Operators
and
Matrices
Volume 16, Number 1 (2022), 95–99
doi:10.7153/oam-2022-16-09

A NOTE ON A SPECTRAL CONSTANT ASSOCIATED WITH AN ANNULUS

GEORGIOS TSIKALAS

(*Communicated by G. Misra*)

Abstract. Fix $R > 1$ and let $A_R = \{1/R \leqslant |z| \leqslant R\}$ be an annulus. Also, let $K(R)$ denote the smallest constant such that A_R is a $K(R)$-spectral set for the bounded linear operator $T \in \mathscr{B}(H)$ whenever $||T|| \leqslant R$ and $||T^{-1}|| \leqslant R$. We show that $K(R) \geqslant 2$, for all $R > 1$. This improves on previous results by Badea, Beckermann and Crouzeix.

1. Background

Let X be a closed set in the complex plane and let $\mathscr{R}(X)$ denote the algebra of complex-valued bounded rational functions on X, equipped with the supremum norm

$$||f||_X = \sup\{|f(x)| : x \in X\}.$$

Suppose that T is a bounded linear operator acting on the (complex) Hilbert space H. Suppose also that the spectrum $\sigma(T)$ of T is contained in the closed set X. Let $f = p/q \in \mathscr{R}(X)$. As the poles of the rational function f are outside of X, the operator $f(T)$ is naturally defined as $f(T) = p(T)q(T)^{-1}$ or, equivalently, by the Riesz-Dunford functional calculus (see e.g. [4] for a treatment of this topic).

Recall that for a fixed constant $K > 0$, the set X is said to be a K-*spectral set* for T if $\sigma(T) \subseteq X$ and the inequality $||f(T)|| \leqslant K||f||_X$ holds for every $f \in \mathscr{R}(X)$. The set X is a *spectral set* for T if it is a K-spectral set with $K = 1$. Spectral sets were introduced and studied by von Neumann in [8], where he proved the celebrated result that an operator T is a contraction if and only if the closed unit disk is a spectral set for T (we refer the reader to the book [9] and the survey [2] for more detailed presentations and more information on K-spectral sets).

We will be concerned with the case where $X = A_R := \{1/R \leqslant |z| \leqslant R\}$ $(R > 1)$ is a closed annulus, the intersection of the two closed disks $D_1 = \{|z| \leqslant R\}$ and $D_2 = \{|z| \geqslant 1/R\}$. Now, the intersection of two spectral sets is not necessarily a spectral set; counterexamples for the annulus were presented in [7], [10] and [12]. However, the same question for K-spectral sets remains open (the counterexamples for spectral sets

Mathematics subject classification (2020): Primary 47A25; Secondary 47A10.
Keywords and phrases: Spectral set, K-spectral set, intersection of spectral sets, weighted shift.
I thank Professor John E. McCarthy, my thesis advisor, for his invaluable advice during the preparation of this work. I also thank the Onassis Foundation for providing financial support for my PhD studies.

Paper OaM-16-09

show that the same constant cannot be used for the intersection). Regarding the annulus in particular, Shields proved that, given an invertible operator $T \in \mathscr{B}(H)$ with $||T|| \leqslant R$ and $||T^{-1}|| \leqslant R$, A_R is a K-spectral set for T with $K = 2+\sqrt{(R^2+1)/(R^2-1)}$, see [11, Proposition 23]. This bound is large if R is close to 1. In this context, Shields raised the question of finding the smallest constant $K = K(R)$ such that A_R is $K(R)$-spectral, see [11, Question 7]. In particular, he asked whether this optimal constant $K(R)$ would remain bounded.

This question was answered positively by Badea, Beckermann and Crouzeix in [3, Theorem 1.2], where they obtained that (for every $R > 1$)

$$\frac{4}{3} < \gamma(R) := 2(1-R^{-2})\prod_{n=1}^{\infty}\left(\frac{1-R^{-8n}}{1-R^{4-8n}}\right)^2 \leqslant K(R) \leqslant 2+\frac{R+1}{\sqrt{R^2+R+1}} \leqslant 2+\frac{2}{\sqrt{3}}.$$

It should be noted that the quantity $\gamma(R)$ was numerically shown to be greater than or equal to $\pi/2$ (leading to the universal lower bound $\pi/2$ for $K(R)$) and it also tends to 2 as R tends to infinity.

