

SEMIGROUPS GENERATED BY TENSOR SUM OF GENERATORS

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Abstract. In this article, it is shown that the notion of tensor sum semigroups introduced in [8] is not valid. Further, a C_0 -semigroup generated by the tensor sum of generators of two C_0 -semigroups is developed.

1. Introduction

Let \mathcal{X} be a Banach space and $L(\mathcal{X})$ be the space of bounded linear operators on \mathcal{X} . A one parameter semigroup of operators on \mathcal{X} means that a map $T : [0, \infty) \rightarrow L(\mathcal{X})$ such that

1. $T(0) = I$, the identity operator on \mathcal{X} .
2. $T(\delta + \varepsilon) = T(\delta)T(\varepsilon)$ for all $\delta, \varepsilon \geq 0$, the semigroup property.

The linear operator \mathcal{P} whose domain $D(\mathcal{P})$ is given by

$$D(\mathcal{P}) = \left\{ x \in \mathcal{X} : \lim_{\varepsilon \rightarrow 0^+} \frac{T(x)x - x}{\varepsilon} \text{ exists} \right\}$$

such that $\mathcal{P}x = \lim_{\varepsilon \rightarrow 0^+} \frac{T(x)x - x}{\varepsilon} = \frac{d^+}{d\varepsilon}(T(\varepsilon)x) /_{\varepsilon=0}$ for $x \in D(\mathcal{P})$ is called the infinitesimal generator of the semigroup $(T(\varepsilon))_{\varepsilon \geq 0}$ [3].

Tensor product. The tensor product $\mathcal{X} \otimes \mathcal{Y}$ of vector spaces \mathcal{X}, \mathcal{Y} can be constructed as a space of linear functionals on $L(\mathcal{X} \otimes \mathcal{Y})$ in the following way:

For $x \in \mathcal{X}, y \in \mathcal{Y}$, denoted by $x \otimes y$ the functional given by evaluation at the point (x, y) . In other words,

$$(x \otimes y)\mathcal{P} = \langle \mathcal{P}, x \otimes y \rangle = \mathcal{P}(x, y).$$

for each bilinear form \mathcal{P} on $\mathcal{X} \times \mathcal{Y}$.

The tensor product of two operators \mathcal{P} in $L(\mathcal{X})$ and \mathcal{Q} in $L(\mathcal{Y})$ is the transformation $\mathcal{P} \otimes \mathcal{Q} : \mathcal{X} \otimes \mathcal{Y} \rightarrow \mathcal{X} \otimes \mathcal{Y}$ defined by

$$(\mathcal{P} \otimes \mathcal{Q})\sum_i x_i \otimes y_i = \sum_i \mathcal{P}x_i \otimes \mathcal{Q}y_i,$$

for every $\sum_i x_i \otimes y_i \in \mathcal{X} \otimes \mathcal{Y}$ which is an operator in $L[\mathcal{X} \otimes \mathcal{Y}]$ [4].

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Tensor sum. Let \mathcal{P} and \mathcal{Q} be arbitrary operators on Banach spaces \mathcal{X} and \mathcal{Y} respectively. The tensor sum of \mathcal{P} and \mathcal{Q} is the transformation $\mathcal{P} \boxplus \mathcal{Q} : (\mathcal{X} \otimes \mathcal{Y}) \rightarrow (\mathcal{X} \otimes \mathcal{Y})$ defined by

$$\mathcal{P} \boxplus \mathcal{Q} = (\mathcal{P} \otimes I) + (I \otimes \mathcal{Q}),$$

which is an operator in $L[\mathcal{X} \otimes \mathcal{Y}]$ [6]. When the tensor product (in a finite-dimensional setting) is identified with the Kronecker product of matrices, the corresponding tensor sum expression is referred to as the Kronecker sum [1, 2].

Semigroups have become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite dimensional control theory. The tensor product of closed operators on Banach spaces are studied by Reed and Simon in [7]. They used the Gelfand theory and resolvent algebra techniques to prove the spectral mapping theorem for a certain class of rational functions. Tensor product semigroup is introduced and studied by Khalil and others in [5, 9, 10]. They have also discussed the compactness, ergodic limits and other asymptotic behaviours.

