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ON A CHARACTERIZATION OF JERIBI, RAKOČEVIĆ, SCHECHTER, SCHMOEGER AND WOLF ESSENTIAL SPECTRA OF A 3×3 BLOCK OPERATOR MATRICES WITH NON DIAGONAL DOMAIN AND APPLICATION

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Abstract. In this paper, we investigate the stability of some essential spectra of a 3×3 block operator matrices with unbounded entries and with non diagonal domain by using the resolvent of this kind of matrix operator. Furthermore, we give an application from Three-Group transport theory to illustrate the validity of the main results in the Banach space $L_1([-a,a]\times[-1,1];dxdv)$, $a>0$.

1. Introduction

In this work we are concerned with the essential spectra of operators defined by a 3×3 block operator matrices

$$\mathscr{A} := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & L \end{pmatrix} \tag{1.1}$$

where the entries of the matrix are in general unbounded operators. The operator (1.1) is defined on a domain consist of vectors which satisfy certain relations between the components that is:

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(A_m)\times\mathscr{D}(E_m)\times\mathscr{D}(L_m) \ \text{ such that } \begin{array}{l} \phi_1(f)=\psi_2(g)=\psi_3(h) \\ \phi_2(g)=\psi_1(f)=\psi_6(h) \\ \phi_3(h)=\psi_4(f)=\psi_5(g) \end{array} \right\},$$

for continuous linear operators ϕ_i and ψ_j, $i=1,2,3$ and $j\in\{1,2,3,4,5,6\}$.

Note that, the operator $\mathscr{A}$ need to be closed, so in view of the continuity assumptions on the ϕ's and ψ's the domain $\mathscr{D}(\mathscr{A})$ is closed in $\mathscr{D}(A_m)\times\mathscr{D}(E_m)\times\mathscr{D}(L_m)$ with respect to the graph norm. Hence $(\mathscr{A},\mathscr{D}(\mathscr{A}))$ is a closed operator in the product of Banach spaces.

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We need some standard notation from Fredholm theory. Let X and Y be two Banach spaces. By an operator T from X into Y, we mean a linear operator with domain $\mathscr{D}(T) \subset X$ and range $Im(T) \subset Y$. By $\mathscr{C}(X,Y)$ we denote the set of all closed, densely defined linear operators from X into Y, by $\mathscr{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y. If $X = Y$, the sets $\mathscr{L}(X,Y)$ and $\mathscr{C}(X,Y)$ are replaced respectively by $\mathscr{L}(X)$ and $\mathscr{C}(X)$.

If $T \in \mathscr{C}(X)$ then $\rho(T)$ denotes the resolvent set of T, $\alpha(T)$ the dimension of the $\ker(T)$ and $\beta(T)$ the codimension of $Im(T)$ in Y. The classes of Fredholm, upper semi-Fredholm and lower semi-Fredholm operators from X into Y are respectively given by:

$$\Phi(X,Y) = \{T \in \mathscr{C}(X,Y) \text{ such that } \alpha(T) < \infty, \ \beta(T) < \infty \text{ and } Im(T) \text{ is closed in } Y\},$$

$$\Phi_+(X,Y) = \{T \in \mathscr{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } Im(T) \text{ is closed in } Y\},$$

and

$$\Phi_-(X,Y) = \{T \in \mathscr{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } Im(T) \text{ is closed in } Y\}.$$

If $X = Y$, the sets $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are respectively replaced by $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$.

An operator $F \in L(X,Y)$ is called a Fredholm perturbation, upper semi-Fredholm perturbation or lower semi-Fredholm perturbation, if $T+F \in \Phi(X,Y)$, $T+F \in \Phi_+(X,Y)$ or $T+F \in \Phi_-(X,Y)$ whenever $T \in \Phi(X,Y)$, $T \in \Phi_+(X,Y)$ or $T \in \Phi_-(X,Y)$, respectively. The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathscr{F}(X,Y)$, $\mathscr{F}_+(X,Y)$ and $\mathscr{F}_-(X,Y)$, respectively. These classes of operators were introduced and investigated by Gohberg et al [9].

An operator $T \in L(X,Y)$ is said to be weakly compact if $T(M)$ is relatively weakly compact in Y for every bounded subset $M \subset X$. The family of weakly compact operators from X into Y is denoted by $W(X,Y)$ (see [10]).

An operator $T \in \mathscr{L}(X,Y)$ is said to be strictly singular if the restriction of T to any infinite-dimensional subspace of X is not a homeomorphism. the set of strictly singular operators from X to Y is denoted by $\mathscr{S}(X,Y)$ (see [22]).

