

LOCAL ISOMETRIES ON SUBSPACES AND SUBALGEBRAS OF FUNCTION SPACES

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Abstract. Let \mathbb{K} denotes the field of real or complex numbers. For a locally compact Hausdorff space X , we denote by $C_0(X)$ the space of all \mathbb{K} -valued continuous functions on X vanishing at infinity. Let E be a (real or complex) Banach space, K_E be a closed subset of E , and $C_u(K_E)$ be the algebra of all real or complex valued, uniformly continuous bounded functions defined on K_E . Endowed with the supremum norm, both $C_0(X)$ and $C_u(K_E)$ are Banach spaces. In this paper we study the structure of local isometries on subspaces of $C_0(X)$ and various subalgebras of $C_u(K_E)$.

1. Introduction

Let E and F be Banach spaces. We respectively denote by $B(E, F)$ and $\mathcal{G}(E, F)$, the Banach space of all bounded linear operators, and the set of all surjective linear isometries, from E to F . If $E = F$, then $B(E, E)$ is denoted by $B(E)$, and $\mathcal{G}(E, E)$ by $\mathcal{G}(E)$. Let $T \in B(E)$ such that for every $x \in E$, Tx coincides with the action of a surjective linear isometry on x , that is, there exists a $T_x \in \mathcal{G}(E)$ (depending on x , that is why the subscript x and this isometry may vary from point to point) such that $T(x) = T_x(x)$. One may ask under what conditions we have $T \in \mathcal{G}(E)$. Such a T is called a local surjective isometry and we say that T interpolates $\mathcal{G}(E)$. We observe here that any local surjective isometry is in fact an isometry. Indeed, $\|T(x)\| = \|T_x(x)\| = \|x\|$. So, the problem reduces to see whether any local surjective isometry is automatically surjective. Assume that E is finite dimensional, then we know that any injective linear map is automatically surjective. Thus, the above problem has a positive answer in the case of finite dimensional Banach spaces.

Now, let E be an infinite dimensional Hilbert space, and T be any into isometry on E . Let $x, y \in E$ such that $Tx = y$. As $\|x\| = \|y\|$, there exists an operator $S \in \mathcal{G}(E)$ such that $S(x) = y$. Therefore, T is a local surjective isometry on E which is not surjective.

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It is now a natural question to ask what happens in other infinite dimensional Banach spaces. This is a very basic problem in the sense that we want to get a global conclusion from a local hypothesis.

Besides the isometry group, the above problem can be generalized to other important classes of transformations on operator algebras like automorphism group and derivations. Investigations of this kind were initiated by Kadison, Larson and Sourour [5, 6, 7]. Later on, this problem attracted the attention of many researchers, see [3, 4, 8, 11, 12] and the references therein. For a comprehensive account of this theory we refer the reader to Molnar's Monograph [9].

We set some notations and terminologies which will be followed in the rest of the paper. Let \mathbb{K} denotes the field of real or complex numbers. Let X be a locally compact Hausdorff space, and $C_0(X)$ be the Banach space of all \mathbb{K} -valued continuous functions on X vanishing at infinity, equipped with the supremum norm. Let E be a (real or complex) Banach space, K_E be a closed subset of E , and $C_u(K_E)$ be the Banach algebra of all real or complex valued, uniformly continuous bounded functions defined on K_E endowed with the supremum norm.

We consider the following definitions.

DEFINITION 1.1. Let A be a subspace of $C_0(X)$. We say that A is strongly separating if given any pair of distinct points x_1, x_2 of X , then there exists $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$.

DEFINITION 1.2. A closed subalgebra $A_u(K_E)$ of $C_u(K_E)$ is said to be weakly normal if, given any subsets A and B of K_E with a positive distance $d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}$, there is an $f \in A_u(K_E)$ such that $|f(x)| \geq 1$ for every $x \in A$, and $|f(y)| \leq \frac{1}{2}$ for every $y \in B$.