Two subsequent improvements were made to the upper bound for $K(R)$: the first one in [5, Lemma 2.1] by Crouzeix and the most recent one in [6, p. 7] by Crouzeix and Greenbaum, where it was proved that

$$K(R) \leqslant 1+\sqrt{2}, \quad \forall R > 1.$$

As for the lower bound, Badea obtained in [1, p. 242] the statement

$$\frac{3}{2} < 2\frac{1+R^2+R}{1+R^2+2R} \leqslant K(R), \quad \forall R > 1,$$

where the quantity $2(1+R^2+R)/(1+R^2+2R)$ again tends to 2 as R tends to infinity.

We improve the aforementioned estimates by showing that 2 is actually a universal lower bound for $K(R)$:

THEOREM 1.1. *Put $A_R = \{1/R \leqslant |z| \leqslant R\}$, for any $R > 1$. Let $K(R)$ denote the smallest positive constant such that A_R is a $K(R)$-spectral set for the bounded linear operator $T \in \mathscr{B}(H)$ whenever $||T|| \leqslant R$ and $||T^{-1}|| \leqslant R$. Then,*

$$K(R) \geqslant 2, \quad \forall R > 1.$$

2. Proof of Theorem 1.1

Proof. Fix $R > 1$. For every $n \geqslant 2$, define

$$g_n(z) = \frac{1}{R^n}\left(\frac{1}{z^n}+z^n\right) \in \mathscr{R}(A_R).$$

It is easy to see that

$$||g_n||_{A_R} = g_n(R) = 1+\frac{1}{R^{2n}}. \tag{1}$$

To achieve the stated improvement, we will apply g_n to a bilateral shift operator S acting on a particular weighted sequence space $L^2(\beta)$. First, define the sequence $\{\beta(k)\}_{k\in\mathbb{Z}}$ of positive numbers (weights) as follows:

$$\beta(2ln+q) = R^q, \quad \forall q \in \{0,1,\dots,n\}, \forall l \in \mathbb{Z};$$

$$\beta((2l+1)n+q) = R^{n-q}, \quad \forall q \in \{0,1,\dots,n\}, \forall l \in \mathbb{Z}.$$

Consider now the space of sequences $f = \{\hat{f}(k)\}_{k\in\mathbb{Z}}$ such that

$$||f||_\beta^2 := \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2[\beta(k)]^2 < \infty.$$

We shall use the notation $f(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^k$ (formal Laurent series), whether or not the series converges for any (complex) values of z. Our weighted sequence space will be denoted by

$$L^2(\beta) := \{f = \{\hat{f}(k)\}_{k\in\mathbb{Z}} : ||f||_\beta^2 < \infty\}.$$

This is a Hilbert space with the inner product

$$\langle f,g\rangle_\beta := \sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)}[\beta(k)]^2.$$

Consider also the linear transformation (bilateral shift) S of multiplication by z on $L^2(\beta)$:

$$(Sf)(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^{k+1}.$$

In other words, we have

$$\widehat{(Sf)}(k) = \hat{f}(k-1), \quad \forall k \in \mathbb{Z}.$$

Observe that

$$||S|| = \sup_{k\in\mathbb{Z}} \frac{\beta(k+1)}{\beta(k)} = R$$

and

$$||S^{-1}|| = \sup_{k\in\mathbb{Z}} \frac{\beta(k)}{\beta(k+1)} = R.$$

Now, let $m \geqslant 3$ and define $h = \{\hat{h}(k)\}_{k\in\mathbb{Z}} \in L^2(\beta)$ by putting:

$$\hat{h}(2ln) = \frac{1}{m}, \quad \forall l \in \{0,1,2\dots,m^2\};$$

$$\hat{h}(k) = 0, \quad \text{in all other cases.}$$

We calculate

$$||h||_\beta^2 = \sum_{l=0}^{m^2} \frac{1}{m^2}[\beta(2ln)]^2 = \sum_{l=0}^{m^2} \frac{1}{m^2} \cdot 1^2 = \frac{m^2+1}{m^2},$$

