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REFINING AND REVERSING JENSEN'S INEQUALITY

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Abstract. This paper is focused on Jensen's inequality and its variants. Various refinements and reverses of Jensen's inequality, including scalar and operator versions, are given.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the C^*–algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$. In the case when $\dim \mathscr{H} = n$, we identify $\mathscr{B}(\mathscr{H})$ with the matrix algebra $\mathscr{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. As customary, we reserve m, M for scalars and $\mathbf{1}_{\mathscr{H}}$ for the identity operator on $\mathscr{H}$. A self-adjoint operator A is said to be positive (written $A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ holds for all $x \in \mathscr{H}$. An operator A is said to be strictly positive (denoted by $A > 0$) if A is positive and invertible. If A and B are self-adjoint, we write $B \geqslant A$ in case $B - A \geqslant 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$–isomorphism between the C^*–algebra $C(sp(A))$ of continuous functions on the spectrum $sp(A)$ of a selfadjoint operator A and the C^*–algebra generated by A and the identity operator $\mathbf{1}_{\mathscr{H}}$. If $f, g \in C(sp(A))$, then $f(t) \geqslant g(t)$ $(t \in sp(A))$ implies that $f(A) \geqslant g(A)$.

A linear map $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ is positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is said to be unital if $\Phi(\mathbf{1}_{\mathscr{H}}) = \mathbf{1}_{\mathscr{K}}$. A continuous function f defined on the interval J is called an operator convex (resp. concave) function if $f((1-v)A + vB) \leqslant$ (resp. $\geqslant$) $(1-v)f(A) + vf(B)$ for every $0 < v < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are in J. For instance, the function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leqslant r \leqslant 2$ or $-1 \leqslant r \leqslant 0$. Also, the function $f(t) = t^r$ is operator concave on $(0, \infty)$ if $0 \leqslant r \leqslant 1$.

The classical Jensen's inequality for convex functions states that if f is a convex function on an interval $[m, M]$, then

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant \sum_{i=1}^{n} w_i f(x_i) \tag{1.1}$$

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for all $x_i \in [m,M]$ and $w_i \in [0,1]$ $(i=1,\dots,n)$ with $\sum_{i=1}^n w_i = 1$. There is an extensive amount of literature devoted to Jensen's inequality concerning different generalizations, refinements, and converse results; see, for example [5, 10, 11, 12].

Mercer [6] established a variant of the Jensen's inequality (1.1) as follows: If f is a convex function on $[m,M]$, then

$$f\left(M+m-\sum_{i=1}^n w_i x_i\right) \leqslant f(M)+f(m)-\sum_{i=1}^n w_i f(x_i) \tag{1.2}$$

for all $x_i \in [m,M]$ and $w_i \in [0,1]$ $(i=1,\dots,n)$ with $\sum_{i=1}^n w_i = 1$.

Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator with spectra in $[m,M]$, and let $f(t)$ be a convex function on $[m,M]$, then from [8], we have for any unit vector $x \in \mathscr{H}$,

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle. \tag{1.3}$$

For more information of (1.3) see [9, Lemma 1]. The well known operator Jensen's inequality states

$$f(\Phi(A)) \leqslant \Phi(f(A)). \tag{1.4}$$

It holds for every operator convex function $f: J \to \mathbb{R}$, self-adjoint operator A with spectra in J, and unital positive linear map Φ [1, 2]. See [4] for a comprehensive account on this inequality.

In [4], the authors gave a generalization of the operator Jensen's inequality (1.4) as follows: Let $A_i \in \mathscr{B}(\mathscr{H})$ be self-adjoint operators with spectra in $[m,M]$, let $\Phi_i \in P_N[\mathscr{B}(\mathscr{H}),\mathscr{B}(\mathscr{K})]$ be normalized positive linear maps and let $f(t)$ be operator convex on $[m,M]$. Then

$$f\left(\sum_{i=1}^n w_i \Phi_i(A_i)\right) \leqslant \sum_{i=1}^n w_i \Phi_i(f(A_i)), \tag{1.5}$$

where $w_i \in [0,1]$ $(i=1,\dots,n)$ with $\sum_{i=1}^n w_i = 1$.

