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EXTENSION OF GENERALIZED STRONG DRAZIN INVERSE

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Abstract. As an extension of the generalized strong Drazin inverse, we present a new generalized inverse for Banach algebra elements based on a g–Drazin invertible element rather than on a quasinilpotent element in the definition of the generalized strong Drazin inverse. Because of that, our new inverse will be called an extended gs–Drazin inverse. Some characterizations of this inverse are given using idempotents and tripotents. We also study extensions of Cline's formula to the case of extended gs–Drazin inverse. Applying these results, we introduce and investigate an extended s–Drazin inverse.

1. Introduction

Let $\mathscr{A}$ be a complex Banach algebra with unit 1, and let, for $a \in \mathscr{A}$, $\sigma(a)$, $r(a)$ and acc $\sigma(a)$ be the spectrum of a, the spectral radius of a and the set of all accumulation points of $\sigma(a)$, respectively. We use $\mathscr{A}^{-1}$, $\mathscr{A}^{nil}$ and $\mathscr{A}^{qnil}$ to denote the sets of all invertible, nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathscr{A}$, respectively. If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, we denote by $\sigma_{\mathscr{B}}(a)$ the spectrum of $a \in \mathscr{B}$ with respect to $\mathscr{B}$, and by $a_{\mathscr{B}}^{-1}$ the inverse of a in $\mathscr{B}$. An element $a \in \mathscr{A}$ is tripotent if $a^3 = a$, and a is idempotent if $a^2 = a$.

The notion of a strongly nil-clean element was defined for an element of an associative ring in [5]. Wang [16] introduced a strong Drazin inverse as a class of new generalized inverse corresponding to the strong nil-cleanness. Several recent results related to nil-clean elements and strong Drazin inverses can be found in [1, 2, 4, 8, 9].

In [12], a generalized strong Drazin inverse was introduced in a Banach algebra: an element $a \in \mathscr{A}$ is called generalized strongly Drazin invertible (or gs–Drazin invertible) if there exists an element $x \in \mathscr{A}$ such that

$$xax = x, \quad ax = xa \quad \text{and} \quad a - ax \in \mathscr{A}^{qnil}.$$

The gs–Drazin inverse x of a is unique if it exists. If $a - ax \in \mathscr{A}^{nil}$ in the above definition, then x is the strong Drazin inverse (or s–Drazin inverse) of a. For more details concerning generalized strong Drazin inverse see [6].

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In the case that $a(1-ax) \in \mathscr{A}^{qnil}$ instead of $a-ax \in \mathscr{A}^{qnil}$ in the definition of the gs–Drazin inverse, a is g–Drazin invertible. The g–Drazin inverse $x = a^d$ of a is unique, if it exists [7]. Recall that a^d exists if and only if $0 \notin \text{acc}\,\sigma(a)$. By $\mathscr{A}^d$ will be denoted the set of all g–Drazin invertible elements of $\mathscr{A}$. For $a \in \mathscr{A}^d$, $a^\pi = 1 - aa^d$ is the spectral idempotent of a corresponding to the set $\{0\}$. The g–Drazin inverse of a doubly commutes with a, that is, a^d commutes with every element of $\mathscr{A}$ that commutes with a (that is, $ab = ba$ implies $a^d b = ba^d$) [7]. It is well-known that $\mathscr{A}^{qnil} \subseteq \mathscr{A}^d$, since the g–Drazin inverse of a quasinilpotent element exists and it is equal to zero. Some interesting results related to g–Drazin inverses were proved in [13, 14, 15, 18].

If $a - axa \in \mathscr{A}^{nil}$ in the definition of the g–Drazin inverse, then $a^d = a^D$ is the Drazin inverse of a. The group inverse is a particular case of the Drazin inverse for which $a = axa$ holds instead of $a - axa \in \mathscr{A}^{nil}$. The group inverse of a will be denoted by $a^\#$. The sets $\mathscr{A}^D$ and $\mathscr{A}^\#$ consist of all Drazin invertible and group invertible elements of $\mathscr{A}$, respectively.