For a weakly normal closed subalgebra $A_u(K_E)$ of $C_u(K_E)$, we denote by $A_u^0(K_E)$ the subalgebra of $A_u(K_E)$ whose elements vanish at $0 \in K_E$, that is,

$$A_u^0(K_E) = \{f \in A_u(K_E) : f(0) = 0\}.$$

Let $\mathcal{S} \subset B(E)$. We define the algebraic closure of \mathcal{S} as

$$\overline{\mathcal{S}}^a = \{T \in B(E) : Tx \in \mathcal{S}x, \forall x \in E\},$$

where $\mathcal{S}x = \{Sx : S \in \mathcal{S}\}$. If $T \in \overline{\mathcal{S}}^a$, we say that T interpolates \mathcal{S} or T is locally in \mathcal{S} . Clearly, $\mathcal{S} \subseteq \overline{\mathcal{S}}^a$. The subset \mathcal{S} is called algebraically reflexive if $\mathcal{S} = \overline{\mathcal{S}}^a$. When $\mathcal{S} = \mathcal{G}(E)$, then $\mathcal{G}(E)$ is called algebraically reflexive $\overline{\mathcal{G}(E)}^a = \mathcal{G}(E)$, that is, if every local surjective isometry is surjective. From whatever we mentioned in the first and second paragraphs of this article, we can say that, if E is finite dimensional, then $\mathcal{G}(E)$ is algebraically reflexive, and if E is an infinite dimensional Hilbert space, then $\mathcal{G}(E)$ fails to be algebraically reflexive.

A natural setting for studying the algebraic reflexivity of the isometry group of a Banach space is where a complete description is available. The study of isometries between Banach spaces is one of the most important research areas in functional analysis.

One of the most classical results in this area is the Banach-Stone theorem describing surjective linear isometries between Banach spaces of complex-valued continuous functions on compact Hausdorff spaces. This classical theorem has been generalized by many authors in several directions, for example, by considering into linear isometries or replacing $C_0(X)$ by its subspaces and subalgebras. In [1], Araujo and Fonf described the structure of surjective linear isometries on strongly separating subspaces of $C_0(X)$, thereby extending many earlier results available in this direction. Similarly, in [2], the authors characterized surjective linear isometries on the subalgebras $A_u(K_E)$ and $A_u^0(K_E)$ of $C_u(K_E)$.

Motivated by these results, in this paper we establish the algebraically reflexivity of the following three important sets:

1. The set of all surjective linear isometries between strongly separating subspaces of $C_0(X)$;
2. The set of all surjective linear isometries between weakly normal closed subalgebras of $C_u(K_E)$; and
3. The set of all surjective linear isometries between the subalgebra $A_u^0(K_E)$ of $A_u(K_E)$.

Our proofs follow many ideas presented in the papers [1] and [2], but the local structure of the isometries considered made the proofs much easier.

2. Preliminaries and basic results

In this section we mention few definitions and recall some results which will be used later. We begin with the following definition.

DEFINITION 2.1. Let A be a subspace of $C_0(X)$. A subset U of X is said to be a boundary for A if each function in A attains its maximum on U . The Shilov boundary of A , denoted ∂A , is the unique minimal closed boundary for A .

REMARK 2.2. Let us define the sets $\sigma A = \{x_0 \in X : \text{for each neighbourhood } U \text{ of } x_0, \exists f \in A \text{ such that } |f(x)| < \|f\|, \forall x \in X - U\}$, and $\sigma_0 A = \sigma A \cap \{x \in X : \exists f \in A \text{ with } f(x) \neq 0\}$. It is known that if A is a subspace of $C_0(X)$, then $\partial A = \sigma A$ [1, Lemma 2.1].

REMARK 2.3. The closed subalgebras $A_u(K_E)$ and $A_u^0(K_E)$ can be identified respectively with closed subalgebras $A(E)$ and $A_0(E)$ of $C(\gamma E)$, where γE is a compactification of K_E defined as the quotient space $\gamma E := \beta K_E / \mathcal{R}$. Here, βK_E is the Stone-Cech compactification of K_E , and \mathcal{R} is the equivalence relation defined as $x_1 \mathcal{R} x_2$ if $f(x_1) = f(x_2)$ for every $f \in A_u(K_E)$. It is known that $A(E)$ and $A_0(E)$ separate strongly the points of γE . Moreover, $K_E \subseteq \partial A(E)$ and $\partial A(E) = \gamma E$. Furthermore, $K_E \setminus \{0\} \subseteq \partial A_0(E)$ and $\partial A_0(E) \setminus \{0\} = \gamma E \setminus \{0\}$. For more details see [2].