The main aim of this paper is to give an improvement of the Jensen's inequality (1.1). The advantage of our method is that it can also be used to obtain a converse of (1.1). A refinement and a reverse for the Jensen-Mercer inequality (1.2) are given as well. We extend our method to self-adjoint operators by reversing and improving the inequalities (1.3), (1.4) and (1.5).

2. Main results

Our refinement and reverse of the inequality (1.1) are presented in the following theorem.

THEOREM 2.1. *If f is a convex function on an interval $[m,M]$, then for every $x_1,\dots,x_n \in [m,M]$ and every positive real numbers $w_1,\dots,w_n$ with $\sum\limits_{i=1}^n w_i = 1$,*

$$f\left(\sum_{i=1}^n w_i x_i\right) \leqslant 2\sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right) - f\left(\sum_{i=1}^n w_i x_i\right) \leqslant \sum_{i=1}^n w_i f(x_i), \tag{2.1}$$

and

$$\sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant 2\left(\sum_{i=1}^{n} w_i f(x_i) - \sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)\right). \tag{2.2}$$

Both inequalities (2.1) *and* (2.2) *hold in the reverse direction if f is concave.*

Proof. We prove the left-hand side of (2.1). To do this end, we have

$$\begin{aligned}
&2\sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) - f\left(\sum_{i=1}^{n} w_i x_i\right)\\
&\geqslant 2f\left(\sum_{i=1}^{n} w_i \left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)\right) - f\left(\sum_{i=1}^{n} w_i x_i\right) \quad \text{(by (1.1))}\\
&= 2f\left(\sum_{i=1}^{n} w_i x_i\right) - f\left(\sum_{i=1}^{n} w_i x_i\right)\\
&= f\left(\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&2\sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) - f\left(\sum_{i=1}^{n} w_i x_i\right)\\
&\leqslant 2\sum_{i=1}^{n} w_i \frac{f\left(\sum_{j=1}^{n} w_j x_j\right) + f(x_i)}{2} - f\left(\sum_{i=1}^{n} w_i x_i\right) \quad \text{(by (1.1))}\\
&= f\left(\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right)\\
&= \sum_{i=1}^{n} w_i f(x_i)
\end{aligned}$$

and the right-hand side of (2.1) is proved.

The inequality

$$\sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) \leqslant \frac{1}{2}\left(f\left(\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i f(x_i)\right)$$

implies (2.2). □

The operator versions of inequalities (2.1) and (2.2) can be stated as follow. The proof is similar to the above, so the details are left to the reader.

PROPOSITION 2.1. *Let $A_i \in \mathscr{B}(\mathscr{H})\,(i=1,\cdots,n)$ be self-adjoint operators with $sp(A_i) \subset [m,M]$ for some scalars $m<M$ and $\Phi_i \in P_N[\mathscr{B}(\mathscr{H}),\mathscr{B}(\mathscr{K})]$ normalized positive linear maps $(i=1,\cdots,k)$. If f is operator convex on an interval $[m,M]$, then for every positive real numbers $w_1,\ldots,w_n$ with $\sum_{i=1}^n w_i = 1$,*

$$\begin{aligned} f\left(\sum_{i=1}^n w_i\Phi_i(A_i)\right) &\leqslant 2\sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j\Phi_j(A_j)+\Phi_i(A_i)}{2}\right) - f\left(\sum_{i=1}^n w_i\Phi_i(A_i)\right) \\ &\leqslant \sum_{i=1}^n w_i\Phi_i(f(A_i)), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\sum_{i=1}^n w_i\Phi_i(f(A_i)) - f\left(\sum_{i=1}^n w_i\Phi_i(A_i)\right) \\ &\leqslant 2\left(\sum_{i=1}^n w_i f(\Phi_i(A_i)) - \sum_{i=1}^n w_i f\left(\frac{\sum_{j=1}^n w_j\Phi_j(A_j)+\Phi_i(A_i)}{2}\right)\right). \end{aligned} \tag{2.4}$$

Both inequalities (2.3) and (2.4) hold in the reverse direction if operator f is concave.