Cline [3] proved that if ab is Drazin invertible, then so is ba and $(ba)^D = b((ab)^D)^2 a$. This equality is so-called Cline's formula and it was extended to various generalized inverses under various conditions. Motivated by [17], a new generalization of Cline's formula was studied in [19] under assumptions $acd = dbd$ and $dba = aca$.

In [11], the notion of g–Drazin inverse was extended using a corresponding g–Drazin invertible element rather than a quasinilpotent element in the definition of g–Drazin inverse. An element $a \in \mathscr{A}$ is called extended g–Drazin invertible (or eg–Drazin invertible) if there exists an element $x \in \mathscr{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - axa \in \mathscr{A}^d.$$

In this case, x is an extended g–Drazin inverse (or eg–Drazin inverse) of a and it is not uniquely determined. Recall that a is extended g–Drazin invertible if and only if a is g–Drazin invertible. If we replace $a - axa \in \mathscr{A}^d$ with $a - axa \in \mathscr{A}^D$ in the definition of eg–Drazin inverse, then x is an extended Drazin inverse (or e–Drazin inverse) of a. Denote by $\mathscr{A}^{ed}$ and $\mathscr{A}^{eD}$, respectively, the sets of all eg–Drazin invertible and e–Drazin invertible elements of $\mathscr{A}$.

Our goal is to continue studying generalized strong Drazin inverses and proposed a wider class of generalized strong Drazin inverses. Inspired by extension of the g–Drazin inverse to the extended g–Drazin inverse, we replace the condition $a - ax \in \mathscr{A}^{qnil}$ in the definition of generalized strong Drazin inverse with $a - ax \in \mathscr{A}^d$ to introduce a new generalized inverse in a Banach algebra. Since this new inverse is an extension of gs–Drazin inverse, it will be called the extended gs–Drazin inverse. Using idempotents and tripotents, we characterize extended gs–Drazin invertible elements. We show that an element $a \in \mathscr{A}$ is extended gs–Drazin invertible if and only if a is extended g–Drazin invertible if and only if a is g–Drazin invertible. We investigated generalizations of Cline's formula for extended gs–Drazin inverse whenever $acd = dbd$ and $dba = aca$. As a consequence of these results, we define and study an extension of strong Drazin inverse.

2. Extended *gs*–Drazin inverse

Using the condition $a - ax \in \mathscr{A}^d$ instead of $a - ax \in \mathscr{A}^{qnil}$ in the definition of *gs*–Drazin inverse, we extend the concept of the *gs*–Drazin inverse and define a new generalized inverse in a Banach algebra.

DEFINITION 1. An element $a \in \mathscr{A}$ is called extended *gs*–Drazin invertible (or *egs*–Drazin invertible) if there exists an element $x \in \mathscr{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - ax \in \mathscr{A}^d.$$

In this case, x is an extended *gs*–Drazin inverse (or *egs*–Drazin inverse) of a.

Notice that, by $\mathscr{A}^{qnil} \subseteq \mathscr{A}^d$, if $a \in \mathscr{A}$ is *gs*–Drazin invertible, then a is *egs*–Drazin invertible. If we assume that $a - ax \in \mathscr{A}^D$ in the above definition, we introduce an extension of the strong Drazin inverse.

DEFINITION 2. An element $a \in \mathscr{A}$ is called extended *s*–Drazin invertible (or *es*–Drazin invertible) if there exists an element $x \in \mathscr{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - ax \in \mathscr{A}^D.$$

In this case, x is an extended *s*–Drazin inverse (or *es*–Drazin inverse) of a.

We denote by $\mathscr{A}^{esd}$ and $\mathscr{A}^{esD}$, respectively, the sets of all *egs*–Drazin invertible and *es*–Drazin invertible elements of $\mathscr{A}$. Clearly, $\mathscr{A}^{esD} \subseteq \mathscr{A}^{esd}$.

