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ON THE SPECTRUM OF COMPLEX SYMMETRIC TOEPLITZ OPERATORS

MARCOS S. FERREIRA

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Abstract. In this paper we focus on the spectrum of Toeplitz operators T_ϕ on the Hardy space H^2 of the unit disk $\mathbb{D}$ when ϕ is not continuous and especially when T_ϕ is a complex symmetric operator. Moreover, we show a necessary condition for T_ϕ to be complex symmetric, for ϕ belongs to subalgebra $H^\infty + \mathscr{C}(\mathbb{T})$ of L^∞ and we present a case of the continuity of the spectral function restricted to the space of Toeplitz operators.

1. Introduction

A *conjugation* C on a separable complex Hilbert space $\mathscr{H}$ is an antilinear operator $C : \mathscr{H} \to \mathscr{H}$ such that C is *involutive* and *isometric*, that is, $C^2 = I$ and $\langle Cf, Cg\rangle = \langle g, f\rangle$, forall $f, g \in \mathscr{H}$. A bounded linear operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathscr{H}$ such that $CT = T^*C$. In this case we say that T is an operator C-symmetric. The concept of complex symmetric operators on separable Hilbert spaces is a natural generalization of complex symmetric matrices, and their general study was initiated by Garcia, Putinar, and Wogen [5, 6, 7, 8]. The class of complex symmetric operators includes other basic classes of operators such as normal, Hankel, compressed Toeplitz, and some Volterra operators.

Let L^2 be the Hilbert space of square-integrable functions on the unit circle $\mathbb{T}$, L^∞ the space of essentially bounded functions on $\mathbb{T}$ and $\mathscr{C}(\mathbb{T})$ the space of continuous functions on $\mathbb{T}$. For each $\phi \in L^\infty$, the *Toeplitz operator* $T_\phi : H^2 \to H^2$, with symbol ϕ, is defined by

$$T_\phi f = P(\phi f),$$

for all $f \in H^2$, where $P : L^2 \to H^2$ is the orthogonal projection.

The concept of Toeplitz operators generalizes the concept of Toeplitz matrices and their general algebraic properties were studied by Brown and Halmos first addressed by [2]. The study of complex symmetric Toeplitz operators is relatively recent and provides deep and important connections in several problems in quantum mechanics [1, 9]. One of the first examples of complex symmetric Toeplitz operator is due to Guo

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and Zhu [10]. In this work, Guo and Zhu raised the question of characterizing complex symmetric Toeplitz operators on the Hardy space H^2.

In an attempt to resolve the question of Guo and Zhu, Ko and Lee [13] considered the family of conjugations C_λ on H^2 defined by

$$C_\lambda f(z) = \overline{f(\lambda \overline{z})}$$

for $\lambda \in \mathbb{T}$ and proved the following:

THEOREM 1. *If $\phi(z) = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) z^n \in L^\infty$, then T_ϕ is C_λ-symmetric if, and only if, $\widehat{\phi}(-n) = \lambda^n \widehat{\phi}(n)$, for all $n \in \mathbb{Z}$, with $\lambda \in \mathbb{T}$.*

If $\gamma : \mathbb{T} \to \mathbb{C}$ is a continuous function and z is a point that is not in the range of γ, then the *index* of the z with respect to γ is defined as

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi} \int_\gamma \frac{d\xi}{\xi - z}.$$

We will often say that $\mathrm{Ind}_\gamma(z)$ is the *winding number* of γ about z. Recently, Noor [16] studied the complex symmetry of Toeplitz operators with continuous symbols. Noor considered *nowhere winding curves* that are closed curves $\gamma : \mathbb{T} \to \mathbb{C}$ such that $\mathrm{Ind}_\gamma(z) = 0$, for all z out of the range of γ, and proved that:

THEOREM 2. *If $\phi \in \mathscr{C}(\mathbb{T})$ and T_ϕ is complex symmetric, then ϕ is a nowhere winding curve.*

The plan of this paper is to obtain characterizations for nowhere winding curves and then generalize the Theorem 2. Then, we determine the spectrum of complex symmetric Toeplitz operators for general cases and as a consequence we obtain a case of continuity of the spectral function $\sigma : \mathscr{I} \to \mathscr{K}$, that maps each Toeplitz operator T_ϕ to its spectrum $\sigma(T_\phi)$.