Clearly, (2.3) is a refinement of (1.5) and (2.4) is a reverse of (1.5).

REMARK 2.1. If we choose f as in the following: $f: X \to \mathbb{R}$, $f(x) = x^p$ ($\|\cdot\|$ is a norm on X, $p \geqslant 1$); $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp x$; $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, etc., we can obtain refinements and reverses of some well-known inequalities for vectors or real numbers as in [3], but we omit the details.

We prove a refinement and a reverse of the inequality (1.2) in the following theorem.

THEOREM 2.2. *If f is a convex function on $[m,M]$, then for every $x_1,\ldots,x_n \in [m,M]$ and every positive real numbers $w_1,\ldots,w_n$ with $\sum_{i=1}^n w_i = 1$,*

$$\begin{aligned} &f\left(M+m-\sum_{i=1}^n w_i x_i\right) \\ &\leqslant 2\sum_{i=1}^n w_i f\left(M+m-\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right) - f\left(M+m-\sum_{i=1}^n w_i x_i\right) \\ &\leqslant f(M)+f(m)-\sum_{i=1}^n w_i f(x_i), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &f(M)+f(m)-\sum_{i=1}^n w_i f(x_i) - f\left(M+m-\sum_{i=1}^n w_i x_i\right) \\ &\leqslant 2\left(f(M)+f(m)-\sum_{i=1}^n w_i f(x_i) - \sum_{i=1}^n w_i f\left(M+m-\frac{\sum_{j=1}^n w_j x_j + x_i}{2}\right)\right). \end{aligned} \tag{2.6}$$

Proof. Compute

$$
\begin{aligned}
&2\sum_{i=1}^{n} w_i f\left(M+m-\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right)\\
&\geqslant 2f\left(\sum_{i=1}^{n} w_i \left(M+m-\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right) \quad \text{(by (1.1))}\\
&= 2f\left(M+m-\sum_{i=1}^{n} w_i x_i\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right)\\
&= f\left(M+m-\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&2\sum_{i=1}^{n} w_i f\left(M+m-\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right)\\
&= 2\sum_{i=1}^{n} w_i f\left(\frac{M+m-\sum_{j=1}^{n} w_j x_j + M+m-x_i}{2}\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right)\\
&\leqslant \sum_{i=1}^{n} w_i \left(f\left(M+m-\sum_{i=1}^{n} w_i x_i\right) + f\left(M+m-x_i\right)\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right) \quad \text{(by (1.1))}\\
&= f\left(M+m-\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i f\left(M+m-x_i\right) - f\left(M+m-\sum_{i=1}^{n} w_i x_i\right)\\
&= \sum_{i=1}^{n} w_i f\left(M+m-x_i\right)\\
&\leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(x_i) \quad \text{(by [6, Lemma 1.3])}.
\end{aligned}
$$

This proves (2.5). The inequality (2.6) follows from the following fact

$$
\begin{aligned}
&\sum_{i=1}^{n} w_i f\left(M+m-\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)\\
&\leqslant \frac{1}{2}\left(f\left(M+m-\sum_{i=1}^{n} w_i x_i\right) + f(M)+f(m)-\sum_{i=1}^{n} w_i f(x_i)\right).
\end{aligned}
$$

Thus, the proof is completed. □

We continue this section by improving and reversing the inequality (1.3).