LEMMA 1. *If $a \in \mathscr{A}^{esd}$, then $a \in \mathscr{A}^{ed}$. In addition, if x is an egs–Drazin inverse of a, then x is an eg–Drazin inverse of a.*

Proof. Let x be an *egs*–Drazin inverse of a. Since $1 - ax$ is an idempotent, then $1 - ax \in \mathscr{A}^{\#} \subseteq \mathscr{A}^d$. Notice that $a - ax \in \mathscr{A}^d$ and $(a - ax)(1 - ax) = (1 - ax)(a - ax)$. By [7, Theorem 5.5], $a - axa = a(1 - ax) = (a - ax)(1 - ax) \in \mathscr{A}^d$. Thus, x is an *eg*–Drazin inverse of a. □

Remark that $\mathscr{A}^{esd} \subseteq \mathscr{A}^{ed} = \mathscr{A}^d$, by Theorem 1 and [11, Theorem 2.2]. We now verify some characterizations of *egs*–Drazin invertible elements and prove that $\mathscr{A}^{esd} = \mathscr{A}^{ed} = \mathscr{A}^d$. Also, notice that the *egs*–Drazin inverse is not uniquely determined.

THEOREM 1. *Let $a \in \mathscr{A}$. The following statements are equivalent:*

(i) *a is egs–Drazin invertible;*

(ii) *a is eg–Drazin invertible;*

(iii) *a is g–Drazin invertible;*

(iv) *there exists an idempotent $p \in \mathscr{A}$ commuting with a such that $ap \in (p\mathscr{A}p)^{-1}$ and $a - p \in \mathscr{A}^d$;*

(v) *there exists an idempotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap+1-p \in \mathscr{A}^{-1}$ *and* $a-p \in \mathscr{A}^d$.

In this case, we have that 0 *and* $(ap)^{-1}_{p\mathscr{A}p} = (ap+1-p)^{-1}p$ *are egs–Drazin inverses of* a.

Proof. (i) $\Rightarrow$ (ii): By Lemma 1, this implication is clear.

(ii) $\Leftrightarrow$ (iii): It follows by [11, Theorem 2.2].

(iii) $\Rightarrow$ (i): In the case that $a \in \mathscr{A}^d$, we observe that 0 is an *egs*–Drazin inverse of a.

(i) $\Rightarrow$ (iv) $\wedge$ (v): Let x be an *egs*–Drazin inverse of a and $p = ax$. Then we get $p^2 = p$, $pa = ap$ and $a-p = a-ax \in \mathscr{A}^d$. Since $apx = a^2x^2 = ax = p = xap$, we have that ap is invertible in the Banach algebra $p\mathscr{A}p$ and $x = (ap)^{-1}_{p\mathscr{A}p}$. We can also verify that $(ap)^{-1}_{p\mathscr{A}p}+1-p$ is the inverse of $ap+1-p$.

(iv) $\Rightarrow$ (i): Assume that there exists an idempotent $p \in \mathscr{A}$ commuting with a such that $ap \in (p\mathscr{A}p)^{-1}$ and $a-p \in \mathscr{A}^d$. Set $x = (ap)^{-1}_{p\mathscr{A}p}$. Now, $xa = (ap)^{-1}_{p\mathscr{A}p}a = (ap)^{-1}_{p\mathscr{A}p}pa = (ap)^{-1}_{p\mathscr{A}p}ap = p = ap(ap)^{-1}_{p\mathscr{A}p} = a(ap)^{-1}_{p\mathscr{A}p} = ax$, $xax = px = x$ and $a-ax = a-p \in \mathscr{A}^d$. So, x is an *egs*–Drazin inverse of a.

(v) $\Rightarrow$ (i): From $(ap+1-p)p = ap$, we obtain $p = (ap+1-p)^{-1}ap$. Denote by $x = (ap+1-p)^{-1}p$. Hence, by $ax = xa = p$, $xax = xp = x$ and $a-ax = a-p \in \mathscr{A}^d$, x is an *egs*–Drazin inverse of a. $\square$

Using tripotents, we characterize *egs*–Drazin invertible elements in the next way.