hence

$$||h||_\beta = \frac{\sqrt{m^2+1}}{m}. \tag{2}$$

Also, put $f = (S^{-n} + S^n)h$ and notice that

$$\begin{aligned} ||(S^{-n}+S^n)h||_\beta^2 = ||f||_\beta^2 &\geqslant \sum_{l=1}^{m^2} |\hat{f}((2l-1)n)|^2 [\beta((2l-1)n)]^2 \\ &= \sum_{l=1}^{m^2} \left(\frac{2}{m}\right)^2 R^{2n} = 4R^{2n}. \end{aligned}$$

Thus,

$$||(S^{-n}+S^n)h||_\beta \geqslant 2R^n. \tag{3}$$

Using (1), (2) and (3), we can now write

$$\begin{aligned} K(R) &\geqslant \frac{||g_n(S)||}{||g_n||_{A_R}} = \frac{1}{R^n} \cdot \frac{||S^{-n}+S^n||}{1+R^{-2n}} \\ &\geqslant \frac{1}{R^n+R^{-n}} \cdot \frac{||(S^{-n}+S^n)h||_\beta}{||h||_\beta} \\ &\geqslant \frac{1}{R^n+R^{-n}} \cdot \frac{2R^n}{\frac{\sqrt{m^2+1}}{m}}. \end{aligned}$$

Letting $m \longrightarrow \infty$, we obtain

$$K(R) \geqslant \frac{1}{R^n+R^{-n}} \cdot \frac{2R^n}{1} = \frac{2R^n}{R^n+R^{-n}} \xrightarrow{n\to\infty} 2, \quad \text{as } R > 1.$$

The proof is complete. □

REFERENCES

[1] CATALIN BADEA, *K-spectral sets: an asymptotic viewpoint*, J. Appl. Funct. Anal. **9** (3-4) (2014), pp. 239–250.

[2] CATALIN BADEA AND BERNHARD BECKERMANN, *Spectral Sets*, Handbook of Linear Algebra, (L. Hogben, ed.), second edition, CRC Press, 2013.

[3] CATALIN BADEA, BERNHARD BECKERMANN AND MICHEL CROUZEIX, *Intersections of several disks of the Riemann sphere as K-spectral sets*, Commun. Pure Appl. Anal. **8** (1) (2009), pp. 37–54.

[4] JOHN B. CONWAY, *A course in functional analysis*, second vol. **96**, Graduate Texts in Mathematics, Springer-Verlag, New York, 1990, pp. xvi+399.

[5] MICHEL CROUZEIX, *The annulus as a K-spectral set*, Commun. Pure Appl. Anal. **11** (6) (2012), pp. 2291–2303.

[6] MICHEL CROUZEIX AND ANNE GREENBAUM, *Spectral sets: numerical range and beyond*, SIAM J. Matrix Anal. Appl. **40** (3) (2019), pp. 1087–1101.

[7] MISRA GADADHAR, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory **11** (2) (1984), pp. 305–317.

[8] JOHANN VON NEUMANN, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), pp. 258–281, ISSN: 0025-584X.

[9] VERN PAULSEN, *Completely bounded maps and operator algebras*, vol. **78**, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2002, pp. xii+300.

[10] VERN I. PAULSEN, *Toward a theory of K-spectral sets*, Surveys of some recent results in operator theory, vol. I, **171**, Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1988, pp. 221–240.

[11] ALLEN L. SHIELDS, *Weighted shift operators and analytic function theory*, 1974, 49–128, Math. Surveys, no. 13.

[12] JAMES P. WILLIAMS, *Minimal spectral sets of compact operators*, Acta Sci. Math. (Szeged) **28** (1967), pp. 93–106.

(Received June 1, 2021)

Georgios Tsikalas
Department of Mathematics and Statistics
Washington University in St. Louis
St. Louis, MO, 63136
e-mail: gtsikalas@wustl.edu