}}(T^*), \quad \zeta \in \hat{\rho}(T) \tag{2}$$

remains constant, on each connected component of $\hat{\rho}(T)$ (cf. [9, Thm. 3.7.4]).

The *resolvent* set of a closed relation T is defined by

$$\rho(T) := \{\zeta \in \mathbb{C} : (T-\zeta I)^{-1} \in \mathscr{B}(\mathscr{H})\},$$

where $\mathscr{B}(\mathscr{H})$ denotes the class of bounded operators defined in the whole space $\mathscr{H}$. The resolvent set is open and consists of all connected components of $\hat{\rho}(T)$, in which (2) is equal zero. Also, we consider the following sets:

$$\begin{aligned}
&\sigma(T) := \mathbb{C}\backslash\rho(T) && \text{(spectrum)}\\
&\hat{\sigma}(T) := \mathbb{C}\backslash\hat{\rho}(T) && \text{(spectral core)}\\
&\sigma_p(T) := \{\zeta \in \mathbb{C} : \dim \mathbf{N}_\zeta(T) \neq 0\} && \text{(point spectrum)}\\
&\sigma_c(T) := \{\zeta \in \mathbb{C} : \operatorname{ran}(T-\zeta I) \neq \overline{\operatorname{ran}(T-\zeta I)}\} && \text{(continuous spectrum)}\\
&\sigma_r(T) := \sigma(T)\backslash\hat{\sigma}(T) && \text{(residual spectrum)}
\end{aligned}$$

Analogously to the case of operators, it follows

$$\sigma_p(T) \cup \sigma_c(T) = \hat{\sigma}(T). \tag{3}$$

REMARK 1. For a closed relation T, one has that $\sigma(T^*)$ is the complex conjugate of $\sigma(T)$ [31, Prop. 2.5]. The same holds for the continuous spectrum, since $\operatorname{ran}(T-\zeta I)$ and $\operatorname{ran}(T^*-\overline{\zeta}I)$ are simultaneously closed (cf. [15, Lem. 2.3]).

PROPOSITION 1. *Let T be a closed relation. If ζ belongs to $\sigma_r(T)$ then $\overline{\zeta}$ belongs to $\sigma_p(T^*)\backslash\sigma_c(T^*)$.*

Proof. Since $\zeta \in \sigma_r(T)$, one has that $(T-\zeta I)^{-1}$ is closed and bounded, which is not defined on the whole space. Thus, $\operatorname{ran}(T-\zeta I)$ is closed as well as $\operatorname{ran}(T^*-\overline{\zeta}I)$ and $\ker(T^*-\overline{\zeta}I) = [\operatorname{ran}(T-\zeta I)]^\perp \neq \{0\}$. These facts imply the required. $\square$

Before proceeding to the theory of invariant subspaces, we shall set the following. For a relation T in $\mathscr{H}\oplus\mathscr{H}$ and a linear set $\mathscr{K}$ in $\mathscr{H}$, we denote

$$T_{\mathscr{K}} := T \cap (\mathscr{K}\oplus\mathscr{K}),$$

where $\mathscr{K}\oplus\mathscr{K}$ is the orthogonal sum of $\mathscr{K}$ with itself. It is clear that $T_{\mathscr{H}} = T$ and $T_{\{0\}} = \{0\}\oplus\{0\}$.

DEFINITION 1. A subspace $\mathscr{K} \subset \mathscr{H}$ is called *invariant* for a relation T (briefly T-invariant), when the following conditions are true:

(i) $\operatorname{dom} T = (\operatorname{dom} T \cap \mathscr{K}) \oplus (\operatorname{dom} T \cap \mathscr{K}^{\perp})$.

(ii) $\operatorname{mul} T = (\operatorname{mul} T \cap \mathscr{K}) \oplus (\operatorname{mul} T \cap \mathscr{K}^{\perp})$.

(iii) $\operatorname{dom} T_{\mathscr{K}} = \operatorname{dom} T \cap \mathscr{K}$.

We see at once that $\mathscr{H}$ and $\{0\}$ are invariant, for any linear relation.

DEFINITION 2. A subspace $\mathscr{K} \subset \mathscr{H}$ is said *to reduce* a relation T if

$$T = T_{\mathscr{K}} \oplus T_{\mathscr{K}^{\perp}}.$$

The subspaces $\mathscr{K}$ and $\mathscr{K}^{\perp}$ reduce T simultaneously and in this case

$$\begin{aligned} \operatorname{dom} T &= \operatorname{dom} T_{\mathscr{K}} \oplus \operatorname{dom} T_{\mathscr{K}^{\perp}}, & \ker T &= \ker T_{\mathscr{K}} \oplus \ker T_{\mathscr{K}^{\perp}}, \\ \operatorname{ran} T &= \operatorname{ran} T_{\mathscr{K}} \oplus \operatorname{ran} T_{\mathscr{K}^{\perp}}, & \operatorname{mul} T &= \operatorname{mul} T_{\mathscr{K}} \oplus \operatorname{mul} T_{\mathscr{K}^{\perp}}. \end{aligned} \tag{4}$$

REMARK 2. The existence of relations $T_1 \subset \mathscr{K} \oplus \mathscr{K}$, $T_2 \subset \mathscr{K}^{\perp} \oplus \mathscr{K}^{\perp}$, for which $T = T_1 \oplus T_2$, implies that $\mathscr{K}$ reduces T and $T_1 = T_{\mathscr{K}}$, $T_2 = T_{\mathscr{K}^{\perp}}$.

