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BANACH WEAK TOPOLOGY ON HILBERT C^*–MODULES

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Abstract. In this paper we study the Banach weak topology on Hilbert C^*-modules. For a Hilbert C^*-module, we show that every sequence that is convergent in the weak module topology generated by the inner product is also convergent in the Banach weak topology generated by continuous linear functionals and the converse is true for the Hilbert C^*-modules that their underlying C^*-algebras are either of the form of a c_0-direct sum of matrices or of the form of a finite-dimensional matrix algebra.

1. Introduction and preliminaries

Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{M}$ be a right $\mathscr{A}$-module such that $\lambda(xa) = x(\lambda a) = (\lambda x)a$, for all $x \in \mathscr{M}, a \in \mathscr{A}$ and $\lambda \in \mathbb{C}$. An $\mathscr{A}$-valued inner product in $\mathscr{M}$ is a mapping $\langle \cdot, \cdot \rangle : \mathscr{M} \times \mathscr{M} \to \mathscr{A}$ which satisfies the following properties for all $x, y, z \in \mathscr{M}$, $\lambda \in \mathbb{C}$ and $a \in \mathscr{A}$:

(1) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;
(2) $\langle x, x \rangle \geqslant 0$;
(3) $\langle x, x \rangle = 0$ if and only if $x = 0$;
(4) $\langle x, y \rangle^* = \langle y, x \rangle$;
(5) $\langle x, ya \rangle = \langle x, y \rangle a$.

If $\mathscr{M}$ is complete with respect to the norm defined by $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}$ for any $x \in \mathscr{M}$, then $\mathscr{M}$ is called a Hilbert $\mathscr{A}$-module or right Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. Left Hilbert C^*-modules are defined in a similar manner.

In order to extend the Banach-Saks and Schur properties from the situation of Banach spaces to the situation of certain Hilbert C^*-modules, for a Hilbert C^*-module $\mathscr{M}$ over a C^*-algebra $\mathscr{A}$, the weak module topology $\mathscr{T}_{\mathscr{M}'}$ generated by $\mathscr{A}$-linear bounded functionals and the weak module topology $\mathscr{T}_{\hat{\mathscr{M}}}$ generated by the inner product were introduced in [8] (see also [1]) with emphasizing more on $\mathscr{T}_{\hat{\mathscr{M}}}$. More precisely, the $\mathscr{M}'$-weak module topology $\mathscr{T}_{\mathscr{M}'}$ on $\mathscr{M}$ is generated by the family of semi-norms

$$\{\vartheta_f\}_{f \in \mathscr{M}'}, \text{ where } \vartheta_f(x) = \|f(x)\|, \ (x \in \mathscr{M}),$$

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in which $\mathscr{M}'$ is the dual module of $\mathscr{M}'$, i.e., the set of all $\mathscr{A}$-linear bounded maps from $\mathscr{M}$ to $\mathscr{A}$, and the $\hat{\mathscr{M}}$-weak module topology $\mathscr{T}_{\hat{\mathscr{M}}}$ on $\mathscr{M}$ is generated by the family of semi-norms

$$\{\omega_y\}_{y\in\mathscr{M}}, \text{ where } \omega_y(x) = \|\langle y,x\rangle\|, \ (x\in\mathscr{M}).$$

Notice that $\mathscr{T}_{\hat{\mathscr{M}}} \subseteq \mathscr{T}_{\mathscr{M}'}$ and these two topologies coincide as $\mathscr{M}$ is self-dual. Generally, two weak module topologies $\mathscr{T}_{\mathscr{M}'}$ and $\mathscr{T}_{\hat{\mathscr{M}}}$ are distinct (see [8, Example 3.2]).

In addition to the above two weak module topologies, one can consider the usual Banach weak topology $\mathscr{T}_w$ on Hilbert C*-modules, i.e., the weak topology induced by the continuous linear functionals on $\mathscr{M}$. In this paper, we study the Banach weak topology $\mathscr{T}_w$ of Hilbert C*-modules. We give some relations between the topologies $\mathscr{T}_w$ and $\mathscr{T}_{\hat{\mathscr{M}}}$. We show that every sequence that is convergent in the weak module topology generated by the inner product is also convergent in the Banach weak topology generated by continuous linear functionals and the converse is true for the Hilbert C*-modules that their underlying C*-algebras are either of the form of a c_0-direct sum of matrices or of the form of a finite-dimensional matrix algebra.

