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ERRATUM TO "A SURJECTIVITY PROBLEM FOR 3 BY 3 MATRICES, OPERATOR AND MATRICES, 13, NO. 1, (2019) 111—119"

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Abstract. Let P be a complex polynomial. We prove that the associated polynomial matrix-valued function $\tilde{P}$ is surjective if for each $\lambda \in \mathbb{C}$ the polynomial $P-\lambda$ has at least a simple zero and it is not surjective if it does not have the double zero property.

1. Natural powers for matrices of order three

In this paper we correct slight inaccuracies in some statements in [1]. One of the main results in [1] is Theorem 2.1 and there is no change there in the statement and proof. Recently it was brought to our attention that a particular case of Theorem 2.1 in [1] was considered in [2] (in a general setting); note the result in Theorem 2.1 in [1] takes care of all cases and an added advantage is that the proof in [1] is quite elementary.

Corollary 1.1 in [1] is restated as follows

COROLLARY 1.1. *Let $P(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients, and let x, a, b be given complex numbers. For*

$$A_1 = A_1(x,a,b) := \begin{pmatrix} x & a & 0 \\ 0 & x & b \\ 0 & 0 & x \end{pmatrix}, \tag{1.1}$$

$\tilde{P}(A_1) := a_n A_1^n + a_{n-1} A_1^{n-1} + \cdots + a_1 A_1 + a_0 I_3$, *is given by*

$$\begin{pmatrix} P(x) & aP'(x) & \frac{1}{2!}abP''(x) \\ 0 & P(x) & bP'(x) \\ 0 & 0 & P(x) \end{pmatrix}. \tag{1.2}$$

Proof. It is enough to see that

$$A_1^n = \begin{pmatrix} x^n & anx^{n-1} & \frac{1}{2!}n(n-1)abx^{n-2} \\ 0 & x^n & bnx^{n-1} \\ 0 & 0 & x^n \end{pmatrix}. \tag{1.3}$$

The details are omitted. □

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2. Global problems in the space of matrices

Lemma 2.1 in [1] is restated as follows.

LEMMA 2.1. *If the polynomial $P \in \mathbb{C}[z]$ has no zeros of multiplicity less than 3 then the matrix equation*

$$\tilde{P}(X) = Y := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

has no solutions in $\mathscr{M}(3,\mathbb{C})$.

Proof. We argue by contradiction. Suppose that there exists a $A \in \mathscr{M}(3,\mathbb{C})$ such that $\tilde{P}(A) = Y$. Then $P(\sigma(A)) = \sigma(\tilde{P}(A)) = \{0\}$, i.e. the eigenvalues of A are zeros of the polynomial P. On the other hand, since each root of P has multiplicity at least 3, the minimal polynomial m_A is a divisor of P, this yields $\tilde{P}(A) = 0_3$, and this is a contradiction. □

Note Lemma 2.1 in [1] was stated incorrectly; note the polynomial $P(\lambda) := \lambda^2$ has no simple zeros and

$$\tilde{P}\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = Y.$$

We note that Lemma 2.1 in [1] was not used in the proof of Theorem 2.1.

Proposition 2.1 in [1] then is restated as follows.

PROPOSITION 2.1. *Let $P \in \mathbb{C}[z]$ be a polynomial having the property that there exists a $m \in \mathbb{C}$ such that $Q := P - m$ has no zeros of multiplicity less than 3. Then the map $X \mapsto \tilde{P}(X) : \mathscr{M}(3,\mathbb{C}) \to \mathscr{M}(3,\mathbb{C})$ is not surjective.*

Proof. In view of Lemma 2.1, the equation

$$\tilde{P}(X) = mI_3 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

has no solutions in $\mathscr{M}(3,\mathbb{C})$. □

DEFINITION 2.1. **(i)** We say that a polynomial $P \in \mathbb{C}[z]$ has the simple zero property and we write (**SZP**) if for every $m \in \mathbb{C}$ the polynomial $Q := P - m$ has at least a simple zero.

(ii) We say that a polynomial $P \in \mathbb{C}[z]$ has the double zero property and we write (**DZP**) if for every $m \in \mathbb{C}$ the polynomial $Q := P - m$ has at least a zero of multiplicity less than or equal to 2.

Clearly every polynomial which has simple zero property has the double zero property as well.

It seems that Theorem 2.2 (in [1]) needs to be rephrased slightly. We restate it as a Corollary.

COROLLARY 2.1. *Let $P \in \mathbb{C}[z]$ be a scalar polynomial and let us consider the map*

$$X \mapsto \tilde{P}(X) : \mathscr{M}(3,\mathbb{C}) \to \mathscr{M}(3,\mathbb{C}). \tag{2.3}$$

Thus the following two statements hold.

1. *If P has the simple zero property then the map $\tilde{P}$ is surjective.*

2. *If P does not have the double zero property then the map $\tilde{P}$ is not surjective.*

The proof of the second part is an easy consequence of Proposition 2.1.

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