Operators and Matrices
Volume 16, Number 1 (2022), 75–85 doi:10.7153/oam-2022-16-07

SURGERY OF FRAMES IN HILBERT SPACES

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(*Communicated by D. Han*)

Abstract. Frames which are tight or full spark might be considered optimally conditioned in applications. This leads to the question of perfect preconditioning of frames. In this paper, we consider the surgery of frames such that given frames can be manipulated to tight or full spark frames by removing and adding some elements. We give a necessary and sufficient condition such that a (r,k)-surgery on a frame results in a tight frame. We also provide a necessary and sufficient condition such that a $(1,k)$-surgery on a tight frame resulting in a tight frame with same bound. Finally, we characterize that a (r,k)-surgery on a frame resulting in a full spark frame is possible. We obtain a necessary and sufficient condition such that a (r,k)-surgery on a given frame results in a full spark frame.

1. Introduction

A frame is a sequence of vectors in $\mathscr{H}$ such that every element in $\mathscr{H}$ has a representation as a linear combination of the frame elements and its element are not necessarily linearly independent. We also call a frame a redundant basis. Due to redundant, frames have wide applications in coding theory [11], signal processing [6, 13], quantum information [3, 5], filter bank theory [8] and neural networks [17].

Frames with nice geometric structures, such as tight or full spark, play an important role in signal processing because they provide more robust to erasures, additive noise, distortions and they also provide stable reconstruction formula. Hence, it is necessary to manipulate a given frame to a tight frame or full spark frame. There are several methods to manipulate frames to tight frames in Hilbert spaces. Kutyniok et al. introduced and characterized scalable frames such that the vectors can be rescaled to yield a tight frame [9]. However, not every frame is scalable. For example, a basis in $\mathbb{R}^2$ which is not an orthogonal basis is not scalable. An another technique for modification of a given frame to a tight frame is surgery which is first provided by Han et al. [4]. It is possible to generate a new tight frame by adding or removing some elements of a given frame. Li and Sun proved that every frame can be expanded to a tight frame by adding an element [10]. Sivaram et al. expanded a frame to a tight frame from the view of length of elements [14]. They called this surgery the length surgery. Copenhaver et al. generalized the rusults on surgery from [14] and answered the question of when length surgery resulting in a tight frame for a finite dimensional Hilbert space is possible [2].

Mathematics subject classification (2020): 42C15, 15A03, 05B20.
Keywords and phrases: Frames, surgery, full spark, tight.

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Paper OaM-16-07

Full spark frames are increasing interest in applications because they provide maximum robustness to erasures [1, 16]. However, at least to the author, there are no convenient techniques such that a frame can be modified to a full spark frames.

Our goal of this paper is to study surgery on frames such that a frame can be modified to a tight or full spark frame by removing some vectors and replacing this set with other a set of vectors.

Throughout this paper, let $\mathscr{H}$ be an n-dimensional Hilbert space. A sequence $\{x_i\}_{i=1}^m$ is called a frame for $\mathscr{H}$ if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|x\|^2 \leqslant \sum_{i=1}^m |\langle x, x_i\rangle|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{H}.$$

Here A is the greatest lower frame bound and B is the least upper frame bound. When $A = B$, $\{x_i\}_{i=1}^m$ is called a tight frame with bound A. A uniform frame is a frame in which all the vectors have equal norm. A unit-norm frame is a frame such that each frame element has norm one.

Let $\{x_i\}_{i=1}^m$ be a sequence of vectors in $\mathscr{H}$. The linear map $\Theta : \mathscr{H} \to \ell^2(\{1,\cdots,m\})$ defined by $(\Theta x)_i = \langle x, x_i\rangle$ is called the analysis operator. The adjoint Θ^* such that $\Theta : \ell^2(\{1,\cdots,m\}) \to \mathscr{H}$ is called the synthesis operator. The frame operator S of a sequence of vectors $\{x_i\}_{i=1}^m$ (not necessarily a frame) is defined as $\Theta^*\Theta$. For all $f \in \mathscr{H}$,

$$Sx = \Theta^*\Theta x = \sum_{i=1}^m \langle x, x_i\rangle x_i.$$

We can verify that $S = \sum_{i=1}^m x_i \otimes x_i$, where $x \otimes y$ is the elementary tensor rank-one operator defined by $(x \otimes y)z = \langle z, y\rangle x$ for $z \in \mathscr{H}$.