2. Preliminaries

2.1. The Hardy space H^2

The Hardy space H^2 consists of the all holomorphic functions f on the unit disk $\mathbb{D}$ such that

$$\|f\| = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} < \infty.$$

The Hardy space is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n},$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

For each $\alpha \in \mathbb{D}$, let k_α denote the reproducing kernel for H^2 given by

$$k_\alpha(z) = \frac{1}{1-\overline{\alpha}z},$$

which satisfies $f(\alpha) = \langle f, k_\alpha \rangle$, for all f in H^2.

2.2. Complex symmetric Toeplitz operators

The spectrum of a Toeplitz operator with continuous symbol is well known. If fact, if $\phi \in \mathscr{C}(\mathbb{T})$ then

$$\sigma(T_\phi) = \operatorname{Ran}(\phi) \cup \left\{\lambda \in \mathbb{C} : \operatorname{Ind}_\gamma(\lambda) \neq 0\right\}, \tag{1}$$

where $\gamma = \operatorname{Ran}(\phi)$ is the range of ϕ.

In particular, if $\phi \in \mathscr{C}(\mathbb{T})$ is a nowhere winding curve, then

$$\sigma(T_\phi) = \operatorname{Ran}(\phi). \tag{2}$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is called a *Fredholm operator* if the range of the T is closed and $\dim \operatorname{Ker} T$ and $\dim \operatorname{Ker} T^*$ are finite. In this case, the *classical index* of T is given by

$$j(T) = \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^*.$$

The *essential spectrum* of T is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Fredholm operator}\}$$

and evidently $\sigma_e(T) \subseteq \sigma(T)$.

Now note that if T is a complex symmetric operator with conjugation C, then the natural map $C : \operatorname{Ker} T \to \operatorname{Ker} T^*$ is an antilinear isometric isomorphism and so we have:

REMARK 1. If $T \in \mathscr{L}(\mathscr{H})$ is a complex symmetric Fredholm operator, then $j(T) = 0$.

Now notice that, if ϕ is non-constant and T_ϕ is C-symmetric, then for all $\alpha, \beta \in \mathbb{C}$ we have

$$CT_{\alpha\phi+\beta} = C(\alpha T_\phi + \beta) = (\overline{\alpha} T_\phi^* + \overline{\beta})C = T_{\alpha\phi+\beta}^* C \tag{3}$$

and therefore $T_{\alpha\phi+\beta}$ is also C-symmetric.

The Coburn Alternative ([3, Proposition 7.24]) states that if ϕ is a function in L^∞ other than 0, then at least one of T_ϕ and T_ϕ^* is injective. As a consequence, we have the following invertibility criteria for Toeplitz operators:

PROPOSITION 1. ([3, Corollary 7.25]) *If ϕ is a function in L^∞ such that T_ϕ is a Fredholm operator, then T_ϕ is invertible if, and only if, $j(T_\phi) = 0$.*

We end this section with the important characterization for Fredholm's Toeplitz operator:

THEOREM 3. ([3, Theorem 7.26]) *If* ϕ *is a continuous function on* $\mathbb{T}$, *then* T_ϕ *is a Fredholm operator if, and only if,* ϕ *does not vanish and in this case* $j(T_\phi) = -\operatorname{Ind}_\gamma(0)$, *where* $\gamma = \operatorname{Ran}(\phi)$.

3. Nowhere winding curves and Toeplitz operators

The main goal of this section is to obtain operator theoretic conditions generalizations for nowhere winding curves. We begin with two characterizations of nowhere winding curves.

PROPOSITION 2. *Let* $\phi \in \mathscr{C}(\mathbb{T})$. *The following statements are equivalent:*

(i) ϕ *is a nowhere winding curve.*

(ii) $T_{\phi-\lambda}$ *and* $T^*_{\phi-\lambda}$ *are injective, for all* $\lambda \notin \operatorname{Ran}(\phi)$.

(iii) $\sigma(T_\phi) = \sigma_e(T_\phi)$.

