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# PRODUCT OF A NILPOTENT AND A UNIPOTENT MATRIX OVER AN ALGEBRAICALLY CLOSED FIELD

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*Abstract.* In this note, we give a proof that a matrix of determinant 0 on any algebraically closed field is the product of a nilpotent matrix and a unipotent matrix which only uses elementary facts.

*"Nous sommes toujours étonnés que les autres ignorent ce que nous savons depuis cinq minutes."*
Marie Valyère, *Nuances Morales*

## 1. Introduction

In this note, all fields considered are commutative. Let $\mathbb{K}$ be an arbitrary field. $0_{k,l}$ denotes the zero matrix of $M_{k,l}(\mathbb{K})$. If $A \in M_n(\mathbb{K})$ we denote $\chi_A(X) = \det(XI_n - A)$ the characteristic polynomial of $A$ (with this definition $\chi_A(X)$ is a monic polynomial).

The multiplicative form of the well-known theorem of Jordan-Chevalley states that an invertible and triangularizable matrix over $\mathbb{K}$ can be written in a unique way as the product of a diagonalizable matrix and a unipotent matrix (for a proof of this classical result see for example [2] Theorem 21.24). Here, we want to find a similar decomposition in the case of a non invertible matrix. We have the following results which can help us to express a matrix into a product of two matrices with prescribed eigenvalues:

THEOREM 1. (A. R. Sourour, K.Tang, [3] Theorem 1) *Let $A$ be an $n \times n$ singular matrix over an arbitrary commutative field $\mathbb{F}$ and let $\beta_j$ and $\gamma_j$ $(1 \leqslant j \leqslant n)$ be elements of $\mathbb{F}$. If $A$ is not a nonzero $2 \times 2$ nilpotent matrix, then $A$ can be factored as a product $BC$ where the eigenvalues of $B$ and $C$ are $\beta_1, \dots, \beta_n$ and $\gamma_1, \dots, \gamma_n$ respectively, if and only if the number of zeros $m$ among $\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ is not less than the dimension of the null space of $A$. If $A$ is a nonzero $2 \times 2$ nilpotent matrix then $A$ can be factored as above if and only if $1 \leqslant m \leqslant 3$.*

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In particular, an easy corollary of this theorem is that on an arbitrary commutative field a non invertible matrix is the product of a nilpotent matrix and a unipotent matrix. However, the proof of theorem 1 is quite dificult and uses the nilpotent factorization theorem (see [4] for the complex case and [1] Theorem 4 for the general result). Here, we want to give a proof of this result in the case of an algebraically closed field which only uses elementary facts. Hence, we prove the following result:

THEOREM 2. *Let $\mathbb{K}$ be an algebraically closed field and $A \in M_n(\mathbb{K})$ such that $A$ is not invertible. $A$ is the product of a nilpotent matrix and a unipotent matrix.*

Unfortunately this decomposition is not unique and we give an example at the end of this note of an non-invertible matrix which has two decompositions of this type.

## 2. Proof of the result

### 2.1. Preliminary lemma

In this subpart, $\mathbb{K}$ is an arbitrary commutative field. We need the following preliminary result:

LEMMA 1. *Let* $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in M_n(\mathbb{K})$ *with* $a_{i,i} \neq 0$ *and* $P \in \mathbb{K}[X]$ *a monic polynomial of degree* $n$. *It exists* $(\alpha_1,\dots,\alpha_n) \in \mathbb{K}^n$ *such that the characteristic polynomial of the matrix* $B = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \dots & \alpha_n \\ a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & \dots & a_{2,n} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n} \end{pmatrix}$ *is equal to* $P$.

*Proof.* The characteristic polynomial of $B$ is

$$\chi_B(X) = \begin{vmatrix} X-\alpha_1 & -\alpha_2 & \dots & \dots & -\alpha_n \\ -a_{1,1} & X-a_{1,2} & \dots & \dots & -a_{1,n} \\ 0 & -a_{2,2} & \dots & \dots & -a_{2,n} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & -a_{n-1,n-1} & X-a_{n-1,n} \end{vmatrix}.$$