We also note here that $A(E)$ is a uniform algebra, that is, a closed separating subalgebra of $C(\gamma E)$ which contains the constants. This means that $A(E)$ is nowhere vanishing, that is, for every $\xi \in \gamma E$, there exists $f \in A(E)$ such that $f(\xi) \neq 0$. It follows from Remark 2.2 that

$$\begin{aligned}\sigma_0 A(E) &= \sigma A(E) \cap \{\xi \in \gamma E : \exists f \in A(E) \text{ with } f(\xi) \neq 0\} \\ &= \partial A(E) \cap \gamma(E) \text{ (since } A(E) \text{ is nowhere vanishing)} \\ &= \partial A(E) \\ &= \gamma E.\end{aligned}$$

Further, since $A_0(E)$ separates strongly the points of γE , for $\xi \in \gamma E$ such that $\xi \neq 0$, there exists an $f \in A_0(E)$ such that $|f(\xi)| \neq |f(0)|$. Since $f(0) = 0$, we have $|f(\xi)| \neq 0$ and hence $f(\xi) \neq 0$. Therefore, $\{\xi \in \gamma E : \exists f \in A_0(E) \text{ with } f(\xi) \neq 0\} = \gamma E \setminus \{0\}$. This implies that $\sigma_0 A_0(E) = \gamma E \setminus \{0\}$.

The structure of into and onto isometries on strongly separating subspaces of $C_0(X)$ are given in the next two theorems.

THEOREM 2.4. [1, Theorem 3.1] *Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ into $C_0(Y)$. Then there are a subset Y_0 of Y , which is a boundary for $T(A)$, a continuous map h from Y_0 onto $\sigma_0 A$ and a continuous map $a : Y_0 \rightarrow \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in Y_0$, and*

$$Tf(y) = a(y)f(h(y)) \text{ for all } y \in Y_0 \text{ and all } f \in A.$$

Furthermore, if $\sigma_0 A$ is compact, then Y_0 is closed.

THEOREM 2.5. [1, Theorem 4.1] *Let T be a linear isometry of a strongly separating linear subspace A of $C_0(X)$ onto such a subspace B of $C_0(Y)$. Then there exist a homeomorphism h of $\sigma_0 B$ onto $\sigma_0 A$ and a continuous map $a : \sigma_0 B \rightarrow \mathbb{K}$, such that $|a(y)| = 1$ for all $y \in \sigma_0 B$, and*

$$Tf(y) = a(y)f(h(y)) \text{ for all } y \in \sigma_0 B \text{ and all } f \in A.$$

The next two theorems characterizes surjective linear isometries on the subalgebras $A_u(K_E)$ and $A_u^0(K_E)$ of $C_u(K_E)$

THEOREM 2.6. [2, Theorem 4.3] *Let X and Y be Banach spaces and let $T : A_u(K_X) \rightarrow A_u(K_Y)$ be a linear surjective isometry. Then there exists a uniform homeomorphism h of K_Y onto K_X and a function $a \in C_u(K_Y)$, such that $|a(y)| = 1$ for all $y \in K_Y$, and $Tf(y) = a(y)f(h(y))$ for all $y \in K_Y$ and for all $f \in A_u(K_X)$.*

THEOREM 2.7. [2, Theorem 4.8] *Let X and Y be Banach spaces and let $T : A_u^0(K_X) \rightarrow A_u^0(K_Y)$ be a linear surjective isometry. Then there exists a uniform homeomorphism h of K_Y onto K_X with $h(0) = 0$. Furthermore, there is a function $a \in C(K_Y \setminus \{0\})$, with $|a(y)| = 1$ for all $y \in K_Y \setminus \{0\}$, such that, for all $f \in A_u^0(K_X)$*

$$Tf(y) = \begin{cases} a(y)f(h(y)), & y \in K_Y \setminus \{0\}, \\ 0, & y = 0. \end{cases}$$

3. Algebraic reflexivity of isometries between subspaces of continuous functions

In this section we prove the algebraic reflexivity of the set of all surjective linear isometries between strongly separating subspaces of $C_0(X)$. Our proof rely on the assumption that such subspaces support an injective function. Using the local structure of the isometry in question along with this assumption, the proof becomes much easier compared to the proof of Theorem 2.5 in [1]. Our main result is the following.