We review some basics which will be needed later.

Any C*-algebra $\mathscr{A}$ can be considered as a Hilbert C*-module over itself under $\mathscr{A}$-valued inner product $\langle a,b\rangle = a^*b$ and a left Hilbert C*-module over itself under $\langle a,b\rangle = ab^*$. In these cases, by the C*-identity, the Hilbert module norm coincides with C*-norm on $\mathscr{A}$. Also, when $(H,\langle\cdot,\cdot\rangle_H)$ is a Hilbert space, H can be considered as a left Hilbert C*-module over $K(H)$, the C*-algebra of all compact operators on the Hilbert space H, if one defines

$$\begin{cases} \cdot : K(H)\times H \to H \\ f\cdot x = f(x) \end{cases}$$

for all $f\in K(H)$ and $x\in H$ and the inner product as

$$\begin{cases} \langle\cdot,\cdot\rangle_{K(H)} : H\times H \to K(H) \\ \langle x,y\rangle_{K(H)} = \xi_{x,y} \end{cases}$$

in which $\xi_{x,y}(z) = \langle z,y\rangle_H x$, for all $x,y,z\in H$. Note that for every $x\in H$, $\|x\|_H$ is equal to the Hilbert $K(H)$-module norm and $\|\xi_{x,y}\| = \|x\|\|y\|$, for all $x,y\in H$.

For a Hilbert $\mathscr{A}$-module $\mathscr{M}$ the closure of the linear span of all $\langle x,y\rangle$, where $x,y\in\mathscr{M}$, denoted by $\langle\mathscr{M},\mathscr{M}\rangle$, is obviously a two sided ideal of $\mathscr{A}$. A Hilbert $\mathscr{A}$-module is called full if $\langle\mathscr{M},\mathscr{M}\rangle = \mathscr{A}$. One can always consider any Hilbert module as a full Hilbert module over the C*-algebra $\langle\mathscr{M},\mathscr{M}\rangle$. The Hilbert $\mathscr{A}$-module $\mathscr{M}$ is called self-dual if the isometric module embedding $\wedge : \mathscr{M}\to\mathscr{M}'$ defined as $\wedge : x\mapsto\hat{x}$, where $\hat{x} : \mathscr{M}\to\mathscr{A}$ and $\hat{x}(y) = \langle x,y\rangle$, for all $y\in\mathscr{M}$ is surjective.

We refer the reader to [11] and [12] for more information about Hilbert C*-modules. The reader is also referred to [8] for the properties "Schur, module Schur, Banach-Saks, weak Banach-Saks" and "module Banach-Saks".

2. Main results

In this section, we give some relations between the weak topologies $\mathscr{T}_w$ and $\mathscr{T}_{\widehat{\mathscr{M}}}$. First, it should be noticed that when $\mathscr{M}$ is a full Hilbert C*-module over a finite dimensional C*-algebra, the convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathscr{M}$ in one of three weak topologies implies the convergence of $(x_n)_{n\in\mathbb{N}}$ in two others. In fact, if $x_n \to x_0$ in $\mathscr{T}_w$, since the mapping $\langle\cdot,y\rangle : \mathscr{M} \to \mathscr{A}$ is continuous, it is also ($\mathscr{T}_w, \mathscr{T}_w$)-continuous (see [3, Theorem 1.1]) and therefore $\langle x_n,y\rangle \to \langle x_0,y\rangle$, for all $y \in \mathscr{M}$. Hence, $x_n \to x_0$ in $\mathscr{T}_{\widehat{\mathscr{M}}}$. Now assume that $x_n \to x_0$ in $\mathscr{T}_{\widehat{\mathscr{M}}}$ and $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. Then $\langle x_{n_k},y\rangle \to \langle x_0,y\rangle$ for all $y \in \mathscr{M}$. Therefore, by [8, Proposition 3.3], $(x_{n_k})_{k\in\mathbb{N}}$ is norm bounded and since $\mathscr{M}$ is reflexive as a Banach space (see [9, Lemma 2.1] or [6, Corollary 4.3]), $(x_{n_k})_{k\in\mathbb{N}}$ has a subsequence converging to x_0 in $\mathscr{T}_w$. Thus, every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a subsequence which is convergent to x_0 in $\mathscr{T}_w$. This implies that $x_n \to x_0$ in $\mathscr{T}_w$. Finally assuming $x_n \to x_0$ in $\mathscr{T}_{\widehat{\mathscr{M}}}$, since $\mathscr{A}$ is finite-dimensional, $\mathscr{M}$ is self-dual (see [13]), and we have $x_n \to x_0$ in $\mathscr{T}_{\mathscr{M}'}$.