Our aim of this paper is to show that the tensor sum semigroup proposed in [8] is no longer valid. From the above literature survey, to the best of our knowledge, no work has been reported on the tensor sum of generators of C_0 -semigroups. Motivated by this fact, in this manuscript we develop a C_0 -semigroup generated by tensor sum of generators of two other C_0 -semigroups. In the first part of the paper, a counter example is given to show that the lemma 2.3 of [8] is no longer valid. Further a corrected version of the lemma is provided. In the second part, the corollary of Chernoff product formula [3] is used for the main result.

2. On tensor sums

Let \mathcal{X} and \mathcal{Y} be Banach spaces and let $\mathcal{X} \otimes \mathcal{Y}$ be the tensor product of \mathcal{X} and \mathcal{Y} . Let $\mathcal{X} \otimes_{\alpha} \mathcal{Y}$ be the completion of $\mathcal{X} \otimes \mathcal{Y}$ with respect to uniform cross norm α . In fact $\mathcal{P} \boxplus \mathcal{Q}$ acts on $\mathcal{X} \otimes \mathcal{Y}$ but not on $\mathcal{X} \boxplus \mathcal{Y}$ as given in [8]. Note that $\mathcal{X} \boxplus \mathcal{Y}$ is not defined. So lemma 2.1 of [8] is restated as follows:

LEMMA 1. *Let \mathcal{X} and \mathcal{Y} be Banach spaces and α be a uniform cross norm on $\mathcal{X} \otimes \mathcal{Y}$. If $\|(\mathcal{P}_i \boxplus \mathcal{Q}_i)z - (\mathcal{P} \boxplus \mathcal{Q})z\| \rightarrow 0$ as $i \rightarrow \infty$ for all $z \in \mathcal{X} \otimes \mathcal{Y}$ and none of the sequences (\mathcal{P}_i) , (\mathcal{Q}_i) has a subsequence that converges to zero pointwise, then $\|(\mathcal{P}_i \boxplus \mathcal{Q}_i)\|$ is uniformly bounded. Moreover each of $(\mathcal{P}_i)_i$, $(\mathcal{Q}_i)_i$ is uniformly bounded.*

In [8] lemma 2.3 is stated as, let \mathcal{X} , \mathcal{Y} be Banach spaces, α any cross norm on $\mathcal{X} \boxplus \mathcal{Y}$. Let $p, r \in \mathcal{X}$, $q, s \in \mathcal{Y}$ be non zero vectors. If $p \boxplus q = r \boxplus s$, then there exists a non zero scalar β such that $p = \beta r$, $q = \frac{1}{\beta} s$.

The results of the paper [8] depends mainly on this lemma. The following example is given to show that the lemma is no longer valid.

EXAMPLE 1. Let $M_k(\mathbb{R})$ be the vector space of matrices of order k by k . The norm of the matrix $\mathcal{P} \in M_k(\mathbb{R})$ is defined as $\|\mathcal{P}\|_{M_k} := \sup_{x \in \mathbb{R}^k} \|\mathcal{P}x\|_{\mathbb{R}^k}$ where $\|x\|_{\mathbb{R}^k} = 1$ and $\|\cdot\|_{\mathbb{R}^k}$ denotes any norm in \mathbb{R}^k . Also $M_k(\mathbb{R})$ is a Banach space.

Let $X = Y$ be the Banach space of 2×2 real matrices.

$$\text{Let } p = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, r = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, s = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$p \otimes I + I \otimes q = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$r \otimes I + I \otimes s = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Here we get, $p \boxplus q = r \boxplus s$, but \exists no non zero scalar β such that, $p = \beta r$ and $q = \frac{1}{\beta} s$.