PROPOSITION 2. *A subspace $\mathscr{K}$ reduces T if and only if $\mathscr{K}$ and $\mathscr{K}^{\perp}$ are T-invariant.*

Proof. If $\mathscr{K}$ reduces T, then by inclusion in both directions, one arrives at

$$\begin{aligned} \operatorname{dom} T_{\mathscr{K}} &= \operatorname{dom} T \cap \mathscr{K}, & \operatorname{ran} T_{\mathscr{K}} &= \operatorname{ran} T \cap \mathscr{K}, \\ \ker T_{\mathscr{K}} &= \ker T \cap \mathscr{K}, & \operatorname{mul} T_{\mathscr{K}} &= \operatorname{mul} T \cap \mathscr{K}. \end{aligned}$$

Hence, by (4), $\mathscr{K}$ is T-invariant. This also holds for $\mathscr{K}^{\perp}$, since it reduces T.

The converse follows once we show that $T \subset T_{\mathscr{K}} \oplus T_{\mathscr{K}^{\perp}}$. By inclusion, if $\binom{f}{g} \in T$ then the first condition of T-invariant implies that there exist

$$\binom{a}{s} \in T_{\mathscr{K}}; \quad \binom{b}{t} \in T_{\mathscr{K}^{\perp}}, \tag{5}$$

such that $f = a + b$. Thus, $\binom{f}{s+t} \in T$, which yields $\binom{0}{g-(s+t)} \in T$. The second condition of T-invariant indicates the existence of

$$\binom{0}{h} \in T_{\mathscr{K}}; \quad \binom{0}{k} \in T_{\mathscr{K}^{\perp}}, \tag{6}$$

such that $g - (s+t) = h + k$. Therefore, (5) and (6) produce

$$\binom{f}{g} = \binom{a}{s+h} + \binom{b}{t+k} \in T_{\mathscr{K}} \oplus T_{\mathscr{K}^{\perp}},$$

as required. $\square$

If $\mathscr{K}$ reduces T, then a simple computation shows that

$$\overline{T} = \overline{T}_{\mathscr{K}} \oplus \overline{T}_{\mathscr{K}^{\perp}}. \tag{7}$$

This implies that T is closed if and only if both $T_{\mathscr{K}}$, $T_{\mathscr{K}^{\perp}}$ are closed.

THEOREM 1. *If $\mathscr{K}$ reduces T, then $\mathscr{K}$ reduces T^* and the following holds*

$$(T_{\mathscr{K}} \oplus T_{\mathscr{K}^\perp})^* = (T_{\mathscr{K}})^* \oplus (T_{\mathscr{K}^\perp})^* . \tag{8}$$

Proof. By hypothesis $T = T_{\mathscr{K}} \oplus T_{\mathscr{K}^\perp}$. Besides, the first property of (1) yields

$$-(\overline{T}_{\mathscr{K}})^{-1} \oplus (T_{\mathscr{K}})^* = \mathscr{K} \oplus \mathscr{K}\,; \quad -(\overline{T}_{\mathscr{K}^\perp})^{-1} \oplus (T_{\mathscr{K}^\perp})^* = \mathscr{K}^\perp \oplus \mathscr{K}^\perp . \tag{9}$$

Thus, in view of (7) and (9),

$$\begin{aligned}
-(\overline{T})^{-1} \oplus [(T_{\mathscr{K}})^* \oplus (T_{\mathscr{K}^\perp})^*] &= -[\overline{T}_{\mathscr{K}} \oplus \overline{T}_{\mathscr{K}^\perp}]^{-1} \oplus [(T_{\mathscr{K}})^* \oplus (T_{\mathscr{K}^\perp})^*] \\
&= [-(\overline{T}_{\mathscr{K}})^{-1} \oplus (T_{\mathscr{K}})^*] \oplus [-(\overline{T}_{\mathscr{K}^\perp})^{-1} \oplus (T_{\mathscr{K}^\perp})^*] \\
&= (\mathscr{K} \oplus \mathscr{K}) \oplus (\mathscr{K}^\perp \oplus \mathscr{K}^\perp) \\
&= \mathscr{H} \oplus \mathscr{H} = -(\overline{T})^{-1} \oplus T^* ,
\end{aligned}$$

whence it yields (8). Also, (9) implies $(T_{\mathscr{K}})^* \subset \mathscr{K} \oplus \mathscr{K}$ and $(T_{\mathscr{K}^\perp})^* \subset \mathscr{K}^\perp \oplus \mathscr{K}^\perp$. Hence, one infers from Remark (2) that $\mathscr{K}$ reduces T^*. $\square$

REMARK 3. As a consequence of Theorem 1, if $\mathscr{K}$ reduces T then

$$(T_{\mathscr{K}})^* = (T^*)_{\mathscr{K}} \quad \text{and} \quad (T_{\mathscr{K}^\perp})^* = (T^*)_{\mathscr{K}^\perp} . \tag{10}$$

We shall introduce a version of the Cayley transform for linear relations (q.v. [18]).