PROPOSITION 2.1. *Let $\mathscr{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{M}$ such that $x_n \xrightarrow{\mathscr{T}_{\widehat{\mathscr{M}}}} x_0$, then $x_n \xrightarrow{\mathscr{T}_w} x_0$.*

Proof. Without loss of generality, we can assume that $x_0 = 0$. Since every $\mathscr{T}_{\widehat{\mathscr{M}}}$-convergent sequence is norm bounded (see [8, Proposition 3.3]), $(x_n)_{n\in\mathbb{N}}$ is norm bounded by some $M_0 > 0$. Suppose that there is $f \in \mathscr{M}^*$ such that $f(x_n) \nrightarrow 0$. We can assume, by passing to a subsequence of $(x_n)_{n\in\mathbb{N}}$ if necessary, that $\|f(x_n)\| \geqslant \varepsilon_0$, for some $\varepsilon_0 > 0$. Consider a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ as follows:

$$
\begin{aligned}
&y_1 := x_{n_1} = x_1 \\
&y_2 := x_{n_2} \qquad \text{with } \|\langle x_{n_1},x_{n_2}\rangle\| \leqslant \tfrac{1}{n_1 n_2} \\
&\vdots \\
&y_k := x_{n_k} \qquad \text{with } \|\langle x_{n_i},x_{n_k}\rangle\| \leqslant \tfrac{1}{n_i n_k} \quad \text{for all } i<k.
\end{aligned}
$$

There is a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ and an $\alpha \in \mathbb{C}$ such that $\|\alpha\| = 1$ and

$$\mathrm{Re}(f(\alpha y_{n_k})) \geqslant \frac{\varepsilon_0}{3},$$

for all k. If $z_k = \alpha y_{n_k}$, then

$$
\begin{cases}
\mathrm{Re}(f(z_n)) \geqslant \frac{\varepsilon_0}{3} & n \in \mathbb{N}, \\
\|z_n\| \leqslant M_0 & n \in \mathbb{N}, \\
\|\langle z_m,z_n\rangle\| \leqslant \frac{1}{mn} & m,n \in \mathbb{N},\ m \neq n.
\end{cases}
$$

Now, we show $\sum_{i=1}^{\infty} \frac{z_i}{i}$ is convergent. For any $\varepsilon > 0$ there is a number N such that

$$\frac{1}{p^2} + \frac{1}{(p+1)^2} + \ldots + \frac{1}{q^2} < \varepsilon,$$

for all $q \geqslant p \geqslant N$. Therefore,

$$\begin{aligned}\Big\|\frac{z_p}{p}+\frac{z_{p+1}}{p+1}+\ldots+\frac{z_q}{q}\Big\|^2 &= \Big\|\Big\langle\frac{z_p}{p}+\frac{z_{p+1}}{p+1}+\ldots+\frac{z_q}{q},\frac{z_p}{p}+\frac{z_{p+1}}{p+1}+\ldots+\frac{z_q}{q}\Big\rangle\Big\| \\ &\leqslant \Big(\frac{M_0^2}{p^2}+\frac{M_0^2}{(p+1)^2}+\ldots+\frac{M_0^2}{q^2}\Big) \\ &\quad + \Big(\frac{1}{p^2(p+1)^2}+\frac{1}{p^2(p+2)^2}+\ldots+\frac{1}{p^2q^2}\Big) \\ &\quad + \Big(\frac{1}{(p+1)^2p^2}+\frac{1}{(p+1)^2(p+2)^2}+\ldots+\frac{1}{(p+1)^2q^2}\Big) \\ &\quad + \ldots \leqslant M_0^2\varepsilon+\frac{1}{p^2}\varepsilon+\frac{1}{(p+1)^2}\varepsilon+\ldots+\frac{1}{q^2}\varepsilon \\ &\leqslant \varepsilon M_0^2+\varepsilon^2.\end{aligned}$$