If a sequence of vectors $\{x_i\}_{i=1}^m$ is a frame for $\mathscr{H}$, its frame operator S is a positive invertible bounded linear operator on $\mathscr{H}$. Let $\{\lambda_i\}_{i=1}^n$ (counted with multiplicity and arranged in non-increasing order) be the eigenvalues of S, then $B = \lambda_1 \geqslant \cdots \geqslant \lambda_n = A$. The frame operator $S = AI$ if and only if $\{x_i\}_{i=1}^m$ is a tight frame with bound A.

2. Surgery for tight frames

We first give a simple definition of the (r,k)-surgery.

DEFINITION 2.1. A (r,k)-surgery on a finite sequence of vectors X in $\mathscr{H}$ removes r vectors from X and replaces them with k vectors.

For two sequences of vectors $X = \{x_i\}_{i=1}^m$, $Y = \{y_i\}_{i=1}^k$, we say that a (r,k)-surgery on X with Y means removing k vectors from X and replacing them with Y.

For the convenience, we always assume that the last r vectors are removed from X throughout this paper.

LEMMA 2.1. *Given two sequences of vectors $X = \{x_i\}_{i=1}^m$, $Y = \{y_i\}_{i=1}^k$, a (r,k)-surgery on X with Y resulting in a tight frame is possible if*

$$\frac{1}{n}\Big(\sum_{i=1}^{m-r}\|x_i\|^2+\sum_{i=1}^{k}\|y_i\|^2\Big)\geqslant\lambda_1,$$

where λ_1 is the largest eigenvalue of the frame operator of $\{x_i\}_{i=1}^{m-r}$.

Proof. Let S_X and S_Y be the frame operators of $X = \{x_i\}_{i=1}^{m-r}$ and $Y = \{y_i\}_{i=1}^k$, respectively, and let

$$S = S_X + S_Y = \sum_{i=1}^{m-r} x_i\otimes x_i + \sum_{i=1}^{k} y_i\otimes y_i.$$

Next, we show that it is possible to find a sequence of vectors $Y = \{y_i\}_{i=1}^k$ and a nonnegative constant A such that $S = AI$.

We first consider trace of S. Since trace is additive, we have

$$nA = \mathrm{trace}(S) = \mathrm{trace}(S_X) + \mathrm{trace}(S_Y) = \sum_{i=1}^{m-r}\|x_i\|^2+\sum_{i=1}^{k}\|y_i\|^2. \tag{2.1}$$

Let $S_Y = AI - S_X$. In this case, we calculate eigenvalues of S_Y as following:

$$\det(S_Y-\lambda_y I) = \det(AI - S_X - \lambda_y I) = \det((A-\lambda_y)I - S_X) = \det(\lambda_x I - S_X) = 0, \tag{2.2}$$

where $\lambda_x = A-\lambda_y$. From the right side of (2.2), $\det(\lambda_x I - S_X) = 0$, we have that the eigenvalues of S_X are $\{\lambda_x\}$.

We assume that the eigenvalues of S_X are $\{\lambda_x\}$, From (2.2), we have $\lambda_y = A-\lambda_x$. Next, we show $A > 0$. Since S is positive semi-definite, S_X has n nonnegative real eigenvalues (counted with multiplicity and arranged in non-increasing order). Hence, λ_y is always real. Therefore, we can choose a nonnegative constant A such that $\lambda_y = A-\lambda_x \geqslant 0$. Thus there exists a constant A such that $A \geqslant \lambda_1$. From (2.1), we have that

$$\frac{1}{n}\Big(\sum_{i=1}^{m-r}\|x_i\|^2+\sum_{i=1}^{k}\|y_i\|^2\Big)\geqslant\lambda_1. \qquad\square$$

Note that a (r,k)-surgery on X with Y resulting in a tight frame with bound A in Lemma 2.1 if and only if $S_Y = AI - S_X$. Next we give a necessary and sufficient condition such that $S_Y = AI - S_X$ from the view of majorization. We first give the following conception.