THEOREM 2.3. *Let f be a convex function on an interval $[m,M]$ and $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator with spectra in $[m,M]$. Then for any unit vector $x \in \mathscr{H}$,*

$$
f(\langle Ax,x\rangle) \leqslant 2\left\langle f\left(\frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}\right)x,x\right\rangle - f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle, \tag{2.7}
$$

and

$$\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \leqslant 2\left(\langle f(A)x,x\rangle - \left\langle f\left(\frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}\right)x,x\right\rangle\right). \tag{2.8}$$

Both inequalities (2.7) and (2.8) hold in the reverse direction if f is concave.

Proof. The first inequality in (2.7) follows directly from the inequality (1.3),

$$\begin{aligned}2\left\langle f\left(\frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}\right)x,x\right\rangle - f(\langle Ax,x\rangle) &\geqslant 2f\left(\left\langle \frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}x,x\right\rangle\right) - f(\langle Ax,x\rangle)\\ &= f(\langle Ax,x\rangle).\end{aligned}$$

The second inequality in (2.7) follows from the following inequality

$$\left\langle f\left(\frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}\right)x,x\right\rangle \leqslant \frac{1}{2}\left(\langle f(A)x,x\rangle + f(\langle Ax,x\rangle)\right) \tag{2.9}$$

for any unit vector $x \in \mathscr{H}$. In fact, if $f: J \to \mathbb{R}$ is a convex function and $a,b \in J$ we have,

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}.$$

Applying functional calculus for the operator A, we get

$$f\left(\frac{A+b\mathbf{1}_{\mathscr{H}}}{2}\right) \leqslant \frac{f(A)+f(b)\mathbf{1}_{\mathscr{H}}}{2}.$$

This implies for any unit vector $x \in \mathscr{H}$,

$$\left\langle f\left(\frac{A+b\mathbf{1}_{\mathscr{H}}}{2}\right)x,x\right\rangle \leqslant \frac{1}{2}\left(\langle f(A)x,x\rangle + f(b)\right).$$

By replacing b by $\langle Ax,x\rangle$, we infer (2.9).

The inequality (2.8) follows from (2.9). This completes the proof of Theorem 2.3. □

Theorem 2.3 implies the following refinement and reverse of Hölder-McCarthy inequality [7] (see also [4, Theorem 1.4]):

COROLLARY 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and $x \in \mathscr{H}$ be a unit vector. Then for any $r > 1$,*

$$\langle Ax,x\rangle^r \leqslant 2\left\langle \left(\frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}\right)^r x,x\right\rangle - \langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle, \tag{2.10}$$

and

$$\langle A^r x,x\rangle - \langle Ax,x\rangle^r \leqslant 2\left(\langle A^r x,x\rangle - \left\langle \left(\frac{A+\langle Ax,x\rangle \mathbf{1}_{\mathscr{H}}}{2}\right)^r x,x\right\rangle\right). \tag{2.11}$$

The above inequalities hold for $r < 0$, whenever A is positive and invertible.

Both inequalities (2.10) and (2.11) hold in the reverse direction if $0 < r < 1$.

The next elementary lemma will be used in the proof of Theorem 2.4, whose proof is given for the completeness since we were unable to find a suitable reference.

LEMMA 2.1. *Let* Φ *be a positive linear map and let* f *be a real valued continuous function on the interval* J. *If* $\Phi(A) = A$ *for any self-adjoint operator* A *with spectrum in* J, *then* $\Phi(f(A)) = f(A)$.