DEFINITION 3. For a relation T and $\zeta \in \mathbb{C}$, we define the *Z transform* of T by

$$\mathsf{Z}_\zeta(T) := \left\{ \begin{pmatrix} g - \overline{\zeta} f \\ \overline{\zeta} g - |\zeta|^2 f \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\} .$$

The Z transform is a linear relation with

$$\begin{aligned}
\operatorname{dom} \mathsf{Z}_\zeta(T) &= \operatorname{ran}(T - \overline{\zeta} I), \quad \operatorname{ran} \mathsf{Z}_\zeta(T) = \operatorname{ran}(T - \zeta I), \\
\operatorname{mul} \mathsf{Z}_\zeta(T) &= \ker(T - \overline{\zeta} I), \quad \ker \mathsf{Z}_\zeta(T) = \ker(T - \zeta I).
\end{aligned} \tag{11}$$

For $\zeta \in \mathbb{C}$, the following holds (q.q. [15, Lems. 2.6, 2.7] and [18, Props. 3.6, 3.7]).

(i) $\mathsf{Z}_\zeta(\mathsf{Z}_\zeta(T)) = T$.

(ii) $\mathsf{Z}_\zeta(T) \subset \mathsf{Z}_\zeta(S) \Leftrightarrow T \subset S$.

(iii) $\mathsf{Z}_{-\zeta}(T) = -\mathsf{Z}_\zeta(-T)$.

(iv) $\mathsf{Z}_\zeta(T^{-1}) = \mathsf{Z}_{\overline{\zeta}}(T) = (\mathsf{Z}_\zeta(T))^{-1}$, if $|\zeta| = 1$.

Besides, for $\zeta \in \mathbb{C} \backslash \mathbb{R}$,

(v) $\mathsf{Z}_\zeta(T \dotplus S) = \mathsf{Z}_\zeta(T) \dotplus \mathsf{Z}_\zeta(S)$.

(vi) $\mathbf{Z}_{\pm i}(T \oplus S) = \mathbf{Z}_{\pm i}(T) \oplus \mathbf{Z}_{\pm i}(S)$.

(vii) $\mathbf{Z}_{\zeta}(T^*) = (\mathbf{Z}_{\overline{\zeta}}(T))^*$.

(viii) $\overline{\mathbf{Z}_{\zeta}(T)} = \mathbf{Z}_{\zeta}(\overline{T})$.

REMARK 4. For any linear set $\mathscr{K} \subset \mathscr{H}$, one has that $\mathbf{Z}_{\zeta}(\mathscr{K} \oplus \mathscr{K}) = \mathscr{K} \oplus \mathscr{K}$, for all $\zeta \in \mathbb{C}$. Indeed, it is straightforward to see that $\mathbf{Z}_{\zeta}(\mathscr{K} \oplus \mathscr{K}) \subset \mathscr{K} \oplus \mathscr{K}$ and the other inclusion follows applying the property (i) of the Z transform.

THEOREM 2. *For $\zeta \in \{i, -i\}$, a subspace $\mathscr{K}$ reduces a relation T if and only if it reduces $\mathbf{Z}_{\zeta}(T)$.*

Proof. If $\mathscr{K}$ reduces T, then $T = T_{\mathscr{K}} \oplus T_{\mathscr{K}^\perp}$ and $\mathbf{Z}_{\zeta}(T) = \mathbf{Z}_{\zeta}(T_{\mathscr{K}}) \oplus \mathbf{Z}_{\zeta}(T_{\mathscr{K}^\perp})$. Besides, since $T_{\mathscr{K}} \subset \mathscr{K} \oplus \mathscr{K}$ and $T_{\mathscr{K}^\perp} \subset \mathscr{K}^\perp \oplus \mathscr{K}^\perp$, one has by Remark 4 that

$$\mathbf{Z}_{\zeta}(T_{\mathscr{K}}) \subset \mathscr{K} \oplus \mathscr{K} \quad \text{and} \quad \mathbf{Z}_{\zeta}(T_{\mathscr{K}^\perp}) \subset \mathscr{K}^\perp \oplus \mathscr{K}^\perp,$$

i.e., $\mathscr{K}$ reduces $\mathbf{Z}_{\zeta}(T)$. The converse follows replacing T by $\mathbf{Z}_{\zeta}(T)$, in the above reasoning. $\square$

3. The canonical decomposition of dissipative relations

We begin this section with a brief exposition of contractions. We recall that a linear operator V in $\mathscr{H} \oplus \mathscr{H}$ (seen as a linear relation) is a *contraction* if it is bounded with $\|V\| \leqslant 1$. Particularly, V is an *isometry* if $V^{-1} \subset V^*$ or *unitary* whenever $V^{-1} = V^*$. In both cases, its norm is equal one. Besides, V is a *maximal contraction* if it does not admit proper contractive extensions. This property means that V belongs to $\mathscr{B}(\mathscr{H})$.