DEFINITION 2.2. [7] Let $a = \{a_i\}_{i=1}^m$, $b = \{b_i\}_{i=1}^k$ be non-increasing summable sequences of nonnegative numbers, and let $t = \min\{m, k\}$. We say that b majorizes a, noted $b \succ a$, if

$$\sum_{i=1}^{j} b_i \geqslant \sum_{i=1}^{j} a_i, \quad \text{for } 1\leqslant i\leqslant j \text{ and } \sum_{i=1}^{m} b_i = \sum_{i=1}^{k} a_i.$$

THEOREM 2.1. [12] *Let $a=\{a_i\}_{i=1}^m$ be a non-increasing sequence of positive numbers and let T be a bounded positive semi-definite operator on $\mathscr{H}$ with eigenvalues (counted with multiplicity and arranged in non-increasing order) $\Lambda=\{\lambda_i\}_{i=1}^n$. Then the following statements are equivalent:*

1. *$a \prec \Lambda$.*
2. *There exists a Bessel sequence $Y=\{y_i\}_{i=1}^m \subset \mathscr{H}$ such that $\|y_i\|^2=a_i$ for $1 \leqslant i \leqslant m$ and $S_Y=T$.*

THEOREM 2.2. *Given two sequences of vectors $X=\{x_i\}_{i=1}^m$, $Y=\{y_i\}_{i=1}^k$ (arranged in non-increasing oder under norm), a (r,k)-surgery on X with Y resulting in a tight frame with bound A if and only if $a \prec \Lambda$, where $a=\{\|y_i\|^2\}_{i=1}^k$, $\Lambda=\{A-\lambda_{n-i+1}\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ are the eigenvalues (arranged in non-increasing order) of $S_X=\sum_{i=1}^{m-r} x_i \otimes x_i$.*

Proof. Let S_X and S_Y be the frame operators of $X=\{x_i\}_{i=1}^{m-r}$ and $Y=\{y_i\}_{i=1}^k$, respectively. Assume that $\{x_i\}_{i=1}^{m-r} \cup \{y_i\}_{i=1}^k$ is a tight frame with bound A. Let S be the frame operator of $\{x_i\}_{i=1}^{m-r} \cup \{y_i\}_{i=1}^k$, we have $S=S_X+S_Y=AI$. Then $S-S_X=S_Y \geqslant 0$. We see that the eigenvalues of S_Y arranged in non-increasing order are $A-\lambda_n \geqslant \cdots \geqslant A-\lambda_1 \geqslant 0$. By Theorem 2.1 we have

$$(A-\lambda_n \geqslant \cdots \geqslant A-\lambda_1) \succ (\|y_1\|^2 \cdots \|y_k\|^2),$$

thus $a \prec \Lambda$.

Conversely, let $a \prec \Lambda$, by Definition 2.2, we have

$$A=\frac{1}{n}(\sum_{i=1}^{m-r}\|x_i\|^2+\sum_{i=1}^{k}\|y_i\|^2) \geqslant \lambda_1.$$

From (2) of Theorem 2.1, we know that there exists a sequence $S_Y=\{y_i\}_{i=1}^k$ with frame operator $S_Y=S-S_X$ and we are done. □

Note that if $a \prec \Lambda$ in Theorem 2.1, by Definition 2.2, we have

$$A=\frac{1}{n}(\sum_{i=1}^{m-r}\|x_i\|^2+\sum_{i=1}^{k}\|y_i\|^2) \geqslant \lambda_1,$$

and

$$A \geqslant \frac{1}{t}\sum_{i=1}^{t}(\|y_i\|^2+\lambda_{n-i+1}), \quad 1 \leqslant t \leqslant \min\{n,k\}.$$

Then we can get a result for a (r,k)-surgery on a sequence of unit norm vectors.

COROLLARY 2.3. *Given two sequences of unit norm vectors $X=\{x_i\}_{i=1}^m$, $Y=\{y_i\}_{i=1}^k$. A (r,k)-surgery on X with Y resulting in a tight frame with bound A if and only if*

$$A \geqslant \max\{\lambda_1, 1+\frac{1}{t}\sum_{i=1}^{t}\lambda_{n-i+1}\},$$

where $1 \leqslant t \leqslant \min\{n,k\}$ *and* $\{\lambda_i\}_{i=1}^n$ *are the eigenvalues (arranged in non-increasing order) of* $S_X = \sum_{i=1}^{m-r} x_i \otimes x_i$.