Since ε was arbitrary, $\sum_{i=1}^{\infty}\frac{z_i}{i}$ is convergent. But

$$\mathrm{Re}(f(\sum_{i=1}^{\infty}\frac{z_i}{i})) = \sum_{i=1}^{\infty}\frac{1}{i}\mathrm{Re}(f(z_i)) \geqslant \sum_{i=1}^{\infty}\frac{1}{i}\frac{\varepsilon_0}{3}$$

leads us to get a contradiction. □

Assume that $\mathscr{A}$ is an infinite-dimensional unital C*-algebra. Since $\mathscr{A}$ lacks the Schur property (see [10, Lemma 3.8]), there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{A}$ such that $x_n \to 0$ in $\mathscr{T}_w$ but $1\cdot x_n = x_n \nrightarrow 0$ in norm. Therefore, the converse of Proposition 2.1 is not true for the Hilbert $\mathscr{A}$-module $\mathscr{A}$.

According to the fact that the converse of Proposition 2.1 is not true in general, we would say that a Hilbert C*-module $\mathscr{M}_{\mathscr{A}}$ (over a C*-algebra $\mathscr{A}$) has the property $(\star)$ if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}_{\mathscr{A}}$ which is convergent to an x_0 in $\mathscr{T}_w$, it is also convergent to x_0 in $\mathscr{T}_{\mathscr{M}}$. Furthermore, a C*-algebra has the property $(\star)$ if, as a Hilbert C*-module over itself, it has the property $(\star)$, or equivalently, for any sequence $(x_n)_{n\in\mathbb{N}}$ which is $\mathscr{T}_w$-convergent to 0, we have $yx_n \to 0$, for all $y \in \mathscr{A}$.

PROPOSITION 2.2. *Any C^*-algebra $\mathscr{A}$ of the form of a c_0-direct sum of matrices has the property $(\star)$.*

Proof. Consider $\mathscr{A}$ as a c_0-direct sum $\oplus_{\lambda\in\Lambda}M_{k_\lambda\times k_\lambda}(\mathbb{C})$ of matrices. Note that for every $y=\{y^\lambda\}_{\lambda\in\Lambda}\in\mathscr{A}$ there exists a countable subset $\{\lambda_1,\lambda_2,\ldots\}\subseteq\Lambda$ such that $y^\lambda=0$ for all $\lambda\notin\{\lambda_1,\lambda_2,\ldots\}$ and $\lim_{n\to\infty}y^{\lambda_n}=0$. Now, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{A}$ such that $x_n \xrightarrow{\mathscr{T}_w} 0$. We have $\lim_{n\to\infty}x_n^\lambda=0$, for every $\lambda\in\Lambda$. Let $\varepsilon>0$ be given. There exists a number N_1 such that $\|y^{\lambda_n}\|<\varepsilon$, for all $n>N_1$. Also, there exists a number $N_2\geqslant N_1$ such that $\|x_n^\lambda\|<\varepsilon$, for all $n\geqslant N_2$ and $\lambda\in\{\lambda_1,\lambda_2,\ldots,\lambda_{N_1}\}$. Therefore,

$$\begin{aligned}\|yx_n\| = \|\{y^\lambda x_n^\lambda\}_{\lambda\in\Lambda}\| = \sup\{\|y^{\lambda_k}x_n^{\lambda_k}\| : k\in\mathbb{N}\} &\leqslant \sup\{\|y^{\lambda_k}x_n^{\lambda_k}\| : k\leqslant N_2\} \\ &\quad + \sup\{\|y^{\lambda_k}x_n^{\lambda_k}\| : k\geqslant N_2\} \leqslant \varepsilon\|y\|+\varepsilon\sup_{n\in\mathbb{N}}\|(x_n)\|,\end{aligned}$$

for all $n \geqslant N_2$. This implies that $yx_n \to 0$ and hence $\mathscr{A}$ has $(\star)$. □

PROPOSITION 2.3. *Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{M}$ be a full Hilbert $\mathscr{A}$-module.*

(i) If every $\mathscr{T}_w$-null net in $\mathscr{A}$ is a $\mathscr{T}_{\hat{\mathscr{A}}}$-null net, then every $\mathscr{T}_w$-null net in $\mathscr{M}$ is a $\mathscr{T}_{\hat{\mathscr{M}}}$-null net.