If $X = Y$, the sets $W(X,Y)$ and $\mathscr{S}(X,Y)$ are replaced respectively by $W(X)$ and $\mathscr{S}(X)$. Let $T \in \mathscr{C}(X)$, various notions of essential spectra have been defined in the literature. In this work, we are concerned with the following spectrum and the essential spectra:

$$\begin{aligned}
\sigma(T) &:= \{\lambda \in \mathbb{C} : \ T - \lambda \text{ is not boundedly invertible}\},\\
\sigma_{e4}(T) &:= \{\lambda \in \mathbb{C} \ : \ T - \lambda \notin \Phi(X)\},\\
\sigma_{e5}(T) &:= \mathbb{C} \setminus \rho_{e5}(T),\\
\sigma_{eap}(T) &:= \mathbb{C} \setminus \rho_{eap}(T),\\
\sigma_{e\delta}(T) &:= \mathbb{C} \setminus \rho_{e\delta}(T),\\
\sigma_J(T) &:= \bigcap_{K \in \mathscr{W}_*(X)} \sigma(T+K)
\end{aligned}$$

where $\mathscr{W}_*(X)$ stands for each one of sets $\mathscr{W}(X)$ and $\mathscr{S}(X)$ and

$$\begin{aligned}\rho_{e5}(T) &:= \{\lambda\in\mathbb{C} : T-\lambda\in\Phi(X,Y),\ i(T-\lambda)=0\},\\ \rho_{eap}(T) &:= \{\lambda\in\mathbb{C} : T-\lambda\in\Phi_+(X,Y),\ i(T-\lambda)\leqslant 0\},\\ \rho_{e\delta}(T) &:= \{\lambda\in\mathbb{C} : T-\lambda\in\Phi_-(X,Y),\ i(T-\lambda)\geqslant 0\}.\end{aligned}$$

We call σ_J, σ_{eap}, $\sigma_{e\delta}$, σ_{e5} and σ_{e4} the Jeribi, Rakočević, Schmoeger, Schechter and Wolf essential spectra, respectively (see for instance[3, 12, 15, 13, 14, 18, 27, 29]).

During the last years, e.g. the papers [2, 7, 8, 30] were devoted to the study of the essential spectra of operators defined by a 2×2 block operator matrix acts on the product $X\times Y$ of Banach spaces. An account of the research and a wide panorama of methods to investigate some essential spectra of block operator matrices are presented by Tretter in [31].

Systems of linear evolution equations as well as linear initial value problems with more than one set of initial data lead, in a natural way, to an abstract Cauchy problem involving an operator matrix defined on a product of n Banach spaces. In the theory of unbounded block operator matrices, The Frobenuis-Schur factorization is a basic tool to study the some essential spectra and various spectral properties. In fact, the characterization and the investigation of some essential spectra of 3×3 block operator matrices with diagonal domain case, have drawn the attention of several authors involving the corresponding Schur complements. In [21, 32], A. Jeribi, N. Moalla, and I. Walha treated a 3×3 block operator matrix (1.1) on a product of Banach spaces. They supposed that the entries of the kind matrix are generally unbounded operators. The operator (1.1) is defined on $\big(\mathscr{D}(A)\cap\mathscr{D}(D)\cap\mathscr{D}(G)\big)\times\big(\mathscr{D}(B)\cap\mathscr{D}(E)\cap\mathscr{D}(H)\big)\times\big(\mathscr{D}(C)\cap\mathscr{D}(F)\cap\mathscr{D}(L)\big)$. Notice that this operator doesnt need to be closed. It was shown that, under certain conditions, this block operator matrix defines a closable operator and its essential spectra are determined. Recently in [4], Ben Amar, Jeribi and Krichen and in [5], Ben Amar, Jeribi and Moalla have studied the spectral properties of a 3×3 block operator matrices (1.1), with unbounded entries and the domain is defined by additional relations of the form $\Gamma_X x=\Gamma_Y y=\Gamma_Z z$ between the three components of its elements. They focused on the description of the Jeribi, Rakočević, Schechter, Schmoeger and Wolf essential spectra of $\mathscr{A}$.