Proof. The proof is straightforward. □

The author in [15, Lemma 2.3] considered a (r,k)-surgery ($r > k$) on a unit norm tight frame resulting in a unit norm tight frame. Next, we provide a necessary and sufficient condition such that a $(1,k)$-surgery on a tight frame resulting in a tight frame with same bound.

THEOREM 2.4. *Let* $X = \{x_i\}_{i=1}^m$ *be a tight frame with bound* A *for* $\mathscr{H}$ *and let* $Y = \{y_i\}_{i=1}^k \subset \mathscr{H}$. *A* $(1,k)$*-surgery on* X *with* Y *resulting in a tight frame with bound* A *if and only if*

$$\|x_m\|^2 = \sum_{i=1}^{k} \|y_i\|^2 \quad \text{and} \quad \operatorname{span}\{x_m\} = \operatorname{span}\{y_i\}_{i=1}^k.$$

Proof. If X and $\{x_i\}_{i=1}^{m-1} \cup \{y_i\}_{i=1}^k$ are A-tight frames for $\mathscr{H}$, we have

$$AI = \sum_{i=1}^{m} x_i \otimes x_i, \tag{2.3}$$

and

$$AI = \sum_{i=1}^{m-1} x_i \otimes x_i + \sum_{i=1}^{k} y_i \otimes y_i. \tag{2.4}$$

By (2.3)–(2.4) and changing sides, we have

$$x_m \otimes x_m = \sum_{i=1}^{k} y_i \otimes y_i. \tag{2.5}$$

By taking trace of both sides of (2.5), we have

$$\|x_m\|^2 = \sum_{i=1}^{k} \|y_i\|^2.$$

We now prove $\operatorname{span}\{x_m\} = \operatorname{span}\{y_i\}_{i=1}^k$. Let

$$T = x_m \otimes x_m = \sum_{i=1}^{k} y_i \otimes y_i.$$

We have rank $(T) = 1 \leqslant n$. As T is positive, there exists a basis for $\mathscr{H}$ with respect to which T is an $n \times n$ diagonal matrix with entries $\lambda_1 > \lambda_2 = \cdots = \lambda_n = 0$ along its diagonal. The a-th entry along the diagonal of T is given by

$$\lambda_a = x_{i_a}^2 = \sum_{i=1}^{k} y_{i_a}^2,$$

where x_{i_a} and y_{i_a} are the a-th entries of x_i and y_i, respectively. As $\lambda_a = 0$ for $2 \leqslant a \leqslant n$, $x_{i_a} = y_{i_a} = 0$ for $2 \leqslant a \leqslant n$. Hence, $\dim \operatorname{span}\{x_m\} \leqslant 1$ and $\dim \operatorname{span}\{y_i\}_{i=1}^{k} \leqslant 1$. Since T is diagonal, there exists a vector $z \in \mathscr{H}$ such that $Tz \neq 0$. As $Tz \in \operatorname{span}\{x_m\}$ and $Tz \in \operatorname{span}\{y_i\}_{i=1}^{k}$,

$$\operatorname{span}\{x_m\} = \operatorname{span}\{Tz\} = \operatorname{span}\{y_i\}_{i=1}^{k}.$$

Conversely, if we want to prove that $\{x_i\}_{i=1}^{m-1} \cup \{y_i\}_{i=1}^{k}$ is a tight frame with bound A, we only need to show

$$x_m \otimes x_m = \sum_{i=1}^{k} y_i \otimes y_i.$$

Assume that

$$\operatorname{span}\{x_m\} = \operatorname{span}\{y_i\}_{i=1}^{k},$$

then $y_1 \cdots y_k \in \operatorname{span}\{x_m\}$. Therefore, there exist $c_1, \cdots, c_k \in \mathbb{R}$ such that for $a = 1, \cdots, n$, $y_{i_a} = c_i x_{m_a}$ $(i = 1, \cdots, k)$. From $\|x_m\|^2 = \sum_{i=1}^{k} \|y_i\|^2$, we have

$$\|x_m\|^2 = \sum_{i=1}^{k} c_i^2 \|x_m\|^2,$$

and then $c_1^2 + c_1^2 + \cdots + c_1^k = 1$. The (p,q) entry of the matrix form of $x \otimes x$ is $x_p x_q$. Then $x_m \otimes x_m = \sum_{i=1}^{k} y_i \otimes y_i$ is equivalent to following,

$$x_{m_p} x_{m_q} = y_{1_p} y_{1_q} + y_{2_p} y_{2_q} + \cdots + y_{k_p} y_{k_q}.$$