Proof. $(i) \Rightarrow (ii)$ Since ϕ is a nowhere winding curve, we have by (2) that $\sigma(T_\phi) = \operatorname{Ran}(\phi)$. Thus if $\lambda \notin \operatorname{Ran}(\phi)$, then $T_{\phi-\lambda}$ is invertible and therefore $T^*_{\phi-\lambda}$ is injective. $(ii) \Rightarrow (iii)$ If $\lambda \notin \sigma_e(T_\phi) = \operatorname{Ran}(\phi)$, then by Theorem 3 $T_{\phi-\lambda}$ is a Fredholm operator and moreover $T_{\phi-\lambda}$ and $T^*_{\phi-\lambda}$ are injective. Thus $T_{\phi-\lambda}$ is surjective and therefore $\lambda \notin \sigma(T_\phi)$. $(iii) \Rightarrow (i)$ Let $\lambda \notin \operatorname{Ran}(\phi)$. Once $\sigma(T_\phi) = \operatorname{Ran}(\phi)$, we have

$$j\left(T_{\phi-\lambda}\right) = \operatorname{dimKer}\left(T_{\phi-\lambda}\right) - \operatorname{dimKer}\left(T^*_{\phi-\lambda}\right) = 0.$$

Now since

$$j\left(T_{\phi-\lambda}\right) = -\operatorname{Ind}_{\phi-\lambda}(0) = -\operatorname{Ind}_\phi(\lambda),$$

follows that $\operatorname{Ind}_\phi(\lambda) = 0$, and so ϕ is a nowhere winding curve. $\square$

Now we present a generalization of Theorem 2. Before, we need a definition. For $\phi(e^{it}) = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) e^{int} \in L^\infty$ and $0 < r < 1$, let $\widetilde{\phi}$ the harmonic extension of ϕ on $\mathbb{D}$ given by

$$\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) r^{|n|} e^{int}.$$

Then we can define the continuous function $\phi_r : \mathbb{T} \to \mathbb{C}$ given by

$$\phi_r(e^{it}) = \widetilde{\phi}(re^{it}).$$

DEFINITION 1. Let $\phi \in H^\infty + \mathscr{C}(\mathbb{T})$. We say that ϕ is a *nowhere winding symbol* if there exists $\delta > 0$ such that for each $1-\delta < r < 1$, holds $\operatorname{Ind}_{\phi_r}(\lambda) = 0$ for all $\lambda \notin \operatorname{Ran}(\phi_r)$.

THEOREM 4. *Let* $\phi \in H^\infty + \mathscr{C}(\mathbb{T})$. *If* T_ϕ *is complex symmetric, then* ϕ *is a nowhere winding symbol.*

Proof. Let $\lambda \notin \operatorname{Ran}(\phi_r)$ with $0 < r < 1$. Since $\phi_r - \lambda \in \mathscr{C}(\mathbb{T}) \subseteq H^\infty + \mathscr{C}(\mathbb{T})$ is invertible, there exist $\delta, \varepsilon > 0$ (see [3, Theorem 6.45]) such that

$$\left|(\phi_r - \widetilde{\lambda)\,(re^{it}})\right| \geqslant \varepsilon,\ 1-\delta < r < 1.$$

Thus $T_{\phi-\lambda}$ is a Fredholm operator (see [3, Theorem 7.36]) and

$$\operatorname{Ind}_{\phi_r}(\lambda) = \operatorname{Ind}_{\phi_r-\lambda}(0) = \operatorname{Ind}_{\widetilde{\phi-\lambda}}(0) = -j(T_{\phi-\lambda}).$$

Now, since $T_{\phi-\lambda}$ is complex symmetric follows that $j(T_{\phi-\lambda}) = 0$, whence $\operatorname{Ind}_{\phi_r}(\lambda) = 0$. $\square$

4. The spectrum of complex symmetric Toeplitz operators

We already know that if $\phi \in \mathscr{C}(\mathbb{T})$ is a nowhere winding curve, then $\sigma(T_\phi) = \sigma_e(T_\phi)$. In the following, we provide a sufficient condition to have this equality without requiring at least that ϕ be continuous.

PROPOSITION 3. *Let* $\phi \in L^\infty$. *If* T_ϕ *is complex symmetric, then* $\sigma(T_\phi) = \sigma_e(T_\phi)$.

Proof. It is sufficient to show the inclusion $\sigma(T_\phi) \subseteq \sigma_e(T_\phi)$. In fact, if $\lambda \notin \sigma_e(T_\phi)$ then $T_{\phi-\lambda}$ is a Fredholm operator. Thus, since $T_{\phi-\lambda}$ is complex symmetric it follows by Remark 1 that $j(T_{\phi-\lambda}) = 0$ and therefore by Proposition 1 we have that $T_{\phi-\lambda}$ is invertible, that is, $\lambda \notin \sigma(T_\phi)$. $\square$

Obviously if ϕ is continuous, the Proposition 3 states that $\sigma(T_\phi) = \operatorname{Ran}(\phi)$, according to [16, Corollary 7].