(ii) $\mathscr{M}$ has $(\star)$ if and only if $\mathscr{A}$ has $(\star)$.

Proof. (i): Suppose that Λ is a directed set, $(x_\lambda)_{\lambda\in\Lambda} \xrightarrow{\mathscr{T}_w} 0$ in $\mathscr{M}$, and $y \in \mathscr{M}$. There exist $y_1 \in \mathscr{M}$ and $a_1 \in \mathscr{A}$ such that $y = y_1 a_1$. Since the mapping $\langle y_1, . \rangle : \mathscr{M} \to \mathscr{A}$ is $(\mathscr{T}_w, \mathscr{T}_w)$-continuous, we have $\langle y_1, x_\lambda \rangle \to 0$ in $\mathscr{T}_w$. Therefore $\langle y, x_\lambda \rangle = a_1^* \langle y_1, x_\lambda \rangle \to 0$ in norm and $x_\lambda \to 0$ in $\mathscr{T}_{\hat{\mathscr{M}}}$.

(ii): Suppose that $\mathscr{A}$ has $(\star)$ and $x_n \xrightarrow{\mathscr{T}_w} 0$ in $\mathscr{M}$. By replacing $\Lambda = \mathbb{N}$ in the above argument, it gives $x_n \to 0$ in $\mathscr{T}_{\hat{\mathscr{M}}}$. Conversely, assume that $\mathscr{M}$ has $(\star)$ and $a_n \xrightarrow{\mathscr{T}_w} 0$ in $\mathscr{A}$. We have

$$\langle m_1, m_2 \rangle a_n \to 0,$$

for all $m_1, m_2 \in \mathscr{M}$. Since $\mathscr{M}$ is full, we also have $ba_n \to 0$, for all $b \in \mathscr{A}$. That is, $\mathscr{A}$ has $(\star)$. □

Notice that when $\mathscr{A}$ is an infinite-dimensional C*-algebra, $\mathscr{A}$ does not have $(\star)$ and therefore for each full Hilbert $\mathscr{A}$-module $\mathscr{M}$, we have $\mathscr{T}_{\hat{\mathscr{M}}} \nsubseteq \mathscr{T}_w$. If $\mathscr{A}$ is a C*-algebra satisfying (i) in Proposition 2.3 (for example, when $\mathscr{A}$ is finite-dimensional), then for every full Hilbert $\mathscr{A}$-module $\mathscr{M}$, we have $\mathscr{T}_{\hat{\mathscr{M}}} \subseteq \mathscr{T}_w$.

PROPOSITION 2.4. *Let H be an infinite-dimensional Hilbert space. Then $K(H)$ fails to have the property $(\star)$.*

Proof. First, notice that since the complex Hilbert space ℓ_2 does not have Schur' property (indeed, for each $y \in H$ we have $\langle y, e_n \rangle \to 0$, where (e_n) is the standard orthonormal basis of ℓ^2; that is $(e_n)_{n\in\mathbb{N}} \to 0$ in $\mathscr{T}_w$ while $e_n \nrightarrow 0$), infinite dimensional Hilbert spaces do not have the Schur property.

Suppose that $(x_n)_{n\in\mathbb{N}} \subseteq H$ is a sequence that is $\mathscr{T}_w$-convergent to 0, but $x_n \nrightarrow 0$. Consider $H_{K(H)}$ as a left Hilbert $K(H)$-module. For any nonzero element $y \in H_{K(H)}$ we have

$$\|\langle y, x_n \rangle_{K(H)}\| = \|\xi_{y,x_n}\| = \|y\|\|x_n\| \nrightarrow 0.$$

Since every left Hilbert $K(H)$-module contains $H_{K(H)}$ as a closed left submodule (see [2, Theorem 3]) and $K(H)$ is a left Hilbert $K(H)$-module over itself, there exist $y' \in K(H)$ and a sequence $(x'_n)_{n\in\mathbb{N}}$ in $K(H)$ such that $x'_n \xrightarrow{\mathscr{T}_w} 0$ and $\langle y', x'{}_n \rangle_{K(H)} = y' x'^*_n \nrightarrow 0$. Hence, $x'^*_n \xrightarrow{\mathscr{T}_w} 0$, but $y' x'^*_n \nrightarrow 0$. This means that the C*-algebra $K(H)$ does not have the property $(\star)$. □

It is worth mentioning that a locally compact Hausdorff space X has discrete topology if and only if every compact subset of X is finite. Furthermore, if X is a locally compact Hausdorff space and $K \subseteq U \subseteq X$, where K is compact and U is open, then there exists a precompact open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$ (see [5, Theorem 4.31]). Note that if $K \neq U$, we can find V such that $\overline{V} \neq U$.