THEOREM 2. *Let* $a \in \mathscr{A}$. *The following statements are equivalent:*

(i) a *is egs–Drazin invertible;*

(ii) *there exists a tripotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap \in (p^2\mathscr{A}p^2)^{-1}$ *and* $a-p^2 \in \mathscr{A}^d$;

(iii) *there exists a tripotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap+1-p^2 \in \mathscr{A}^{-1}$ *and* $a-p^2 \in \mathscr{A}^d$.

In this case, we have that $(ap)^{-1}_{p^2\mathscr{A}p^2}p = (ap+1-p^2)^{-1}p$ *is the egs–Drazin inverse of* a.

Proof. (i) $\Rightarrow$ (iii): Using Theorem 1(v), there exists an idempotent $p \in \mathscr{A}$ commuting with a such that $ap+1-p \in \mathscr{A}^{-1}$ and $a-p \in \mathscr{A}^d$. Therefore, $p = p^2 = p^3$, $ap+1-p^2 \in \mathscr{A}^{-1}$ and $a-p^2 \in \mathscr{A}^d$.

(i) $\Rightarrow$ (ii): By Theorem 1(iv), we show this implication similarly as (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i): Assume that there exists a tripotent $p \in \mathscr{A}$ commuting with a such that $ap \in (p^2\mathscr{A}p^2)^{-1}$ and $a-p^2 \in \mathscr{A}^d$. Since p is a tripotent, then p^2 is an idempotent. Let $x = (ap)^{-1}_{p^2\mathscr{A}p^2}p$. Notice that $xa = (ap)^{-1}_{p^2\mathscr{A}p^2}pa = (ap)^{-1}_{p^2\mathscr{A}p^2}ap = p^2$. From $x = p^2x$, we get $ax = ap^2x = pap(ap)^{-1}_{p^2\mathscr{A}p^2}p = p^2$ and so $ax = xa$. Also, $xax = p^2x = x$ and $a-ax = a-p^2 \in \mathscr{A}^d$, which imply that x is an *egs*–Drazin inverse of a.

(iii) $\Rightarrow$ (i): Because $(ap+1-p^2)p^2 = ap$, we have $p^2 = (ap+1-p^2)^{-1}ap$. Let $x = (ap+1-p^2)^{-1}p$. Then $ax = xa = p^2$, $xax = xp^2 = (ap+1-p^2)^{-1}p^3 = x$ and $a - ax = a - p^2 \in \mathscr{A}^d$ give that x is an *egs*–Drazin inverse of a. $\square$

Remark that, by Theorem 1, the statements of Theorem 2 are characterizations of *eg*–Drazin and *g*–Drazin invertible elements by tripotents. In the following result, we obtain new characterizations of *eg*–Drazin invertible elements by means of tripotents.

THEOREM 3. *Let $a \in \mathscr{A}$. The following statements are equivalent:*

(i) *a is eg–Drazin invertible;*

(ii) *there exists a tripotent $q \in \mathscr{A}$ commuting with a such that $aq \in (q^2\mathscr{A}q^2)^{-1}$ and $a(1-q^2) \in \mathscr{A}^d$;*

(iii) *there exists a tripotent $q \in \mathscr{A}$ commuting with a such that $aq \in \mathscr{A}^\#$ and $a(1-q^2) \in \mathscr{A}^d$;*

(iv) *there exists a tripotent $q \in \mathscr{A}$ commuting with a such that $aq+1-q^2 \in \mathscr{A}^{-1}$ and $a(1-q^2) \in \mathscr{A}^d$.*

In this case, we have that $(aq)^{-1}_{q^2\mathscr{A}q^2}q = (aq)^\# q = (aq+1-q^2)^{-1}q$ is the eg–Drazin inverse of a.