THEOREM 3.1. *Let X and Y be locally compact Hausdorff spaces, and let A and B be strongly separating linear subspaces of $C_0(X)$ and $C_0(Y)$ respectively. If there exists a nonnegative real-valued injective function $g \in A$ and $\sigma_0 A$ is compact, then $\mathcal{G}(A, B)$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(A, B)}^a$. Since T is an into isometry, Theorem 2.4 implies that there exist a subset Y_0 of Y , a continuous onto map $h : Y_0 \rightarrow \sigma_0 A$ and a continuous map $\tau : Y_0 \rightarrow \mathbb{K}$, such that $|\tau(y)| = 1 \ \forall y \in Y_0$, and

$$Tf(y) = \tau(y)f(h(y)), \ \forall y \in Y_0 \text{ and } f \in A. \quad (3.1)$$

To prove the surjectivity of T we will show that h is a homeomorphism and $Y_0 = \sigma_0 B$.

First we show that h is injective. For the map g given in the hypothesis, there exists $T_g \in \mathcal{G}(A, B)$ such that $Tg = T_g g$. Theorem 2.5 implies the existence of a homeomorphism $h_g : \sigma_0 B \rightarrow \sigma_0 A$ and a continuous map $\tau_g : \sigma_0 B \rightarrow \mathbb{K}$, such that $|\tau_g(y)| = 1 \ \forall y \in \sigma_0 B$, and

$$Tg(y) = \tau_g(y)g(h_g(y)), \ \forall y \in \sigma_0 B. \quad (3.2)$$

From the proof of Theorem 2.5 we know that $Y_0 \subseteq \sigma_0 B$. Now, Equations (3.1) and (3.2) imply that $g(h(y)) = g(h_g(y))$, $\forall y \in Y_0$. Thus, $h = h_g$ on Y_0 and hence h is injective. Using [10, Theorem 26.6] we conclude that h is a homeomorphism.

It remains to prove that $\sigma_0 B \subseteq Y_0$. Indeed, for $y \in \sigma_0 B$, we have $h_g(y) \in \sigma_0 A$. As h is onto, there exists $y_0 \in Y_0$ such that $h(y_0) = h_g(y)$. But $h = h_g$ on Y_0 , therefore, $y = y_0$. This completes the proof. \square

4. Algebraic reflexivity of isometries between subalgebras of uniformly continuous functions

In this section we prove the algebraic reflexivity of the set of all surjective linear isometries on weakly normal subalgebras of $C_u(K_E)$ and on the subalgebra $A_u^0(K_E)$. We again assume the these two subalgebras support an injective function.

PROPOSITION 4.1. *Let E and F be Banach spaces. If there exists an injective map $g \in A_u(K_E)$ such that $g(x) \geq 1$ for all $x \in K_E$, then $\mathcal{G}(A_u(K_E), A_u(K_F))$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(A_u(K_E), A_u(K_F))}^u$. Using Remark 2.3 and Theorem 3.1, we conclude that T is a surjective linear isometry between closed subalgebras $A(E)$ and $A(F)$ of $C(\gamma E)$ and $C(\gamma F)$ respectively. Theorem 2.5 implies the existence of a homeomorphism $h: \gamma F \rightarrow \gamma E$ and a continuous map $\tau: \gamma F \rightarrow \mathbb{K}$, such that $|\tau(y)| = 1$ for all $y \in \gamma F$, and

$$Tf(y) = \tau(y)f(h(y)), \forall y \in \gamma F \text{ and } f \in A(E). \quad (4.1)$$

In order to prove that $T: A_u(K_E) \rightarrow A_u(K_F)$ is a surjective linear isometry we need to show that $h: K_F \rightarrow K_E$ is a uniform homeomorphism and $\mu = \tau|_{K_F} \in C_u(K_F)$.

For the first part, considering the map g given in the hypothesis, there exists $T_g \in \mathcal{G}(A_u(K_E), A_u(K_F))$ such that $Tg = T_g g$. Now, Theorem 2.6 implies the existence of a uniform homeomorphism $h_g: K_Y \rightarrow K_X$ and a function $\tau_g \in C_u(K_F)$, such that $|\tau_g(y)| = 1$ for all $y \in K_F$, and

$$Tg(y) = \tau_g(y)g(h_g(y)), \forall y \in K_F. \quad (4.2)$$

Comparing Equations (4.1) and (4.2) and using the injectivity of g we conclude that $h = h_g$ on K_F . Thus, h is a uniform homeomorphism.