From $y_{i_a} = c_i x_{m_a}$, we have

$$\begin{aligned} y_{1_p} y_{1_q} + y_{2_p} y_{2_q} + \cdots + y_{k_p} y_{k_q} &= c_1^2 x_{m_p} x_{m_q} + c_2^2 x_{m_p} x_{m_q} + \cdots + c_k^2 x_{m_p} x_{m_q} \\ &= (c_1^2 + \cdots + c_k^2) x_{m_p} x_{m_q} \\ &= x_{m_p} x_{m_q}. \end{aligned}$$

Thus $x_m \otimes x_m = \sum_{i=1}^{k} y_i \otimes y_i$. Hence $\{x_i\}_{i=1}^{m-1} \cup \{y_i\}_{i=1}^{k}$ is a tight frame for $\mathscr{H}$ with bound A. $\square$

EXAMPLE 1. Let

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad x_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

It is easy to verify that $\{x_1, x_2, x_3\}$ is a tight frame with bound $\frac{3}{2}$. Let

$$y_1 = \begin{bmatrix} -\frac{\sqrt{3}}{6} \\ -\frac{1}{2} \end{bmatrix}, \quad y_2 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{3\sqrt{2}}{6} \end{bmatrix}.$$

By removing x_3 from $\{x_1, x_2, x_3\}$ and adding y_1, y_2, it is easy to verify that $\|x_3\|^2 = \|y_1\|^2 + \|y_2\|^2$ and $\operatorname{span}\{x_3\} = \operatorname{span}\{y_1, y_2\}$, and $\{x_1, x_2, y_1, y_2\}$ is also a tight frame for $\mathbb{R}^2$ with bound $\frac{3}{2}$.

From Theorem 2.4, the following result is straightforward.

COROLLARY 2.5. *Let $X = \{x_i\}_{i=1}^m$ be a tight frame with bound A for $\mathscr{H}$ and let $y \in \mathscr{H}$. A $(r,1)$-surgery on X with y resulting in a tight frame with bound A if and only if*

$$\sum_{i=m-r+1}^{m} \|x_i\|^2 = \|y\|^2 \quad \text{and} \quad \operatorname{span}\{y\} = \operatorname{span}\{x_i\}_{i=m-r+1}^m.$$

3. Surgery for full spark frames

DEFINITION 3.1. [1] A sequence $\{x_i\}_{i=1}^m$ is called a full spark frame for $\mathscr{H}$, if for any $\sigma \subset \{1,\cdots,m\}$ with $|\sigma| = n$, $\{x_i\}_{i\in\sigma}$ is a frame for $\mathscr{H}$.

The following proposition is straightforward but important.

PROPOSITION 3.1. *A sequence $X = \{x_i\}_{i=1}^m$ is a full spark frame for $\mathscr{H}$ if and only if every n elements of X are linearly independent.*

We can see that a frame is full spark if any n of its members make up a basis for $\mathscr{H}$. Hence it is possible to take surgery on a basis resulting in a full spark frame.

PROPOSITION 3.2. *A $(r,0)$-surgery on a full spark frame results a full spark frame if $r \leqslant m-n$.*

Proof. Let $X = \{x_i\}_{i=1}^m$ be a full spark frame, by Proposition 3.1, every n elements of X are linearly independent. If $r \leqslant m-n$, then every n elements of $\{x_i\}_{i=1}^{m-r}$ are linearly independent, and then $\operatorname{span}\{x_i\}_{i=1}^{m-r} = \mathscr{H}$. Thus $\{x_i\}_{i=1}^{m-r}$ is a full spark frame for $\mathscr{H}$. □

LEMMA 3.1. *A (r,k)-surgery on a frame $\{x_i\}_{i=1}^m$ resulting in a full spark frame is always possible if the $\{x_i\}_{i=1}^{m-r}$ is linear independent and $k \geqslant n+r-m$.*

Proof. Since any linear independent set can be expanded to a basis for $\mathscr{H}$, and $k \geqslant n+r-m$, then it is possible to expand $\operatorname{span}\{x_i\}_{i=1}^{m-r}$ to a full spark frame. □

COROLLARY 3.1. *A (r,k)-surgery on a frame $\{x_i\}_{i=1}^m$ resulting in a full spark frame is impossible if the $\{x_i\}_{i=1}^{m-r}$ is a linear dependent set.*

Next, we give a condition such that a (r,k)-surgery on a frame $\{x_i\}_{i=1}^m$ results in a full spark frame.