Proof. Using [11, Theorem 2.1], we verify this result in a similar manner as in the proof of Theorem 2. $\square$

We give some properties of *egs*-Drazin invertible elements in the next result. By a^{esd} and a^{esD} will be denoted an *egs*–Drazin inverse and *es*–Drazin inverse of a, respectively. Let $a\{esd\}$ (or $a\{esD\}$) denote the set of all extended gs-Drazin (es-Drazin) inverses of a.

LEMMA 2. *Let $a \in \mathscr{A}^{esd}$. Then, for arbitrary a^{esd},*

(i) *$a^{esd} \in \mathscr{A}^\#$ and $(a^{esd})^\# = a^2a^{esd}$;*

(ii) *$a^{esd} \in \mathscr{A}^{esd}$ and $a^2a^{esd} \in a^{esd}\{esd\}$.*

Proof. (i) Firstly, we observe that a^{esd} commutes with a^2a^{esd}.

Now, by $(a^2a^{esd})a^{esd}(a^2a^{esd}) = a^2a^{esd}$ and $a^{esd}(a^2a^{esd})a^{esd} = a^{esd}$, we deduce that $a^{esd} \in \mathscr{A}^\#$ and $(a^{esd})^\# = a^2a^{esd}$.

(ii) Notice that a^{esd} commutes with $a - aa^{esd}$, $a - aa^{esd} \in \mathscr{A}^d$ and $a^{esd} \in \mathscr{A}^\#$. Using [7, Theorem 5.5], we have that $a^{esd} - a^{esd}(a^2a^{esd}) = a(a^{esd})^2 - aa^{esd} = -a^{esd}(a - aa^{esd}) \in \mathscr{A}^d$. $\square$

For an idempotent $p \in \mathscr{A}$, it is well-known that an arbitrary element $a \in \mathscr{A}$ can be represented as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$. We now present the matrix representation of an *egs*–Drazin inverse of $a \in \mathscr{A}^d$ relative to idempotent aa^d.

LEMMA 3. *If $a \in \mathscr{A}^d$, then*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{aa^d} \quad \text{and} \quad a^{esd} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_{aa^d},$$

where $a_1 \in (aa^d \mathscr{A} aa^d)^{-1}$, $a_2 \in (a^\pi \mathscr{A} a^\pi)^{qnil}$ and $x_i \in a_i\{esd\}$ for $i = 1,2$.

Proof. Recall that, if $a \in \mathscr{A}^d$, then

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathscr{A}p)^{-1}$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$. In this case, the g–Drazin inverse of a is given by

$$a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Suppose that $x \in \mathscr{A}$ is an *egs*–Drazin inverse of a. Because a^d double commutes with a, then x commutes with p and thus

$$x = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_p.$$

From $ax = xa$ and $xax = x$, for $i = 1,2$, we get $a_i x_i = x_i a_i$ and $x_i a_i x_i = x_i$. Since

$$a - ax = \begin{bmatrix} a_1 - a_1x_1 & 0 \\ 0 & a_2 - a_2x_2 \end{bmatrix}_p$$

is g–Drazin invertible and $\sigma(a-ax) = \sigma_{p\mathscr{A}p}(a_1 - a_1x_1) \cup \sigma_{(1-p)\mathscr{A}(1-p)}(a_2 - a_2x_2)$, we conclude that $a_1 - a_1x_1 \in (p\mathscr{A}p)^d$ and $a_2 - a_2x_2 \in ((1-p)\mathscr{A}(1-p))^d$. Hence, $x_i \in a_i\{esd\}$, for $i = 1,2$. □

We give more characterizations of *egs*–Drazin invertible elements in the following theorem.

THEOREM 4. *Let $a \in \mathscr{A}$ and $n \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egs–Drazin invertible;*

(ii) *there exists an element $y \in \mathscr{A}$ such that $ya^ny = y$, $ya = ay$ and $a - a^ny \in \mathscr{A}^d$;*

(iii) *a^n is egs–Drazin invertible;*

In this case, $a^{n-1}y \in a\{esd\}$.