To prove the second part, suppose on the contrary that μ is not uniformly continuous on K_F . Then there exist $\varepsilon > 0$ and two sequences (x_n) and (y_n) in K_F such that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ and $|\mu(x_n) - \mu(y_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$.

As Tg is uniformly continuous, $\lim_{n \rightarrow \infty} (Tg(x_n) - Tg(y_n)) = 0$ or

$$\lim_{n \rightarrow \infty} (\mu(x_n)g(h(x_n)) - \mu(y_n)g(h(y_n))) = 0. \quad (4.3)$$

Similarly for the map g^2 we will have

$$\lim_{n \rightarrow \infty} (\mu(x_n)g^2(h(x_n)) - \mu(y_n)g^2(h(y_n))) = 0. \quad (4.4)$$

Multiplying Equation (4.3) by $g(h(x_n))$ we get

$$\lim_{n \rightarrow \infty} (\mu(x_n)g^2(h(x_n)) - \mu(y_n)g(h(x_n))g(h(y_n))) = 0. \quad (4.5)$$

Subtracting Equations (4.4) and (4.5) we will get

$$\lim_{n \rightarrow \infty} (\mu(y_n)g^2(h(y_n)) - \mu(y_n)g(h(x_n))g(h(y_n))) = 0.$$

This implies that

$$\lim_{n \rightarrow \infty} (g(h(x_n)) - g(h(y_n))) = 0. \quad (4.6)$$

Lastly multiplying Equation (4.6) by $\mu(x_n)$ and subtracting Equation (4.3) we get

$$\lim_{n \rightarrow \infty} g(h(y_n))(\mu(x_n) - \mu(y_n)) = 0.$$

This is a contradiction. Hence, μ is uniformly continuous on K_F and the proof is complete. \square

PROPOSITION 4.2. *Let E and F be Banach spaces. If there exists a nonnegative real-valued injective function $g \in A_u^0(K_E)$, then $\mathcal{G}(A_u^0(K_E), A_u^0(K_F))$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}(A_u^0(K_E), A_u^0(K_F))}^a$. Using the same arguments of Proposition 4.1, there exist a homeomorphism $h : \gamma F \setminus \{0\} \rightarrow \gamma E \setminus \{0\}$ and a continuous map $\tau : \gamma F \setminus \{0\} \rightarrow \mathbb{K}$, such that $|\tau(y)| = 1$ for all $y \in \gamma F \setminus \{0\}$, and

$$Tf(y) = \tau(y)f(h(y)), \quad \forall y \in \gamma F \setminus \{0\} \text{ and } f \in A_0(E). \quad (4.7)$$

Since $\tau|_{K_F \setminus \{0\}} \in C(K_F \setminus \{0\})$, in order to prove that $T : A_u^0(K_E) \rightarrow A_u^0(K_F)$ is a surjective linear isometry we just need to show that $h : K_F \rightarrow K_E$ is a uniform homeomorphism with $h(0) = 0$.

For the map g in the hypothesis, there exists $T_g \in \mathcal{G}(A_u^0(K_E), A_u^0(K_F))$ such that $Tg = T_g g$. Theorem 2.7 implies the existence of a uniform homeomorphism $h_g : K_F \rightarrow K_E$ with $h_g(0) = 0$ and a function $\tau_g \in C(K_F \setminus \{0\})$, with $|\tau_g(y)| = 1$ for all $y \in K_F \setminus \{0\}$, such that

$$Tg(y) = \begin{cases} \tau_g(y)g(h_g(y)), & y \in K_F \setminus \{0\}, \\ 0, & y = 0. \end{cases} \quad (4.8)$$

From Equations (4.7) and (4.8) we get $h = h_g$ on $K_F \setminus \{0\}$ implying that h is a uniform homeomorphism of $K_F \setminus \{0\}$ onto $K_E \setminus \{0\}$. The map h can be extended to a uniform homeomorphism of K_F onto K_E by defining $h(0) = 0$. This completes the proof. \square

We end this paper by stating the following problem. Are the group of surjective linear isometries on subspaces and subalgebras of vector-valued functions spaces algebraically reflexive?

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