Denote by $\operatorname{essran}(\phi)$ the *essential range* of $\phi \in L^\infty$. We can now describe the spectrum of T_ϕ more generally.

THEOREM 5. *Let* $\phi \in L^\infty$. *If* $T_{\phi-\lambda}$ *is a Fredholm operator for all* $\lambda \notin \operatorname{essran}(\phi)$, *then*

$$\sigma(T_\phi) = \operatorname{essran}(\phi) \cup \left\{\lambda \in \mathbb{C} : j(T_{\phi-\lambda}) \neq 0\right\}.$$

Proof. By spectral inclusion theorem (see [3, Corollary 7.7]) we have $\operatorname{essran}(\phi) \subseteq \sigma(T_\phi)$. Note that if $\alpha \notin \operatorname{essran}(\phi)$, then $T_{\phi-\alpha}$ is a Fredholm operator. Hence if $j(T_{\phi-\alpha}) \neq 0$, follows of the Proposition 1 that $T_{\phi-\alpha}$ is not invertible and so we have

$$\operatorname{essran}(\phi) \cup \left\{\lambda \in \mathbb{C} : j(T_{\phi-\lambda}) \neq 0\right\} \subseteq \sigma(T_\phi).$$

Let now $\beta \in \sigma(T_\phi)$. If $\beta \notin \operatorname{essran}(\phi)$, since $T_{\phi-\beta}$ is a Fredholm operator, we have $j(T_{\phi-\beta}) \neq 0$ and therefore $\beta \in \left\{\lambda \in \mathbb{C} : j(T_{\phi-\lambda}) \neq 0\right\}$. On the other hand, if $\beta \notin$

$\{\lambda \in \mathbb{C} : j(T_{\phi-\lambda}) \neq 0\}$, since $\beta \in \sigma(T_\phi)$, we have $T_{\phi-\beta}$ is not a Fredholm operator. Thus, by hypothesis, $\beta \in \mathrm{essran}(\phi)$ and so

$$\sigma(T_\phi) \subseteq \mathrm{essran}(\phi) \cup \{\lambda \in \mathbb{C} : j(T_{\phi-\lambda}) \neq 0\},$$

as desired. □

Although it seems that we are demanding a lot in the previous theorem, in fact this hypothesis is reasonable. Indeed, if ϕ is continuous, we have $\mathrm{essran}(\phi) = \mathrm{Ran}(\phi)$ and so $\lambda \notin \mathrm{essran}(\phi)$ if, and only if, $T_{\phi-\lambda}$ is a Fredholm operator. Thus, Theorem 5 is a good generalization for (1).

We will now obtain generalizations for Theorems 2 and 4. First, we will say that $\phi \in L^\infty$ is a *spectral symbol* if $\sigma(T_\phi) = \mathrm{essran}(\phi)$.

Note that if ϕ is a continuous spectral symbol, then ϕ is a nowhere winding curve. If fact, if ϕ is a spectral symbol and $\phi \in \mathscr{C}(\mathbb{T})$ we have by Theorem 3 that

$$\sigma(T_\phi) = \mathrm{essran}(\phi) = \mathrm{Ran}(\phi) = \sigma_e(T_\phi)$$

and therefore by Proposition 2 have that ϕ is a nowhere winding curve.

EXAMPLE 1. Let $\phi \in L^\infty$. If $\mathrm{essran}(\phi)$ is convex, then ϕ is a spectral symbol (see [14, Corollary 3.3.7] for more details).

COROLLARY 1. *Let $\phi \in L^\infty$. If T_ϕ is complex symmetric and $\sigma_e(T_\phi) = \mathrm{essran}(\phi)$, then ϕ is a spectral symbol.*

Proof. It follows directly from Proposition 3 and the previous theorem. □

In the next section, we will show that if ϕ is a spectral symbol then the function $\sigma : \mathscr{I} \to \mathscr{K}$ is continuous at T_ϕ.

5. A case of spectral function continuity

In this section, we partially resolve a question concerning the continuity of the spectral function restricted to the space of the Toeplitz operators.