Proof. (i) $\Rightarrow$ (ii): If a is *egs*–Drazin invertible, we denote by $y=(a^{esd})^n$, for arbitrary a^{esd}. We obtain $ya=(a^{esd})^n a = a(a^{esd})^n = ay$, $ya^n y = (a^{esd})^n a^n (a^{esd})^n = (a^{esd}aa^{esd})^n = (a^{esd})^n = y$ and $a-a^n y = a - aa^{esd} \in \mathscr{A}^d$.

(ii) $\Rightarrow$ (i): Let (ii) hold and $x=a^{n-1}y$. Then $ax=a^n y = a^{n-1}ya = xa$, $xax = a^{n-1}ya^n y = a^{n-1}y = x$ and $a-ax=a-a^n y \in \mathscr{A}^d$. So, $a\in \mathscr{A}^{esd}$ and x is an *egs*–Drazin inverse of a.

(i) $\Leftrightarrow$ (iii): By Theorem 1 and [10, Corollary 2.2], $a\in\mathscr{A}^{esd}$ iff $a\in\mathscr{A}^d$ iff $a^n\in\mathscr{A}^d$ iff $a^n\in\mathscr{A}^{esd}$. $\square$

To study Cline's formula for the *egs*–Drazin inverse, we need the following auxiliary result which was proved in [19] for elements of an associative ring $\mathscr{R}$ with the unit 1.

LEMMA 4. [19, Theorem 2.7] *Let $a,b,c,d\in\mathscr{R}$ satisfy $acd=dbd$ and $dba=aca$. Then $bd\in\mathscr{R}^d \Leftrightarrow ac\in\mathscr{R}^d$. In this case, $(bd)^d = b((ac)^d)^2 d$ and $(ac)^d = d((bd)^d)^3 bac$.*

In the case that $acd=dbd$ and $dba=aca$, we present a generalization of Cline's formula for *egs*–Drazin inverse.

THEOREM 5. *Let $a,b,c,d\in\mathscr{A}$ satisfy $acd=dbd$ and $dba=aca$. Then*

$$bd\in\mathscr{A}^{esd} \quad\Leftrightarrow\quad ac\in\mathscr{A}^{esd}.$$

In this case, for arbitrary $(bd)^{esd}$ and $(ac)^{esd}$, we have $b((ac)^{esd})^2 d\in(bd)\{esd\}$ and $d((bd)^{esd})^3 bac\in(ac)\{esd\}$.

Proof. $\Rightarrow$: Suppose that $bd\in\mathscr{A}^{esd}$ and $x=d((bd)^{esd})^3 bac$, for arbitrary $(bd)^{esd}$. Then

$$\begin{aligned} acx &= acd((bd)^{esd})^3 bac = dbd((bd)^{esd})^3 bac = d((bd)^{esd})^3 bdbac \\ &= d((bd)^{esd})^3 bacac = xac \end{aligned}$$

and

$$\begin{aligned} xacx &= d((bd)^{esd})^2 bacx = d((bd)^{esd})^2 bacd((bd)^{esd})^3 bac \\ &= d((bd)^{esd})^2 bdbd((bd)^{esd})^3 bac = d((bd)^{esd})^3 bac = x. \end{aligned}$$

To show that

$$ac-acx = ac - d((bd)^{esd})^2 bac = (1-d((bd)^{esd})^2 b)ac \in \mathscr{A}^d,$$

let $u=(1-d((bd)^{esd})^2 b)a$ and $v=(1-(bd)^{esd})b$. Notice that $vd\in\mathscr{A}^d$,

$$ucd = (1-d((bd)^{esd})^2 b)acd = (1-d((bd)^{esd})^2 b)dbd = d(1-(bd)^{esd})bd = dvd$$

and

$$\begin{aligned} dvu &= d(1-(bd)^{esd})(b-(bd)^{esd}b)a = (d-d((bd)^{esd})^2bd)(1-(bd)^{esd})ba \\ &= (1-d((bd)^{esd})^2b)(dba-dbd((bd)^{esd})^2ba) \\ &= (1-d((bd)^{esd})^2b)(aca-acd((bd)^{esd})^2ba) \\ &= (1-d((bd)^{esd})^2b)ac(1-d((bd)^{esd})^2b)a \\ &= ucu. \end{aligned}$$

By Lemma 4, we obtain $(1-d((bd)^{esd})^2b)ac = uc \in \mathscr{A}^d$. Hence, $ac \in \mathscr{A}^{esd}$ and $d((bd)^{esd})^3bac \in (ac)\{esd\}$.