DEFINITION 4. A contraction V is said to be *completely nonunitary* (c.n.u. for short), whenever there is no nonzero reducing subspace $\mathscr{K}$ for V, in which $V_{\mathscr{K}}$ is unitary.

The following result is an extension of the so-called Sz. Nagy-Foiaş-Langer decomposition (cf. [37, Chap. I, Sec. 3, Thm. 3.2]), which is proven for contractions in $\mathscr{B}(\mathscr{H})$.

THEOREM 3. *For every closed contraction V, there exists a unique reducing subspace $\mathscr{K}$ for V, such that $V_{\mathscr{K}}$ is unitary and $V_{\mathscr{K}^\perp}$ is completely nonunitary.*

Proof. We begin by denoting

$$\hat{V} = V \oplus W, \quad \text{where} \quad W = \left\{ \begin{pmatrix} h \\ 0 \end{pmatrix} : h \in \mathscr{H} \ominus \operatorname{dom} V \right\}. \tag{12}$$

Inasmuch as $\operatorname{dom} V$ is closed, then $\hat{V}$ is a maximal contraction. So, the Sz. Nagy-Foiaş-Langer decomposition asserts that there exists a unique reducing subspace $\mathscr{K}$

for $\hat{V}$, such that $\hat{V}_{\mathscr{K}}$ is unitary and $\hat{V}_{\mathscr{K}^\perp}$ is c.n.u. Thus, for every $\binom{f}{g} \in \hat{V}_{\mathscr{K}} \subset \hat{V}$, in view of (12), there is $\binom{f_1}{g} \in V$ and $f_2 \in (\operatorname{dom} V)^\perp$, such that $f = f_1 + f_2$. Thereby,

$$\|f_1\|^2 \geqslant \|g\|^2 = \|f\|^2 = \|f_1 + f_2\|^2 = \|f_1\|^2 + \|f_2\|^2 ,$$

wherefrom $f_2 = 0$. Hence, $\hat{V}_{\mathscr{K}} \subset V_{\mathscr{K}}$ and they are the same, since $V \subset \hat{V}$. The previous reasoning implies that $\mathscr{K}$ reduces V as well as $W \subset \hat{V}_{\mathscr{K}^\perp}$. Furthermore, $V_{K^\perp} = \hat{V}_{\mathscr{K}^\perp} \ominus W$, which is a c.n.u. contraction. The uniqueness follows directly, bearing in mind that a reducing subspace for V, also reduces $\hat{V}$. $\square$

Let us turn our attention to a particular class of isometries, known as unilateral shifts.

DEFINITION 5. Let V be an isometric operator in $\mathscr{B}(\mathscr{H})$. A subspace $\mathscr{L} \subset \mathscr{H}$ is called *wandering* for V, if $V^m\mathscr{L} \perp V^n\mathscr{L}$, for all $n,m \in \mathbb{N} \cup \{0\}$, with $n \neq m$. Moreover, V is a *unilateral shift* if $\mathscr{L}$ satisfies

$$\mathscr{L} \oplus V\mathscr{L} \oplus V^2\mathscr{L} \oplus \ldots = \mathscr{H} . \tag{13}$$

The wandering space for a unilateral shift V is uniquely determined by means of $\mathscr{L} = \mathscr{H} \ominus \operatorname{ran} V$. Besides, it is straightforward to compute that

$$V^* = V^{-1} \oplus \left\{ \binom{l}{0} : l \in \mathscr{L} \right\}.$$

Let us introduce the following assertion which is well-known as the von Neumann-Wold decomposition [37, Chap. I, Sec. 1, Thm. 1.1].

THEOREM 4. *For every isometric operator V in $\mathscr{B}(\mathscr{H})$, there exists a unique reducing subspace $\mathscr{K}$ for V, such that $V_{\mathscr{K}}$ is unitary and $V_{\mathscr{K}^\perp}$ is a unilateral shift. Namely, if*

$$\mathscr{K} := \bigcap_{n=0}^{\infty} \operatorname{ran} V^n \quad \textit{then} \quad \mathscr{K}^\perp = \bigoplus_{n=0}^{\infty} V^n\mathscr{L} , \quad \textit{where} \quad \mathscr{L} = \mathscr{H} \ominus \operatorname{ran} V . \tag{14}$$

In Theorem 4, the space $\mathscr{K}$ may be trivial or the whole space.