Consider $\mathscr{K}$ the set of all compact subsets of $\mathbb{C}$, equipped with the Hausdorff metric d_H given by

$$d_H(X,Y) = \max\left\{\sup_{z\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\},$$

for all $X,Y \subseteq \mathscr{K}$. Then we can consider the spectrum function $\sigma : \mathscr{L}(\mathscr{H}) \to \mathscr{K}$, mapping each operator $T \in \mathscr{L}(\mathscr{H})$ to its spectrum $\sigma(T)$.

It is well known that the function σ is upper semicontinuous ([15, Theorem 1]) and that in noncommutative subalgebras of $\mathscr{L}(\mathscr{H})$, σ does have points of discontinuity. We are interested in the class $\mathscr{C}$ of operators for which σ becomes continuous when restricted to $\mathscr{C}$. Moreover, due to a Newburgh argument in ([11, Solution 104]), we know that σ is continuous when restricted to the set of normal operators.

Considering $\mathscr{I}$ the subspace of $\mathscr{L}(H^2)$ consisting of all Toeplitz operators, in [4] the authors propose the following question:

QUESTION 1. Is the restriction of σ to the space $\mathscr{I}$ continuous?

Using part of the argument used in the proof of [12, Theorem 10], we obtain the continuity of σ for Toeplitz operators with spectral symbol. First, we need the following lemma:

LEMMA 1. *Let $T_n, T \in \mathscr{L}(\mathscr{H})$, with $n \in \mathbb{N}$. If $T_n \to T$, then*

$$\liminf \sigma(T_n) \subseteq \sigma(T).$$

Proof. In fact, if $\lambda \in \liminf \sigma(T_n)$, then there exist a sequence $\{\lambda_n\}$ such that $\lambda_n \in \sigma(T_n)$, for all $n \in \mathbb{N}$, and $\lambda_n \to \lambda$. Thus, $(T_n - \lambda_n) \to (T - \lambda)$.

Now, since the set of invertible elements in $\mathscr{L}(\mathscr{H})$ is open and each $T_n - \lambda_n$ is not invertible, follows that $T - \lambda$ is not invertible, that is $\lambda \in \sigma(T)$. $\square$

THEOREM 6. *Let $\phi \in L^\infty$. If ϕ is a spectral symbol, then $\sigma : \mathscr{I} \to \mathscr{K}$ is continuous at T_ϕ.*

Proof. By hypothesis we have $\sigma(T_\phi) = \mathrm{essran}(\phi)$. Let $\phi_n, \phi \in L^\infty$ such that

$$\left\|T_{\phi_n} - T_\phi\right\| \to 0.$$

By Lemma 1, $\liminf \sigma(T_{\phi_n}) \subseteq \sigma(T_\phi)$.

Hence is suffices to show that $\sigma(T_\phi) \subseteq \liminf \sigma(T_{\phi_n})$.

Suppose that $\lambda \notin \liminf \sigma(T_{\phi_n})$. Then there exist a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ such that for some $\varepsilon > 0$ we have

$$\mathrm{dist}\left(\lambda, \sigma(T_{\phi_{n_k}})\right) > \varepsilon \text{ for all } k.$$

Now, once $\mathrm{essran}(\phi_{\mathrm{n_k}}) \subseteq \sigma(T_{\phi_{n_k}})$ follows that $\mathrm{dist}\,(\lambda, \mathrm{essran}(\phi_{\mathrm{n_k}})) > \varepsilon$ for all k. Lastly, since $\left\|T_{\phi_n} - T_\phi\right\| \to 0$ implies $\|\phi_n - \phi\|_\infty \to 0$, we have

$$\mathrm{dist}\,(\lambda, \mathrm{essran}(\phi)) \geqslant \varepsilon.$$

Thus $\mathrm{dist}\,(\lambda, \mathrm{essran}(\phi)) > 0$ and so $\lambda \notin \sigma(T_\phi)$. $\square$

In particular, we have the continuity of $\sigma : \mathscr{I} \to \mathscr{K}$ when T_ϕ is complex symmetric.

COROLLARY 2. *Let $\phi \in L^\infty$. If T_ϕ is complex symmetric and $\sigma_e(T_\phi) = \mathrm{essran}(\phi)$, then σ is continuous at T_ϕ.*

Proof. It follows directly from Corollary 1 and the previous theorem. $\square$

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Marcos S. Ferreira
Universidade Estadual de Santa Cruz
Departamento de Ciências Exatas e Tecnológicas
Ilhéus, Bahia, Brazil
e-mail: msferreira@uesc.br