$\Leftarrow$: In a similar manner as in the previous part, we prove this implication. □

If $d=a$ in Theorem 5, we get an extension of Cline's formula for the *egs*–Drazin inverse when $aca=aba$.

COROLLARY 1. *Let $a,b,c \in \mathscr{A}$ satisfy $aca=aba$. Then*

$$ba \in \mathscr{A}^{esd} \quad \Leftrightarrow \quad ac \in \mathscr{A}^{esd}.$$

In this case, for arbitrary $(ba)^{esd}$ and $(ac)^{esd}$, $b((ac)^{esd})^2a \in (ba)\{esd\}$ and $a((ba)^{esd})^2c \in (ac)\{esd\}$.

For $c=b$ in Corollary 1, we obtain that Cline's formula for the *egs*–Drazin inverse holds.

COROLLARY 2. *Let $a,b \in \mathscr{A}$. Then $ba \in \mathscr{A}^{esd} \Leftrightarrow ab \in \mathscr{A}^{esd}$. In this case, for arbitrary $(ab)^{esd}$, $b((ab)^{esd})^2a \in (ba)\{esd\}$.*

3. Extended *s*–Drazin inverse

In this section, we give characterizations of *es*–Drazin invertible elements in a Banach algebra applying the results of previous section. Firstly, as a consequence of Lemma 1, we observe that $\mathscr{A}^{esD} \subseteq \mathscr{A}^{eD}$.

LEMMA 5. *If $a \in \mathscr{A}^{esD}$, then $a \in \mathscr{A}^{eD}$. In addition, if x is an es–Drazin inverse of a, then x is an e–Drazin inverse of a.*

Using Theorem 1 and Theorem 2, we now characterize *es*–Drazin invertible elements by idempotents and tripotents.

COROLLARY 3. *Let $a \in \mathscr{A}$. The following statements are equivalent:*

(i) *a is es–Drazin invertible;*

(ii) *a is e–Drazin invertible;*

(iii) *a is Drazin invertible;*

(iv) *there exists an idempotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap \in (p\mathscr{A}p)^{-1}$ *and* $a - p \in \mathscr{A}^D$*;*

(v) *there exists an idempotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap + 1 - p \in \mathscr{A}^{-1}$ *and* $a - p \in \mathscr{A}^D$*.*

In this case, we have that 0 *and* $(ap)^{-1}_{p\mathscr{A}p} = (ap+1-p)^{-1}p$ *are es–Drazin inverses of* a*.*

COROLLARY 4. *Let* $a \in \mathscr{A}$*. The following statements are equivalent:*

(i) a *is es–Drazin invertible;*

(ii) *there exists a tripotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap \in (p^2\mathscr{A}p^2)^{-1}$ *and* $a - p^2 \in \mathscr{A}^D$*;*

(iii) *there exists a tripotent* $p \in \mathscr{A}$ *commuting with* a *such that* $ap + 1 - p^2 \in \mathscr{A}^{-1}$ *and* $a - p^2 \in \mathscr{A}^D$*.*

In this case, we have that $(ap)^{-1}_{p^2\mathscr{A}p^2}p = (ap+1-p^2)^{-1}p$ *is the es–Drazin inverse of* a*.*

Using tripotents, some new characterizations of *e*–Drazin invertible elements are presented by Theorem 3.