COROLLARY 1. *An isometric operator in $\mathscr{B}(\mathscr{H})$ is a unilateral shift if and only if it is completely nonunitary.*

Proof. We first assume that V is a unilateral shift and suppose that $\mathscr{K}$ is a reducing subspace for V, in which $V_{\mathscr{K}}$ is unitary. Then $\mathscr{K} = V^n\mathscr{K} \subset V^n\mathscr{H}$, for $n = 0, 1, \ldots$ Besides, in view of (13), the wandering space $\mathscr{L}$ for V satisfies

$$V^n\mathscr{L} = V^n\mathscr{H} \ominus V^{n+1}\mathscr{H} .$$

Consequently, $\mathscr{K} \perp V^n\mathscr{L}$, for all $n \in \mathbb{N}\cup\{0\}$. Thus (13) implies $\mathscr{K} = \{0\}$ and hence V is c.n.u. The converse readily follows from Theorem 4. $\square$

Theorems 3 and 4 present two kinds of decompositions that are uniquely determined by their unitary and completely nonunitary parts. In what follows, we shall give the respective decompositions for dissipative relations.

DEFINITION 6. A relation L is called *dissipative* if

$$\operatorname{Im}\langle f,g\rangle \geqslant 0, \quad \text{for all } \begin{pmatrix} f \\ g \end{pmatrix} \in L.$$

Particularly, L is *symmetric* if $L \subset L^*$ and *selfadjoint* when $L = L^*$. Moreover, L is said to be *maximal dissipative* if it does not have proper dissipative extensions.

For the reader's convenience, the following assertion is adapted from [31].

PROPOSITION 3. *Let ζ be in the upper half-plane $\mathbb{C}_+$, such that $|\zeta| = 1$. A linear relation L is (closed, maximal) dissipative (symmetric, selfadjoint) if and only if $\mathsf{Z}_\zeta(L)$ is a (closed, maximal) contraction (isometry, unitary).*

Proposition 3 clarifies that the Z transform gives a one-to-one correspondence between contractions and dissipative relations.

DEFINITION 7. We call a dissipative relation L *completely nonselfadjoint* (briefly c.n.s.), if there is no nonzero reducing subspace $\mathscr{K}$ for L, in which $L_{\mathscr{K}}$ is selfadjoint.

REMARK 5. If a closed dissipative relation L is c.n.s., then it is an operator. Indeed, since $\operatorname{dom} L \subset (\operatorname{mul} L)^\perp$ (cf. [5, Sec. 2]), one has that $\operatorname{mul} L$ is a reducing subspace for L, in which L is selfadjoint. Hence, $\operatorname{mul} L = \{0\}$, viz. L is an operator.

PROPOSITION 4. *A relation L is a completely nonselfadjoint, dissipative relation if and only if $V = \mathsf{Z}_i(L)$ is a completely nonunitary, contraction.*

Proof. We first suppose that L is a c.n.s. dissipative relation. By Proposition 3, one has that $V = \mathsf{Z}_i(L)$ is a contraction. Besides, if there exists a nonzero reducing subspace $\mathscr{K}$ for V, such that $V_{\mathscr{K}}$ is unitary, then Theorem 2 implies that $\mathscr{K}$ also reduces L and Proposition 3 states that $\mathsf{Z}_i(V_{\mathscr{K}}) \subset L$ is selfadjoint. This contradicts our assumption that L is c.n.s. Therefore, V is c.n.u. The proof of the converse is handled in the same lines as above. $\square$

We show in the next theorem the analogue of the Sz. Nagy-Foiaş-Langer decomposition for any closed dissipative relations. Recently, in [10, Thm. 7.6], maximal dissipative operators were decomposed in this way to elucidate the structure of the corresponding dilations.

THEOREM 5. *If L is a closed dissipative relation, then there exists a unique reducing subspace $\mathscr{K}$ for L, such that $L_{\mathscr{K}}$ is selfadjoint and $L_{\mathscr{K}^\perp}$ is completely nonselfadjoint.*

Proof. If L is a closed dissipative relation, then by Proposition 3, one has that $\mathsf{Z}_i(L)$ is a closed contraction. Thus, by virtue of Theorem 3, there exists a unique reducing subspace $\mathscr{K}$ for $\mathsf{Z}_i(L)$, in which $\mathsf{Z}_i(L)_{\mathscr{K}}$ is unitary and $\mathsf{Z}_i(L)_{\mathscr{K}^\perp}$ is c.n.u. In this fashion, Theorem 2 implies that $\mathscr{K}$ reduces L and besides,

$$\begin{aligned} L &= \mathsf{Z}_i(\mathsf{Z}_i(L)) \\ &= \mathsf{Z}_i(\mathsf{Z}_i(L)_{\mathscr{K}} \oplus \mathsf{Z}_i(L)_{\mathscr{K}^\perp}) = \mathsf{Z}_i(\mathsf{Z}_i(L)_{\mathscr{K}}) \oplus \mathsf{Z}_i(\mathsf{Z}_i(L)_{\mathscr{K}^\perp}). \end{aligned} \tag{15}$$