THEOREM 3.2. *Let $X = \{x_i\}_{i=1}^m$ be a frame and let $Y = \{y_i\}_{i=1}^k \subset \mathscr{H}$. A $(m-n,k)$-surgery on X with Y resulting in a full spark frame if and only if $T_{X'}$ is invertible and all minors of $T_{X'}^{-1}T_Y$ are nonzero, where $T_{X'}$ and T_Y are the synthesis operators of $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^k$, respectively.*

Proof. ($\Leftarrow$) Let F be an $n\times(n+k)$ matrix consisting of $\{x_1,\cdots,x_n,y_1,\cdots,y_k\}$, thus $F=[T_{X'}|T_Y]$. If we want to prove that $\{x_1,\cdots,x_n,y_1,\cdots,y_k\}$ is a full spark frame, by Proposition 3.1, it is equivalent to proving that every n vectors of $\{x_1,\cdots,x_n,y_1,\cdots,y_k\}$ are linearly independent. If $T_{X'}$ is invertible, we have

$$T_{X'}^{-1}F=[E|T_{X'}^{-1}T_Y].$$

Next, we show that every n columns of $T_{X'}^{-1}F=[E|T_{X'}^{-1}T_Y]$ are linearly independent if and only if all minors of $T_{X'}^{-1}T_Y$ are nonzero. The idea of the proof is simple and we shall just illustrate it by proving that the top right $l\times l$ submatrix B of $T_{X'}^{-1}T_Y$ is nonsingular, where $1\leqslant l\leqslant \min\{n,k\}$. Take the matrix F' consisting of the last $n-l$ columns of E and the first l columns of $T_{X'}^{-1}T_Y$:

$$F'=\left[\begin{array}{cccc|c} & 0 & & & B \\ \hline 1 & & & & \\ & 1 & & & \\ & & \ddots & & * \\ & & & 1 & \end{array}\right]. \tag{3.1}$$

Then $\det(F')=\det(B)$. Hence, $\det(F')$ is nonzero if and only if $\det(B)$ is nonzero. This means that every n columns of $T_{X'}^{-1}F=[E|T_{X'}^{-1}T_Y]$ are linearly independent if and only if all minors of $T_{X'}^{-1}T_Y$ are nonzero.

($\Rightarrow$) If $\{x_1,\cdots,x_n,y_1,\cdots,y_k\}$ is a full spark frame for $\mathscr{H}$, then $T_{X'}$ is invertible. From [16, Lemma], we deduce that $T_{X'}^{-1}F=[E|T_{X'}^{-1}T_Y]$ is also a full spark frame for $\mathscr{H}$. From (3.1), we know that all minors of $T_{X'}^{-1}T_Y$ are nonzero.

Hence, $\{x_1,\cdots,x_n,y_1,\cdots,y_k\}$ is a full spark frame for $\mathscr{H}$ if and only if $T_{X'}$ is invertible and all minors of $T_{X'}^{-1}T_Y$ are nonzero. $\square$

EXAMPLE 2. We now give an example for Theorem 3.2. Let $\{x_i\}_{i=1}^5$ be a frame for $\mathbb{R}^3$, where

$$x_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad x_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad x_3=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Let

$$y_1=\begin{bmatrix}\frac12\\ \frac13\\ \frac14\end{bmatrix},\quad y_2=\begin{bmatrix}\frac13\\ \frac14\\ \frac15\end{bmatrix},\quad y_3=\begin{bmatrix}\frac14\\ \frac15\\ \frac16\end{bmatrix},\quad y_4=\begin{bmatrix}\frac15\\ \frac16\\ \frac17\end{bmatrix}.$$

It is easy to verify that all minors of T are nonzero, where columns of T are consisting of $\{y_1\}_{i=1}^4$. Then a $(2,4)$-surgery on $\{x_i\}_{i=1}^5$ with $\{y_1\}_{i=1}^4$ results in a full spark frame for $\mathbb{R}^3$. In fact, we can compute that:

(a) $x_i,\ y_j,\ y_k$ are linearly independent, where $i=1,\ 2,\ 3,\ j,\ k=1,\ 2,\ 3,\ 4$ and $j\neq k$;

(b) $x_i,\ x_j,\ y_k$ are linearly independent, where $i,\ j=1,2,3$ and $j\neq j$, $k=1,2,3,4$;

(c) $y_i,\ y_j,\ y_k$ are linearly independent, where $i,\ j,\ k=1,2,3,4$ and $i\neq j\neq k$;

(d) x_1, x_2, x_3 are linearly independent.