COROLLARY 5. *Let* $a \in \mathscr{A}$*. The following statements are equivalent:*

(i) a *is e–Drazin invertible;*

(ii) *there exists a tripotent* $q \in \mathscr{A}$ *commuting with* a *such that* $aq \in (q^2\mathscr{A}q^2)^{-1}$ *and* $a(1-q^2) \in \mathscr{A}^D$*;*

(iii) *there exists a tripotent* $q \in \mathscr{A}$ *commuting with* a *such that* $aq \in \mathscr{A}^{\#}$ *and* $a(1-q^2) \in \mathscr{A}^D$*;*

(iv) *there exists a tripotent* $q \in \mathscr{A}$ *commuting with* a *such that* $aq + 1 - q^2 \in \mathscr{A}^{-1}$ *and* $a(1-q^2) \in \mathscr{A}^D$*.*

In this case, we have that $(aq)^{-1}_{q^2\mathscr{A}q^2}q = (aq)^{\#}q = (aq+1-q^2)^{-1}q$ *is the e–Drazin inverse of* a*.*

According to Lemma 2, Lemma 3 and Theorem 4, several properties of a *es*–Drazin inverse are presented in the following results.

LEMMA 6. *Let* $a \in \mathscr{A}^{esD}$*. Then, for arbitrary* a^{esD}*,*

(i) $a^{esD} \in \mathscr{A}^{\#}$ *and* $(a^{esD})^{\#} = a^2a^{esD}$*;*

(ii) $a^{esD} \in \mathscr{A}^{esD}$ *and* $a^2a^{esD} \in a^{esD}\{esD\}$*.*

LEMMA 7. *If* $a \in \mathscr{A}^D$, *then*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{aa^D} \quad \text{and} \quad a^{esD} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_{aa^D},$$

where $a_1 \in (aa^D \mathscr{A} aa^D)^{-1}$, $a_2 \in (a^\pi \mathscr{A} a^\pi)^{nil}$ *and* $x_i \in a_i\{esD\}$ *for* $i = 1,2$.

COROLLARY 6. *Let* $a \in \mathscr{A}$ *and* $n \in \mathbb{N}$. *The following statements are equivalent:*

(i) a *is es–Drazin invertible;*

(ii) *there exists an element* $y \in \mathscr{A}$ *such that* $ya^n y = y$, $ya = ay$ *and* $a - a^n y \in \mathscr{A}^D$;

(iii) a^n *is es–Drazin invertible;*

In this case, $a^{n-1}y \in a\{esD\}$.

We also have some extensions of Cline's formula to the case of the *es*–Drazin inverse as consequences of Theorem 5, Corollary 1 and Corollary 2.

COROLLARY 7. *Let* $a,b,c,d \in \mathscr{A}$ *satisfy* $acd = dbd$ *and* $dba = aca$. *Then*

$$bd \in \mathscr{A}^{esD} \quad \Leftrightarrow \quad ac \in \mathscr{A}^{esD}.$$

In this case, for arbitrary $(bd)^{esD}$ *and* $(ac)^{esD}$, *we have* $b((ac)^{esD})^2 d \in (bd)\{esD\}$ *and* $d((bd)^{esD})^3 bac \in (ac)\{esD\}$.

COROLLARY 8. *Let* $a,b,c \in \mathscr{A}$ *satisfy* $aca = aba$. *Then*

$$ba \in \mathscr{A}^{esD} \quad \Leftrightarrow \quad ac \in \mathscr{A}^{esD}.$$

In this case, for arbitrary $(ba)^{esD}$ *and* $(ac)^{esD}$, $b((ac)^{esD})^2 a \in (ba)\{esD\}$ *and* $a((ba)^{esD})^2 c \in (ac)\{esD\}$.

COROLLARY 9. *Let* $a,b \in \mathscr{A}$. *Then* $ba \in \mathscr{A}^{esD} \Leftrightarrow ab \in \mathscr{A}^{esD}$. *In addition, for arbitrary* $(ab)^{esD}$, $b((ab)^{esD})^2 a \in (ba)\{esD\}$.

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