Hence, one has from Remark 2 and Proposition 3 that $L_{\mathscr{K}} = \mathsf{Z}_i(\mathsf{Z}_i(L)_{\mathscr{K}})$ is selfadjoint and by virtue of Proposition 4, $L_{\mathscr{K}^\perp} = \mathsf{Z}_i(\mathsf{Z}_i(L)_{\mathscr{K}^\perp})$ is c.n.s. To prove the uniqueness. If $\mathscr{K}'$ holds the same properties of $\mathscr{K}$, for L. Then, Theorem 2 asserts that $\mathscr{K}'$ also reduces $\mathsf{Z}_i(L)$ and

$$\mathsf{Z}_i(L) = \mathsf{Z}_i(L_{\mathscr{K}'} \oplus L_{\mathscr{K}'^\perp}) = \mathsf{Z}_i(L_{\mathscr{K}'}) \oplus \mathsf{Z}_i(L_{\mathscr{K}'^\perp}).$$

Moreover, again by Remark 2 and Propositions 3, 4, one obtains that $\mathscr{K}'$ satisfies the same properties of $\mathscr{K}$, for $\mathsf{Z}_i(L)$. Hence $\mathscr{K}' = \mathscr{K}$, since $\mathscr{K}$ is unique for $\mathsf{Z}_i(L)$. $\square$

REMARK 6. Theorem 5 and Remark 5 claim that the multivalued part of a closed dissipative relation belongs to its selfadjoint part.

In what follows, we shall work with the class of symmetric relations. We follow [31] in assuming that the spectral core of a symmetric relation A satisfies $\hat{\sigma}(A) \subset \mathbb{R}$. Moreover, if A is maximal then $\sigma(A) \subset \mathbb{C}_+ \cup \mathbb{R}$.

REMARK 7. Taking into account (3), if A is a completely nonselfadjoint, symmetric relation, then $\sigma_c(A) = \hat{\sigma}(A)$. Indeed, the linear envelope of every eigenvector of A is a reducing subspace for A, in which A is selfadjoint. Hence, $\sigma_p(A) = \emptyset$, since A is c.n.s.

DEFINITION 8. A symmetric relation A is *elementary-maximal*, if $\mathsf{Z}_i(A)$ is a unilateral shift (q.v. [2, Sec. 82]).

REMARK 8. An elementary-maximal, symmetric relation A is actually maximal, since the unilateral shifts are maximal. This involves that $\dim \mathsf{N}_{\overline{\zeta}}(A^*) = 0$, for all $\zeta \in \mathbb{C}_-$ (q.v. [31]). Thus, the first von Neumann formula for relations (see for instance [15, Thm. 6.1]) implies

$$A^* = A \dotplus \mathsf{N}_{\zeta}(A^*), \quad (\zeta \in \mathbb{C}_-) \tag{16}$$

where for $\zeta = -i$, the direct sum turns to be orthogonal.

LEMMA 1. *A maximal symmetric relation is elementary-maximal if and only if it is completely nonselfadjoint.*

Proof. It follows straightforward from Corollary 1 and Propositions 3, 4. $\square$

The following assertion uses the fact that a maximal dissipative relation L satisfies $\overline{\operatorname{dom} L} = (\operatorname{mul} L)^{\perp}$ (cf. [5, Lem. 2.1]).

THEOREM 6. *If A is an elementary-maximal, symmetric relation, then A is an unbounded densely defined operator, with the following spectral properties:*

$$\begin{aligned} &\sigma_p(A) = \emptyset, \quad &&\sigma_c(A) = \mathbb{R}, \quad &&\sigma_r(A) = \mathbb{C}_+, \\ &\sigma_p(A^*) = \mathbb{C}_-, \quad &&\sigma_c(A^*) = \mathbb{R}, \quad &&\sigma_r(A^*) = \emptyset. \end{aligned} \tag{17}$$

Proof. If A is an elementary-maximal, symmetric relation, then it is a closed operator, by virtue of Lemma 1 and Remark 5. Besides, $\operatorname{dom} A$ is dense but not the whole space, otherwise A is selfadjoint, which contradicts Lemma 1. These facts also imply that A is unbounded.