In the following, for a locally compact Hausdorff space X, the C*-algebra of all continuous complex-valued functions vanishing at infinity is denoted by $C_0(X)$.

LEMMA 2.5. *Let X be a locally compact Hausdorff space. Then $C_0(X)$ has the property ($\star$) if and only if X has discrete topology.*

Proof. Suppose that X does not have discrete topology and $K \subseteq X$ is infinite and compact. Then, there exists an open subset $V \subseteq X$ such that $\overline{V}$ is compact and $K \subseteq V$. There also exists a sequence $(V_i)_{i\in\mathbb{N}}$ of nonempty disjoint open subsets of X such that $V_i \subseteq V$, for all $i \in \mathbb{N}$. Indeed, assume that $V_1, V_2, \ldots, V_k$ are nonempty open subsets of V that are pairwise disjoint. We want to find $k+1$ nonempty open subsets of V that are pairwise disjoint. If there is i_0 such that V_{i_0} is infinite, then there exists a nonempty open subset U such that $\overline{U}$ is compact and $U \subseteq \overline{U} \subsetneqq V_{i_0}$. Now, we have $k+1$ subsets $V_1, \ldots, V_{i_0-1}, V_{i_0} \backslash \overline{U}, U, V_{i_0+1}, \ldots, V_k$. Otherwise, if each V_i is finite, they are closed and $V \backslash (\cup_{i=1}^k V_i)$ is nonempty (because $K \subseteq V$ is infinite) and open. In this case we have the subsets $V_1, \ldots, V_k, (V \backslash \cup_{i=1}^k V_i)$ of V. Now, for each $k \in \mathbb{N}$, choose x_k in V_k. By the locally compact version of Urysohn's lemma (see e.g., [5, Theorem 4.32]), for every $k \in \mathbb{N}$ there is a continuous function f_k with the following properties:

$$\begin{cases} f_k : X \to [0,1] \\ f_k(x_k) = 1 \\ f_k = 0 \text{ on } V_k^c. \end{cases}$$

Note that $f_k \in C_0(X)$ and $\|f_k\| \leqslant 1$, for all $k \in \mathbb{N}$. Furthermore, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded and is convergent pointwise to 0. Therefore, $f_n \xrightarrow{\mathscr{T}_w} 0$. Again, using Urysohn's lemma, let $g \in C_0(X)$ be a function such that $g = 1$ on $\overline{V}$. Since

$$\|gf_n\| \geqslant (gf_n)(x_n) = g(x_n)f_n(x_n) = 1,$$

for all $n \in \mathbb{N}$, we have $gf_n \nrightarrow 0$. Therefore $C_0(X)$ fails to have the property ($\star$).

Conversely, assume that X is infinite with discrete topology. In this case, $C_0(X)$ is equivalent to a c_0-direct sum of $\mathbb{C}$ and therefore by Proposition 2.2 it has ($\star$). □

The fact that "a C*-algebra $\mathscr{A}$ is a C*-algebra of compact operators if and only if the spectrum of every maximal commutative C*-subalgebra of $\mathscr{A}$ is discrete (see [4, 4.7.20])" is crucial for the following result.

PROPOSITION 2.6. Let $\mathscr{M}$ be a full Hilbert C^*-module over a C^*-algebra $\mathscr{A}$. Then, $\mathscr{M}$ has the property ($\star$) if and only if $\mathscr{A}$ is $*$-isomorphic either to a c_0-direct sum of matrices or to a finite-dimensional matrix algebra.