Let \mathcal{X} and \mathcal{Y} be Banach spaces, α any cross norm on $\mathcal{X} \otimes \mathcal{Y}$. Let $p, r \in \mathcal{X}$, $q, s \in \mathcal{Y}$ be non zero vectors. If $p \otimes q = r \otimes s$, it is not necessary that $p = r$ and $q = s$.

LEMMA 2. [5] *Let \mathcal{X} , \mathcal{Y} be Banach spaces, α any cross norm on $\mathcal{X} \otimes \mathcal{Y}$. Let $p, r \in \mathcal{X}$, $q, s \in \mathcal{Y}$ be non zero vectors. If $p \otimes q = r \otimes s$, then there exists a non zero scalar β such that $p = \beta r$ and $q = \frac{1}{\beta} s$.*

The above lemma is used to modify the lemma 2.3 of [8]

LEMMA 3. *Let $p, r \in \mathcal{X}$, $q, s \in \mathcal{Y}$ be non zero vectors. If $p \boxplus q = r \boxplus s$, then there exists a non zero scalar β such that $p - r = \beta u$ and $s - q = \beta v$ where $u \in \mathcal{X}$ and $v \in \mathcal{Y}$ are the unit elements.*

Proof. If $p \boxplus q = r \boxplus s$ then

$$p \otimes v + u \otimes q = r \otimes v + u \otimes s$$

giving, $(p - r) \otimes v = u \otimes (s - q)$ and so $p - r = \beta u$ and $s - q = \beta v$ (by lemma 2.3 [5]). \square

Theorem 2.4 of [8] is crucial for development of tensor sum semigroups proposed in it. But this theorem depends on lemma 2.3 of [8] which is no longer valid as discussed above.

3. Semigroups generated by tensor sums

Since the tensor sum semigroup proposed in [8] is no longer valid, it is motivated to develop a C_0 -semigroup generated by tensor sum of generators of two other C_0 -semigroups, which is the main purpose of this section.

If $T(\varepsilon) : \mathcal{X} \rightarrow \mathcal{X}$ and $S(\varepsilon) : \mathcal{Y} \rightarrow \mathcal{Y}$ are strongly continuous semigroups their tensor product $T(\varepsilon) \otimes S(\varepsilon)$ is a strongly continuous semigroup on $(\mathcal{X} \otimes \mathcal{Y}, \alpha)$ for any uniform crossnorm α . To see this, let $z \in \mathcal{X} \otimes \mathcal{Y}$. Then

$$\begin{aligned} \alpha(T(\varepsilon) \otimes S(\varepsilon)z - z) &\leq \alpha(T(\varepsilon) \otimes S(\varepsilon)z - T(\varepsilon) \otimes Iz) + \alpha(T(\varepsilon) \otimes Iz - z) \\ &\leq \pi(T(\varepsilon) \otimes S(\varepsilon)z - T(\varepsilon) \otimes Iz) + \pi(T(\varepsilon) \otimes Iz - z) \end{aligned}$$

For the first term on the right hand side it follows if $z = \sum_i x_i \otimes y_i$ is any representation of z .

$$\pi(T(\varepsilon) \otimes S(\varepsilon)z - T(\varepsilon) \otimes Iz) \leq \sum_i \|T(\varepsilon)x_i\|_{\mathcal{X}} \cdot \|S(\varepsilon)y_i - y_i\|_{\mathcal{Y}} \rightarrow 0, \quad (\varepsilon \rightarrow 0^+).$$

As an analogous argument shows that the second term goes to zero if $\varepsilon \rightarrow 0^+$, it follows that $T(\varepsilon) \otimes S(\varepsilon)$ is strongly continuous.

LEMMA 4. [4] *Let \mathcal{X}, \mathcal{Y} be Banach spaces and $(T(\delta))_{\delta \geq 0}, (S(\varepsilon))_{\varepsilon \geq 0}$ be one parameter families of operators in $L(\mathcal{X}), L(\mathcal{Y})$ respectively. Then the following are equivalent.*

- (a) $T(\delta)$ is a one parameter semigroup on \mathcal{X} .
- (b) $T(\delta) \otimes I$ is a one parameter semigroup on $\mathcal{X} \overset{\alpha}{\otimes} \mathcal{Y}$.
- (c) $I \otimes T(\delta)$ is a one parameter semigroup on $\mathcal{Y} \overset{\alpha}{\otimes} \mathcal{X}$.