We now proceed to show (17). Since A is c.n.s., it is straightforward to see from Remark 7 that $\sigma_p(A) = \emptyset$. Additionally, Proposition 1 implies $\sigma_r(A^*) \subset \sigma_p(A) = \emptyset$. Besides, (16) yields $\dim \mathbf{N}_\zeta(A^*) \neq 0$, for all $\zeta \in \mathbb{C}_-$, which means $\mathbb{C}_- \subset \sigma_p(A^*)$. To show the other inclusion, if $\zeta \in \sigma_p(A^*)$, then there exists $\binom{f}{\zeta f} \in A^*$, with $\|f\| = 1$ and by (16), there is $\binom{h}{k} \in A$ and $\binom{t}{-it} \in A^*$, such that

$$\binom{f}{\zeta f} = \binom{h}{k} + \binom{t}{-it}. \tag{18}$$

Note, $t \neq 0$, since $\sigma_p(A) = \emptyset$. Moreover, by virtue of A is symmetric, one produces $\langle h, k\rangle \in \mathbb{R}$ and $\langle k, t\rangle = -i\langle h, t\rangle$. Thus, taking into account (18),

$$\begin{aligned} \operatorname{Im} \zeta &= \operatorname{Im} \langle f, \zeta f\rangle \\ &= \operatorname{Im}(\langle h, k\rangle + 2\operatorname{Re}\langle t, k\rangle - i\|t\|^2) < 0. \end{aligned}$$

This proves $\sigma_p(A^*) \subset C_-$ and hence they are equals. Now, the maximality of A implies $\sigma(A) \subset \mathbb{C}_+ \cup \mathbb{R}$. Consequently, by Remark 1 and since $\sigma_r(A^*) = \emptyset$, one has that $\hat{\sigma}(A^*) = \sigma(A^*) \subset \mathbb{C}_- \cup \mathbb{R}$. Then, $\hat{\sigma}(A^*) = \mathbb{C}_- \cup \mathbb{R}$, since $\hat{\sigma}(A^*)$ is closed and contains the lower half-plane. Hence, by virtue of (3), one obtains $\sigma_c(A^*) = \mathbb{R}$. To conclude, again Remark 1 yields $\sigma_c(A) = \mathbb{R}$ and $\sigma(A) = \mathbb{C}_+ \cup \mathbb{R}$, which asserts $\sigma_r(A) = \mathbb{C}_+$. $\square$

The method used in the proof of Theorem 6 can be carried over to unilateral shift operators, holding similar properties to (17). Namely, if V is a unilateral shift, then

$$\begin{aligned} &\sigma_p(V) = \emptyset, \quad &&\sigma_c(V) = \partial\mathbb{D}, \quad &&\sigma_r(V) = \mathbb{D}, \\ &\sigma_p(V^*) = \mathbb{D}, \quad &&\sigma_c(V^*) = \partial\mathbb{D}, \quad &&\sigma_r(V^*) = \emptyset, \end{aligned}$$

where $\mathbb{D}$ is the open unit disc and $\partial\mathbb{D}$ its boundary.

We conclude this section by showing the analogue of the von Neumann-Wold decomposition for symmetric relations.

THEOREM 7. *If A is a maximal symmetric relation, then*

$$\mathscr{K} = \bigoplus_{n=0}^{\infty} \mathbf{Z}_i(A)^n \mathscr{L}, \quad \text{with} \quad \mathscr{L} = \operatorname{dom} \mathbf{N}_{-i}(A^*), \tag{19}$$

is the unique reducing subspace for A, such that $A_{\mathscr{K}^\perp}$ is selfadjoint and $A_{\mathscr{K}}$ is elementary-maximal.

Proof. From Proposition 3 and since A is a maximal symmetric relation, one has that $\mathbf{Z}_i(A)$ is an isometry in $\mathscr{B}(\mathscr{H})$. Then, by Theorem 4, there exists a unique reducing subspace $\mathscr{K}$ for $\mathbf{Z}_i(A)$, such that $\mathbf{Z}_i(A)_{\mathscr{K}^\perp}$ is unitary and $\mathbf{Z}_i(A)_{\mathscr{K}}$ is a unilateral shift. Besides, Theorem 2 shows that $\mathscr{K}$ reduces A. Thus, following the same reasoning of (15), one computes that

$$A = \mathbf{Z}_i(\mathbf{Z}_i(A)_{\mathscr{K}}) \oplus \mathbf{Z}_i(\mathbf{Z}_i(A)_{\mathscr{K}^\perp}).$$

Thereby, Remark 2 and Proposition 3 imply that $A_{\mathscr{K}^\perp} = \mathbf{Z}_i(\mathbf{Z}_i(A)_{\mathscr{K}^\perp})$ is selfadjoint and $A_{\mathscr{K}} = \mathbf{Z}_i(\mathbf{Z}_i(A)_{\mathscr{K}})$ is symmetric, which certainly is elementary-maximal. Therefore, from the properties of Z transform, (11) and (14), one has (19). Uniqueness is proven following the same lines as the proof of Theorem 5. □