Proof. Assume that $\mathscr{M}$ has the property $(\star)$ and $\mathscr{A}$ is not finite-dimensional. By Proposition 2.3, $\mathscr{A}$ and every (maximal) commutative C*-subalgebra $\mathscr{B}$ of $\mathscr{A}$ has the property $(\star)$, too, and so they are non-unital. Therefore, by Lemma 2.5, every (maximal) commutative C*-subalgebra $\mathscr{B}$ of $\mathscr{A}$ has discrete spectrum. Thus, $\mathscr{A}$ is $*$-isomorphic to a C*-algebra of compact operators. This means that there exist Hilbert spaces $(H_i)_{i\in I}$ such that $A \cong \oplus_{i\in I} K(H_i)$, where the (C*-)direct sum consists of elements (T_i) of Cartesian product $\prod_{i\in I} K(H_i)$ with $\|T_i\| \to 0$. By Proposition 2.4, every Hilbert space H_i is finite-dimensional. Therefore, $\mathscr{A}$ is $*$-isomorphic to a c_0-direct sum of matrices.

Conversely, if $\mathscr{A}$ is $*$-isomorphic to a c_0-direct sum of matrices or to a finite-dimensional matrix algebra, by Propositions 2.2 and 2.3, $\mathscr{M}$ has the property $(\star)$. □

It is an immediate consequence of Proposition 2.6 that the convergence of a sequence in a Hilbert C*-module in the topology $\mathscr{T}_w$ and in the topology $\mathscr{T}_{\hat{\mathscr{M}}}$ are the same if and only if the underlying C*-algebra is either a c_0-direct sum of matrices or a finite-dimensional matrix algebra.

Recall that a Hilbert C*-module $\mathscr{M}$ has the Schur property (module Schur property) if every $\mathscr{T}_w$-convergent ($\mathscr{T}_{\hat{\mathscr{M}}}$-convergent) sequence in $\mathscr{M}$ is norm convergent (see [8]). Hence, a Hilbert C*-module has the Schur property if and only if it has the module Schur property and the property $(\star)$.

COROLLARY 2.7. Let $\mathscr{M}$ be a full left Hilbert C^*-module over a C^*-algebra $\mathscr{A}$.

(i) If $\mathscr{A}$ is $*$-isomorphic to a C^*-algebra of compact operators, then $\mathscr{M}$ has module Schur property if and only if the orthogonal dimension of $\mathscr{M}$ is finite.

(ii) $\mathscr{M}$ has the Schur property if and only if $\mathscr{M}$ is of finite dimension as a vector space. In particular, infinite-dimensional left closed ideals of $\mathscr{A}$ do not have the Schur property.

Proof. (i): Suppose that $\mathscr{A}$ is $*$-isomorphic to a C*-algebra of compact operators. There exist Hilbert spaces H_i such that $\mathscr{A} = \oplus_{i\in I} K(H_i)$ and $\mathscr{M} = \oplus_{i\in I} \mathscr{M}_i$, where $\mathscr{M}_i = \overline{K(H_i)\mathscr{M}}$ for all $i \in I$ (see [2]). By [2, Theorem 3], each $\mathscr{M}_i$ is a full Hilbert $K(H_i)$-module of the form

$$\mathscr{M}_i = \bigoplus_{\dim_{K(H_i)} \mathscr{M}_i} H_i$$

in which $\dim_{K(H_i)} \mathscr{M}_i$ is the orthogonal dimension of $\mathscr{M}_i$ over $K(H_i)$. Assume that $\mathscr{M}$ has module Schur property. Suppose that $\dim_{K(H_{i_0})} \mathscr{M}_{i_0}$ is infinite, for some $i_0 \in I$. Choose a nonzero element $h_0 \in H_{i_0}$. Then, for the sequence

$$x_1 = (h_0, 0, 0, \ldots),\ x_2 = (0, h_0, 0, 0, \ldots),\ x_3 = (0, 0, h_0, 0, 0, \ldots),\ \ldots$$

of $\oplus_{\mathbb{N}} H_{i_0}$, we have $\langle y, x_n \rangle \to 0$, for all $y \in \oplus_{\mathbb{N}} H_{i_0}$ and $x_n \nrightarrow 0$. But for every $i \in I$, $\mathscr{M}_i$ and so $\oplus_{\mathbb{N}} H_{i_0}$ has module Schur property. Thus, $\dim_{K(H_i)} \mathscr{M}_i$ is finite for every $i \in I$. On the other hand, as I is infinite, it contains an infinite countable subset $\{i_1, i_2, \ldots\}$