Proof. We know that A can be approximated uniformly by $A = \sum_j t_j E_j$ where $\{E_j\}$ is a decomposition of the unit $\mathbf{1}_{\mathscr{H}}$. Then,

$$\begin{aligned}\Phi(f(A)) &= \Phi\left(\sum_j f(t_j) E_j\right) \\ &= \sum_j f(t_j)\Phi(E_j) = \sum_j f(t_j)E_j = f(A).\end{aligned}$$

This implies the assertion. □

THEOREM 2.4. *Let* $A \in \mathscr{B}(\mathscr{H})$ *be a self-adjoint operator with spectra in* J *and let* $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ *be a unital positive linear mapping which satisfies* $\Phi\left(\overline{A}\right) = \overline{A}$. *Define* $\overline{A} := \Phi(A)$. *If* f *is an operator convex function on* J, *then*

$$f(\Phi(A)) \leqslant 2\Phi\left(f\left(\frac{A+\overline{A}}{2}\right)\right) - f(\Phi(A)) \leqslant \Phi(f(A)), \tag{2.12}$$

and

$$\Phi(f(A)) - f(\Phi(A)) \leqslant 2\Phi\left(f(A) - f\left(\frac{A+\overline{A}}{2}\right)\right). \tag{2.13}$$

Both inequalities (2.12) *and* (2.13) *hold in the reverse direction if* f *is operator concave.*

Proof. We have

$$\begin{aligned}&2\Phi\left(f\left(\frac{A+\overline{A}}{2}\right)\right) - f(\Phi(A)) \\ &\geqslant 2f\left(\Phi\left(\frac{A+\overline{A}}{2}\right)\right) - f(\Phi(A)) \quad \text{(by (1.4))} \\ &= 2f\left(\frac{\Phi(A)+\Phi\left(\overline{A}\right)}{2}\right) - f(\Phi(A)) \quad \text{(by the linearity of } \Phi\text{)} \\ &= 2f\left(\overline{A}\right) - f(\Phi(A)) \\ &= f(\Phi(A)) \quad \text{(by Lemma 2.1)}\end{aligned}$$

then we get the left-hand side of (2.12). On the other hand,

$$\begin{aligned}
&2\Phi\left(f\left(\frac{A+\overline{A}}{2}\right)\right)-f(\Phi(A))\\
&\leqslant 2\Phi\left(\frac{f(A)+f\left(\overline{A}\right)}{2}\right)-f(\Phi(A)) \quad \text{(since } f \text{ is operator convex)}\\
&=\Phi(f(A))+\Phi\left(f\left(\overline{A}\right)\right)-f(\Phi(A)) \quad \text{(by the linearity of } \Phi\text{)}\\
&=\Phi(f(A)) \quad \text{(by Lemma 2.1)}
\end{aligned}$$

then we obtain the right-hand side of (2.12).

The inequality (2.13) follows from the following fact

$$\Phi\left(f\left(\frac{A+\overline{A}}{2}\right)\right)\leqslant\frac{\Phi(f(A))+f(\Phi(A))}{2}.$$

This completes the proof of Theorem 2.4. □

The above result provides a refinement and a new reverse for inequality (1.4) as claimed in the introduction.

REMARK 2.2. Put $\Phi(A)=\frac{1}{n}Tr(A)I_n$ where $A\in\mathscr{M}_n$ is a Hermitian matrix with eigenvalues contained in J and I_n is the identity matrix in $\mathscr{M}_n$. Then, of course, Φ preserves the operator $\Phi(A)$.

COROLLARY 2.2. *Let $A\in\mathscr{B}(\mathscr{H})$ be a positive operator and let $\Phi:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{K})$ be a unital positive linear mapping which satisfies $\Phi\left(\overline{A}\right)=\overline{A}$. Define $\overline{A}:=\Phi(A)$. Then for any $1\leqslant r\leqslant 2$,*

$$\Phi(A)^r\leqslant 2\Phi\left(\left(\frac{A+A}{2}\right)^r\right)-\Phi(A)^r\leqslant\Phi(A^r), \tag{2.14}$$

and

$$\Phi(A^r)-\Phi(A)^r\leqslant 2\Phi\left(A^r-\left(\frac{A+\overline{A}}{2}\right)^r\right). \tag{2.15}$$

The above inequalities hold for $-1\leqslant r\leqslant 0$, whenever A is positive and invertible.

Both inequalities (2.14) and (2.15) hold in the reverse direction if $0\leqslant r\leqslant 1$.

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