We will prove that it exists $(P_1,\dots,P_i) \in \mathbb{K}[X]^i$ such that $P_i$ is a monic polynomial of degree $n-i$ whose coefficients depend only of the coefficients $a_{k,l}$ and such that $\chi_B(X)$

satisfy the following equality:

$$\chi_B(X) = (X-\alpha_1)P_1 + a_{1,1}(-\alpha_2 P_2 + a_{2,2}(-\alpha_3 P_3 + a_{3,3}(\dots + (-1)^{n-i} a_{i,i} \begin{vmatrix} \alpha_{i+1} & \alpha_{i+2} & \dots & \dots & \alpha_n \\ a_{i+1,i+1} & a_{i+1,i+2}-X & \dots & \dots & a_{i+1,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n}-X \end{vmatrix} )\dots). \tag{1}$$

We have

$$\chi_B(X) = (X-\alpha_1) \begin{vmatrix} X-a_{1,2} & \dots & \dots & -a_{1,n} \\ -a_{2,2} & X-a_{2,3} & \dots & -a_{2,n} \\ & \ddots & & \vdots \\ 0 & \dots & -a_{n-1,n-1} & X-a_{n-1,n} \end{vmatrix} + (-1)^{n-1} a_{1,1} \begin{vmatrix} \alpha_2 & \alpha_3 & \dots & \dots & \alpha_n \\ a_{2,2} & a_{2,3}-X & \dots & \dots & a_{2,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n}-X \end{vmatrix}.$$

We set $P_1(X) = \begin{vmatrix} X-a_{1,2} & \dots & \dots & -a_{1,n} \\ -a_{2,2} & X-a_{2,3} & \dots & -a_{2,n} \\ & \ddots & & \vdots \\ 0 & \dots & -a_{n-1,n-1} & X-a_{n-1,n} \end{vmatrix}$. $P_1$ is the characteristic polynomial of the matrix $\begin{pmatrix} a_{1,2} & \dots & \dots & a_{1,n} \\ a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ & \ddots & & \vdots \\ 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \end{pmatrix}$. Hence, $P_1$ is a monic polynomial of degree $n-1$ whose coefficients only depend of the coefficients $a_{k,l}$.

Suppose it exists $i$ such that (1) is true.

$$\begin{aligned} \Delta_i &= (-1)^{n-i} a_{i,i} \begin{vmatrix} \alpha_{i+1} & \alpha_{i+2} & \dots & \dots & \alpha_n \\ a_{i+1,i+1} & a_{i+1,i+2}-X & \dots & \dots & a_{i+1,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n}-X \end{vmatrix} \\ &= (-1)^{n-i} a_{i,i}\alpha_{i+1} \begin{vmatrix} a_{i+1,i+2}-X & \dots & \dots & \dots & a_{i+1,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n}-X \end{vmatrix} \\ &\quad + (-1)^{n-i-1} a_{i+1,i+1} a_{i,i} \begin{vmatrix} \alpha_{i+2} & \dots & \dots & \dots & \alpha_n \\ a_{i+2,i+2} & \dots & \dots & \dots & a_{i+2,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n}-X \end{vmatrix}. \end{aligned}$$

We set $P_{i+1}(X) = \begin{vmatrix} X-a_{i+1,i+2} & \dots & \dots & \dots & -a_{i+1,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & -a_{n-1,n-1} & X-a_{n-1,n} \end{vmatrix}$. $P_{i+1}$ is the characteristic polynomial of the matrix $\begin{pmatrix} a_{i+1,i+2} & \dots & \dots & \dots & a_{i+1,n} \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a_{n-1,n-1} & a_{n-1,n} \end{pmatrix}$. Hence, $P_{i+1}$ is a monic polynomial of degree $n-i-1$ whose coefficients only depend of the coefficients $a_{k,l}$. If we replace in formula (1) the determinant by the preceeding equality we have proved that formula (1) is true for $i+1$.

Hence, formula (1) is proved by induction. Let $P(X) = \sum_{i=0}^{n} \beta_i X^i$, $\beta_n = 1$. We set the coefficients $\alpha_i$ by induction.

Thanks to formula (1), we see that the coefficient of $\chi_B(X)$ of the term of degree $n-1$ is equal to $-\alpha_1 - \sum_{i=1}^{n-1} a_{i,i+1}$. We set $\alpha_1 = -\beta_{n-1} - \sum_{i=1}^{n-1} a_{i,i+1}$.