REMARK 1. [5] In general, if \mathcal{P}, \mathcal{Q} are closable, or even, closed linear operators on the Banach space \mathcal{X} , then $\mathcal{P} + \mathcal{Q}$ need not be closed. But, theorem 1.1 in [4] ensures that $\mathcal{P} \otimes I + I \otimes \mathcal{Q}$, is closable. Moreover, its closure is $\overline{\mathcal{P} \otimes I + I \otimes \mathcal{Q}}$.

The following corollary of Chernoff product formula is needed for the main result.

COROLLARY 1. (Chernoff product formula) [3] *Consider a function $V : R_+ \rightarrow L(\mathcal{X})$ satisfying $V(0) = I$ and $\|[V(\varepsilon)]^k\| \leq Me^{k\omega\varepsilon}$ for all $\varepsilon \geq 0, k \in N$, and some constants $M \geq 1, \omega \in R$. Assume that $\mathcal{P}x := \lim_{\varepsilon \rightarrow 0} \frac{V(\varepsilon)x - x}{\varepsilon}$ exists for all $x \in \mathcal{D} \subset \mathcal{X}$, where \mathcal{D} and $(\lambda_0 - \mathcal{P})\mathcal{D}$ are dense subspaces in \mathcal{X} for some $\lambda_0 > \omega$. Then the closure $\overline{\mathcal{P}}$ of \mathcal{P} generates a strongly continuous semigroup $(T(\varepsilon))_{\varepsilon \geq 0}$ which is given by $T(\varepsilon)x = \lim_{n \rightarrow \infty} [V(\varepsilon/n)]^n x$ for all $x \in \mathcal{X}$ and uniformly for $\varepsilon \in [0, \varepsilon_0]$. Moreover, $(T(\varepsilon))_{\varepsilon \geq 0}$ satisfies the estimate $\|[T(\varepsilon)]\| \leq Me^{\omega\varepsilon}$ for all $\varepsilon \geq 0$.*

The following is the main theorem of this paper.

THEOREM 1. *Let $(T(\varepsilon))_{\varepsilon \geq 0}$ and $(S(\varepsilon))_{\varepsilon \geq 0}$ be C_0 semigroups on \mathcal{X} satisfying the stability condition $\|[T(\varepsilon/n) \otimes S(\varepsilon/n)]^n\| \leq Me^{\omega\varepsilon}$ for all $\varepsilon \geq 0, n \in N$, and for constants $M \geq 1, \omega \in R$. Consider the tensor sum $\mathcal{P} \boxplus \mathcal{Q}$ on $\mathcal{D} := (\mathcal{D}(\mathcal{P}) \otimes \mathcal{Y}) \cap (\mathcal{X} \otimes \mathcal{D}(\mathcal{Q}))$ of the generators $(\mathcal{P}, \mathcal{D}(\mathcal{P}))$ of $(T(\varepsilon))_{\varepsilon \geq 0}$ and $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ of $(S(\varepsilon))_{\varepsilon \geq 0}$, and assume that \mathcal{D} and $(\lambda_0 - \mathcal{P} - \mathcal{Q})\mathcal{D}$ are dense in \mathcal{X} for some $\lambda_0 > \omega$. Then $\overline{\mathcal{R}} := \overline{\mathcal{P} \boxplus \mathcal{Q}}$ generates a C_0 -semigroup $V(\varepsilon)x = \lim_{n \rightarrow \infty} [T(\varepsilon/n) \otimes S(\varepsilon/n)]^n x, x \in \mathcal{X}, n \in N$ with uniform convergence for t in compact intervals.*