4. Example

We consider the Hilbert space of square-summable sequences $l_2(\mathbb{N})$, with canonical basis $\{\delta_k\}_{k\in\mathbb{N}}$. Let $l_2(\text{fin}) \subset l_2(\mathbb{N})$ denote the set of all sequences with only a finite number of nonzero entries. We define the linear operator $\tilde{A}$, whose domain is

$$\operatorname{dom}\tilde{A} := \left\{ \sum_{k\in\mathbb{N}} (f_k - i f_{k-1})\delta_k \,:\, \sum_{k\in\mathbb{N}} f_k \delta_k \in l_2(\text{fin}) \right\},$$

such that

$$\tilde{A}\left(\sum_{k\in\mathbb{N}} (f_k - i f_{k-1})\delta_k \right) = \sum_{k\in\mathbb{N}} (i f_k - f_{k-1})\delta_k .$$

We denote the closure of $\tilde{A}$ by A and one easily checks that $S := \mathbf{Z}_i(A)$ is the shift operator $S\delta_k = \delta_{k+1}$, i.e., A is an elementary-maximal, symmetric operator. By virtue of Theorem 6, A is unbounded, densely defined in $l_2(\mathbb{N})$ and satisfies the spectral properties (17). Since $\operatorname{ran} S = l_2(\mathbb{N}) \ominus \operatorname{span}\{\delta_1\}$, one has by (11) and (1) that

$$\begin{aligned} \operatorname{span}\{\delta_1\} &= l_2(\mathbb{N}) \ominus \operatorname{ran} \mathbf{Z}_i(A) \\ &= l_2(\mathbb{N}) \ominus \operatorname{ran}(A - iI) = \ker(A^* + iI). \end{aligned} \tag{20}$$

Consequently, (16) yields

$$A^* = A \oplus \operatorname{span}\left\{ \begin{pmatrix} \delta_1 \\ -i\delta_1 \end{pmatrix} \right\}. \tag{21}$$

For abbreviation, we set $\mathscr{K} := l_2(\mathbb{N}) \ominus \operatorname{span}\{\delta_1\}$ and $Y := \operatorname{span}\left\{\begin{pmatrix} 0 \\ \delta_1 \end{pmatrix}\right\}$. Besides, we consider $B := A_{\restriction_{\mathscr{K}}}$, which is a closed symmetric operator such that $\mathbf{Z}_i(B) = S_{\restriction_{\mathscr{K}}}$. Moreover, it is straightforward to compute that $B = A \cap Y^*$ and since Y is unidimensional,

$$\begin{aligned} B^* &= (A \cap Y^*)^* \\ &= -\left((A \cap Y^*)^\perp\right)^{-1} \\ &= -\left(\overline{A^\perp \dotplus (Y^*)^\perp}\right)^{-1} \\ &= \overline{(-A^{-1})^\perp \dotplus (-(Y^*)^{-1})^\perp} \\ &= \overline{A^* \dotplus Y} = A^* \dotplus Y\,. \end{aligned} \tag{22}$$

Now, we shall give an example of a maximal symmetric relation, which is not an operator as well as its selfadjoint and elementary-maximal parts. We regard the linear relation

$$A_\infty := B \oplus Y\,,$$

which is a closed symmetric extension of B, with purely multivalued part Y. Since the maximality of A is equivalent to $\dim \mathbf{N}_{\overline{\zeta}}(A^*) = 0$, with $\zeta \in \mathbb{C}_-$ (q.v. [31]), then by (22), $\mathbf{N}_{\overline{\zeta}}(A_\infty^*) \subset \mathbf{N}_{\overline{\zeta}}(B^*) = \mathbf{N}_{\overline{\zeta}}(A^*)$, whence it follows that A_∞ is maximal. Following the same reasoning of (20) and (21), for B in $\mathscr{K}$, one produces

$$B^* = B \oplus \operatorname{span}\left\{\begin{pmatrix} \delta_2 \\ -i\delta_2 \end{pmatrix}\right\}. \tag{23}$$

Thence, it is clear that $\mathscr{K}$ reduces A_∞ and from Theorem 1, (10) and (23), one yields that $(A_\infty)^* = B \oplus \operatorname{span}\left\{\begin{pmatrix} \delta_2 \\ -i\delta_2 \end{pmatrix}\right\} \oplus Y$. Thus, $\operatorname{dom} \mathbf{N}_{-i}((A_\infty)^*) = \operatorname{span}\{\delta_2\}$ and

$$\begin{aligned} \bigoplus_{n=0}^{\infty} \mathbf{Z}_i(A_\infty)^n(\operatorname{span}\{\delta_2\}) &= \bigoplus_{n=0}^{\infty} \mathbf{Z}_i(B)^n(\operatorname{span}\{\delta_2\}) \\ &= \bigoplus_{n=0}^{\infty} S^n(\operatorname{span}\{\delta_2\}) = \mathscr{K}\,. \end{aligned}$$

Hence, by virtue of Theorem 7, we infer that $\mathscr{K}$ is the unique reducing subspace for A_∞, such that $(A_\infty)_{\mathscr{K}^\perp} = Y$ and $(A_\infty)_{\mathscr{K}} = B$ are their selfadjoint and elementary-maximal parts, respectively.

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