Suppose it exists $i$ such that we have defined $\alpha_1, \dots \alpha_{i-1}$. Since the determinant in formula (1) is a polynomial of degree $n-i-1$, the coefficient of $\chi_B(X)$ of the term of degree $n-i$ is equal to the coefficient of the term of degree $n-i$ of

$$Q_i = (X-\alpha_1)P_1 + a_{1,1}(-\alpha_2 P_2 + a_{2,2}(-\alpha_3 P_3 + \dots + a_{i-1,i-1}(-\alpha_i P_i)\dots).$$

Hence, the coefficient of $\chi_B(X)$ of the term of degree $n-i$ is equal to $-\alpha_i \prod_{j=1}^{i-1} a_{j,j} + f(\alpha_1, \dots, \alpha_{i-1}, a_{k,l})$ where $f(\alpha_1, \dots, \alpha_{i-1}, a_{k,l})$ is the coefficient of $Q_i - \prod_{j=1}^{i-1} a_{j,j}(-\alpha_i P_i)$ of the term of degree $n-i$. We set

$$\alpha_i = \frac{-\beta_{n-i} + f(\alpha_1, \dots, \alpha_{i-1}, a_{k,l})}{\prod_{j=1}^{i-1} a_{j,j}} \quad \left(\prod_{j=1}^{i-1} a_{j,j} \neq 0 \text{ since } a_{j,j} \neq 0\right).$$

By induction, we have defined $\alpha_1, \dots \alpha_n$. This choice implies that $\chi_B(X) = P(X)$ and the lemma is proved. □

### 2.2. Proof of Theorem 2

Let $\mathbb{K}$ be an algebraically closed field. We proceed by induction on $n$.

If $n = 1$ then $A = (0) = (0) \times (1)$ and the result is true.

Suppose it exists $n \geqslant 1$ such that any square matrix of size $n$ satisfying the conditions of Theorem 2 is the product of a nilpotent matrix and a unipotent matrix.

Let $A \in M_{n+1}(\mathbb{K})$ satisfying the conditions of the theorem. $A$ is triangularizable and $A$ is not invertible. Hence, $\exists P \in GL_n(\mathbb{K})$, $\exists T \in M_n(\mathbb{K})$ and $\exists C \in M_{n,1}(\mathbb{K})$ such that

$$A = P \begin{pmatrix} T & C \\ 0_{1,n} & 0 \end{pmatrix} P^{-1} \text{ and T triangular.}$$

We have two possibilities:

- $T$ is not invertible. In this case, $T$ is the product of a nilpotent matrix $N$ and a unipotent matrix $U$ (by induction assumption). One has
$$A = \begin{pmatrix} T & C \\ 0_{1,n} & 0 \end{pmatrix} = \begin{pmatrix} NU & C \\ 0_{1,n} & 0 \end{pmatrix} = \begin{pmatrix} N & C \\ 0_{1,n} & 0 \end{pmatrix} \begin{pmatrix} U & 0_{n,1} \\ 0_{1,n} & 1 \end{pmatrix}.$$
$\begin{pmatrix} N & C \\ 0_{1,n} & 0 \end{pmatrix}$ is nilpotent and $\begin{pmatrix} U & 0_{n,1} \\ 0_{1,n} & 1 \end{pmatrix}$ is unipotent. Hence, $A$ is the product of a nilpotent and a unipotent matrix.

- $T$ is invertible. In this case, the diagonal coefficients of $T$ are different from 0. Hence, we can apply lemma 1 that is to say it exists $(\alpha_1, \dots, \alpha_{n+1})$ such that the characteristic polynomial of $B = \begin{pmatrix} \alpha_1 \dots \alpha_n & \alpha_{n+1} \\ T & C \end{pmatrix}$ is $(X-1)^{n+1}$. $B$ is unipotent (since its characteristic polynomial is $(X-1)^{n+1}$) and one has
$$\begin{pmatrix} T & C \\ 0_{1,n} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} B.$$
Hence, $A$ is the product of a nilpotent and a unipotent matrix.

By induction, Theorem 2 is proved. □

### 2.3. Some concluding remarks

In fact the same proof show that on an arbitrary field a triangularizable matrix which is not invertible is the product of a nilpotent matrix and a unipotent matrix.

We conclude this note by the two following remarks:

- In general the decomposition $A = NU$ with $N$ nilpotent and $U$ unipotent is not unique. For example,
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- If $A = NU$ with $N$ nilpotent and $U$ unipotent then $N$ and $U$ don't commute in general. For instance,
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

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