Proof. In order to apply the Chernoff product formula from corollary 1, it suffices to define $V(t) := T(t) \otimes S(t)$, $t \geq 0$, and observe that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{(T(\varepsilon) \otimes S(\varepsilon))(x \otimes y) - (x \otimes y)}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{(T(\varepsilon) \otimes S(\varepsilon))(x \otimes y) - (T(\varepsilon) \otimes I(\varepsilon))(x \otimes y)}{\varepsilon} \\
&\quad + \lim_{\varepsilon \rightarrow 0} \frac{(T(\varepsilon) \otimes I(\varepsilon))(x \otimes y) - (x \otimes y)}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \left[\frac{T(\varepsilon)x \otimes (S(\varepsilon)y - I(\varepsilon)y)}{\varepsilon} \right] \\
&\quad + \lim_{\varepsilon \rightarrow 0} \left[\frac{(T(\varepsilon)x - x) \otimes (I(\varepsilon)y - y)}{\varepsilon} \right] \\
&= (\mathcal{P} \boxplus \mathcal{Q})(x \otimes y) \text{ for all } x \otimes y \in D \subset \mathcal{X}. \quad \square
\end{aligned}$$

COROLLARY 2. Let $(T(\varepsilon))_{\varepsilon \geq 0}$ be a C_0 semigroup with generator \mathcal{P} on \mathcal{X} . If $\mathcal{Q} \in L(\mathcal{X})$, then the semigroup $(V(\varepsilon))_{\varepsilon \geq 0}$ generated by $\mathcal{P} \boxplus \mathcal{Q}$ is given by the Trotter tensor product formula

$$V(\varepsilon)x = \lim_{n \rightarrow \infty} [T(\varepsilon/n) \otimes e^{\varepsilon \mathcal{Q}/n}]^n x, \text{ for all } \varepsilon \geq 0 \text{ and } x \in \mathcal{X}.$$

COROLLARY 3. The tensor sum of the generators of two uniformly continuous semigroups, generates a uniformly continuous semigroup $(V(\varepsilon))_{\varepsilon \geq 0}$ given by

$$V(\varepsilon) = T(\varepsilon) \otimes S(\varepsilon) \text{ for all } \varepsilon \geq 0.$$

Proof.

$$\begin{aligned}
V(\varepsilon) &= [T(\varepsilon/n) \otimes S(\varepsilon/n)]^n \\
&= [e^{(\mathcal{P} \boxplus \mathcal{Q})\varepsilon/n}]^n \text{ [Since } T(\varepsilon) \text{ and } S(\varepsilon) \text{ are uniformly continuous semigroup]} \\
&= e^{(\mathcal{P} \boxplus \mathcal{Q})\varepsilon} \\
&= I + \frac{\varepsilon}{1!}(\mathcal{P} \boxplus \mathcal{Q}) + \frac{\varepsilon^2}{2!}(\mathcal{P} \boxplus \mathcal{Q})^2 + \frac{\varepsilon^3}{3!}(\mathcal{P} \boxplus \mathcal{Q})^3 + \dots \\
&= I + \frac{\varepsilon}{1!}((\mathcal{P} \otimes I) + (I \otimes \mathcal{Q})) + \frac{\varepsilon^2}{2!}((\mathcal{P} \otimes I) + (I \otimes \mathcal{Q}))^2 + \frac{\varepsilon^3}{3!}((\mathcal{P} \otimes I) \\
&\quad + (I \otimes \mathcal{Q}))^3 + \dots \\
&= \left[I + \frac{\varepsilon \mathcal{P}}{1!} + \frac{(\varepsilon \mathcal{P})^2}{2!} + \frac{(\varepsilon \mathcal{P})^3}{3!} + \dots \right] \otimes \left[I + \frac{\varepsilon \mathcal{Q}}{1!} + \frac{(\varepsilon \mathcal{Q})^2}{2!} + \frac{(\varepsilon \mathcal{Q})^3}{3!} + \dots \right] \\
&= e^{\mathcal{P}\varepsilon} \otimes e^{\mathcal{Q}\varepsilon} \\
&= T(\varepsilon) \otimes S(\varepsilon). \quad \square
\end{aligned}$$

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