and for every $n \in \mathbb{N}$ we can choose $m_{i_n} \in \mathscr{M}_{i_n}$ such that $\|m_{i_n}\| = 1$. Now, for the sequence

$$x_1 = (m_{i_1}, 0, 0, \ldots),\ x_2 = (0, m_{i_2}, 0, 0, \ldots),\ x_3 = (0, 0, m_{i_3}, 0, 0, \ldots),\ \ldots$$

we have $\langle y, x_n \rangle \to 0$ for all $y \in \oplus_{i \in \{i_1, i_2, \ldots\}} \mathscr{M}_i$ and $x_n \nrightarrow 0$. Since, $\oplus_{i \in \{i_1, i_2, \ldots\}} \mathscr{M}_i$ has the module Schur property, I is finite. This implies that $\dim_A \mathscr{M} < \infty$.
Now, assume $\dim_A \mathscr{M} < \infty$. If for every $i \in I$, $\mathscr{U}_i$ is an orthogonal basis for $\mathscr{M}_i$, then $\cup_{i \in I} \mathscr{U}_i$ is an orthogonal basis for $\mathscr{M}$. Since two orthogonal bases have same cardinality for $\mathscr{M}$ (see [2] or [1]), I and each $\mathscr{U}_i$ are finite for every $i \in I$. Hence, by [8, Proposition 4.8], it is enough to show that H_i as a full left Hilbert $K(H_i)$-module has module Schur property for every $i \in I$. But if $(x_n)_{n \in \mathbb{N}} \subseteq H_i$ and $\langle y, x_n \rangle = \xi_{y, x_n} \to 0$ for all $y \in H_i$, then $\|\xi_{y, x_n}\| = \|y\| \|x_n\| \to 0$, and so $x_n \to 0$.

(ii): Since $\mathscr{M}$ has Schur property, it has module Schur property and the property $(\star)$. Therefore, by Proposition 2.6, $\mathscr{A}$ is of the form $\mathscr{A} = \oplus_{i \in I} K(H_i)$, where the Hilbert spaces H_i are of finite dimension. By a similar argument as given in the proof of part (i), we get I is finite and $\dim_{K(H_i)} M_i < \infty$ for all $i \in I$. Now, by [2, Theorem 1], $\mathscr{M}_i$ is algebraically finitely generated for all $i \in I$. This together with the fact that H_i's are finite-dimensional imply that the Hilbert modules $\mathscr{M}_i$ and so $\mathscr{M}$ are of finite dimension as vector spaces. □

We end the paper by giving a remark on Banach-Saks property.

Recall that a Banach space X has the Banach-Saks property if every bounded sequence $(x_n)_n$ in X has a subsequence $(x_{n_k})_k$ such that

$$\lim_{k \to +\infty} \left\| \frac{1}{k} \sum_{i=1}^{k} x_{n_i} - x \right\| = 0, \tag{2.1}$$

for some element $x \in X$. And, X has the weak Banach-Saks property if every weakly null sequence $(x_n)_n$ in X has a subsequence $(x_{n_k})_k$ such that the equality (2.1) is fulfilled with $x = 0$ (see [8]). It is obvious that the Banach-Saks property implies the weak Banach-Saks property and by Proposition 2.1, if a full Hilbert C*-module has the weak Banach-Saks property, then it has the module Banach-Saks property. If $\mathscr{A}$ is a finite-dimensional C*-algebra, then all Hilbert $\mathscr{A}$-modules, by [6, Corollary 4.3] (see also [7]), have Banach-Saks property and so they have weak Banach-Saks property. Now assume that $\mathscr{A}$ is either of the form of a c_0-direct sum of matrices or of the form of a finite-dimensional matrix algebra and $\mathscr{M}$ is a full Hilbert $\mathscr{A}$-module. If $(x_n)_n \subseteq \mathscr{M}$ and $x_n \to 0$ in $\mathscr{T}_w$, then by Propositions 2.2 and 2.3, we have $x_n \to 0$ in $\mathscr{T}_{\mathscr{M}}$, and thus by [8, Theorem 5.3], the sequence $(x_n)_n$ has a subsequence $(x_{n_i})_i$ such that $\frac{1}{k} \sum_{i=1}^{k} x_{n_i}$ converges in norm to 0. Hence, every full Hilbert $\mathscr{A}$-module has the weak Banach-Saks property.

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