Note that if $r > m-n$ in Theorem 3.2, and $\{x_i\}_{i=1}^{m-r}$ are unit vectors, then the minors of T_Y may be not all non-zero.

EXAMPLE 3. Let $X = \{x_i\}_{i=1}^{m}$ be a frame for $\mathbb{R}^n$, where $x_1 = (1,0\cdots,0)^T$ and $x_2 = (0,1\cdots,0)^T$. Let T be a Vandermonde matrix as follow:

$$T = \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_k \\ a_1^2 & a_2^2 & \dots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_k^{n-1} \end{bmatrix} = [y_1, y_2, \cdots, y_k].$$

We will show that a $(m-2,k)$-surgery on X with $\{y_i\}_{i=1}^{k}$ results in a full spark frame if $a_1,\dots,a_k$ are k distinct nonzero elements. Let

$$F = [x_1, x_2, y_1, \cdots, y_k] = \begin{bmatrix} 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & a_1 & a_2 & \dots & a_k \\ 0 & 0 & a_1^2 & a_2^2 & \dots & a_k^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_1^{n-1} & a_2^{n-1} & \dots & a_k^{n-1} \end{bmatrix}.$$

Any $n\times n$ matrix, say U, consists of any n columns of F. If U is a submatrix of T, then det $U \neq 0$ because $a_1,\dots,a_k$ are k distinct elements. Next, we will prove that the matrix U contains first p $(0 < p \leqslant 2)$ columns of F. Without loss of generality, assume that U consists of first p columns of F and first $n-p$ columns of T.

Case 1. $p = 1$

Assume that U contains x_1, then

$$\det U = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_1^2 & a_2^2 & \dots & a_{n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1^{n-1} & a_2^{n-1} & \dots & a_{n-1}^{n-1} \end{vmatrix} = \prod_{l=1,\dots,n-1} a_l \prod_{\substack{i,j=1,\dots,n-1 \\ i>j}} (a_i - a_j) \neq 0.$$

Assume that U contains x_2, then

$$\det U = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_1^2 & a_2^2 & \dots & a_{n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1^{n-1} & a_2^{n-1} & \dots & a_{n-1}^{n-1} \end{vmatrix} = -\sum_{i=1}^{n-1} a_1 \dots a_{i-1} a_{i+1} \dots a_{n-1} \prod_{\substack{i,j=1,\dots,n-1 \\ i>j}} (a_i - a_j) \neq 0.$$

Case 2. $p=2$
Assume that U contains x_1,x_2, then

$$\det U = \begin{vmatrix} 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & a_1 & a_2 & \dots & a_{n-2} \\ 0 & 0 & a_1^2 & a_2^2 & \dots & a_{n-2}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_1^{n-1} & a_2^{n-1} & \dots & a_{n-2}^{n-1} \end{vmatrix} = \prod_{l=1,\dots,n-2} a_l^2 \prod_{\substack{i,j=1,\dots,n-2 \\ i>j}} (a_i - a_j) \neq 0.$$

Therefore, all $n\times n$ submatrices of F are non-singular. Thus every n vectors of $\{x_1,x_2,y_1,\cdots,y_k\}$ are linearly independent. Hence, $\{x_1,x_2,y_1,\cdots,y_k\}$ is a full spark frame for $\mathscr{H}$. But there exists a minor of T is zero. Let $a_2=-2$ and $a_3=2$. We can see that a minor of order 2 of T, such as $\begin{vmatrix} 1 & 1 \\ 4 & 4 \end{vmatrix}$, is zero.

Acknowledgements. The research is supported by the Natural Science Foundation of Anhui Province (1908085MF175) and the Fundamental Research Funds for the Central Universities (JZ2019HGBZ0129).

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(Received May 1, 2021)

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