In the present paper we extend these results to 3×3 block operator matrix with unbounded entries and with domain consisting of vectors which satisfy certain relations between their components in the product of Banach spaces (see the expression (2.1)). Comparing with the papers [4, 21], we can determine the essential spectra of the operator $\mathscr{A}$ using its resolvent. This was first recognized by Moalla et al [25] in the case 2×2. For this, to achieve this goal, we determine the expression of the resolvent $(\mathscr{A}-\lambda)^{-1}$ for some convenable λ. More precisely, the idea is to associate to the pair $(\mathscr{A},I)$ a pair $(\mathscr{A}_0,I)$, which is more easier to deal with and we prove that $\sigma_{ek}(\mathscr{A})=\sigma_{ek}(A_0)\cup\sigma_{ek}(E_0)\cup\sigma_{ek}(L_0)$, where $A_0=A_m|_{\ker\phi_1}$, $E_0=E_m|_{\ker\phi_2}$, $L_0=L_m|_{\ker\phi_3}$ and $ek\in\{J,e4,e5,\mathrm{eap},e\delta\}$.(For more details see Theorems 2.8 and 2.9).

An outline of the paper is as follows. The first section consists of two subsections: On the one hand, we establish a decomposition of the operator matrices (1.1) (see

Lemma 2.2) and we give its resolvent as product between two operators matrices one diagonal operator and the other consider an intermediary that one needs to use in the sequel of the second subsection which we will characterize some essential spectra of the kind operator matrices (1.1).

In the last section, we use the notation of the first Section and we apply the main results of Theorem 2.9 and 2.8 to describe the Wolf, Schechter, Rakočević, Schmoeger and Jeribi essential spectra of a three-group transport operator acting in the Banach space $X_1 \times X_1 \times X_1$ where $X_1 := L_1([-a,a] \times [-1,1]; dxdv)$, $a > 0$. (See [11, 16, 17] for more details on transport equations.)

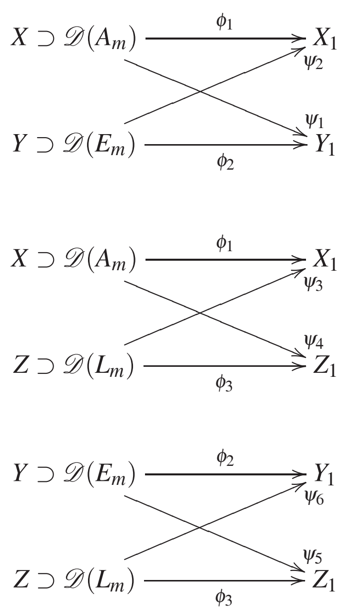
2. Mains results

2.1. The virtual operator matrix $\mathscr{A}$ and its resolvent

Let X, Y and Z be three Banach spaces. We consider the block operator matrix (1.1) in the space $X \times Y \times Z$, that is the linear operator A acts in X, E in Y and L in Z. Further we will consider the following assumptions:

A_m, E_m and L_m are closed, densely defined operators with domains $\mathscr{D}(A_m)$ in X, $\mathscr{D}(E_m)$ in Y and $\mathscr{D}(L_m)$ in Z respectively.

Let X_1, Y_1 and Z_1 be three Banach spaces (called "spaces of boundary conditions"). Endow $\mathscr{D}(A_m)$, $\mathscr{D}(E_m)$ and $\mathscr{D}(L_m)$ with the graph norm and define continuous linear operators ϕ_1, ϕ_2, ϕ_3, and ψ_i; $i = 1,2,3,4,5,6$ as in the diagram:



In addition, we always assume that ϕ_1 , ϕ_2 and ϕ_3 are surjective.

Under the above assumptions, let B from $\mathscr{D}(E_m)$ to X, C from $\mathscr{D}(L_m)$ to X, etc. Consider the operator matrix

$$\mathscr{A}\begin{pmatrix} f\\ g\\ h\end{pmatrix}=\begin{pmatrix} A_m f+Bg+Ch\\ Df+E_m g+Fh\\ Gf+Hg+L_m h\end{pmatrix},\ \text{ for }\ \begin{pmatrix} f\\ g\\ h\end{pmatrix}\in\mathscr{D}(\mathscr{A}).$$

with domain $\mathscr{D}(\mathscr{A})$ consisting of vectors which satisfy certain relations between their components i.e:

$$\mathscr{D}(\mathscr{A})=\left\{\begin{pmatrix} f\\ g\\ h\end{pmatrix}\in\mathscr{D}(A_m)\times\mathscr{D}(E_m)\times\mathscr{D}(L_m)\ \text{ such that }\ \begin{array}{l}\phi_1(f)=\psi_2(g)=\psi_3(h)\\ \phi_2(g)=\psi_1(f)=\psi_6(h)\\ \phi_3(h)=\psi_4(f)=\psi_5(g)\end{array}\right\}. \quad (2.1)$$

DEFINITION 2.1. Consider the restrictions A_0, E_0 and L_0 of A_m, E_m, L_m to $\ker\phi_1$, $\ker\phi_2$ and $\ker\phi_3$ respectively. Then, $\mathscr{A}_0$ denotes the diagonal operator matrix $\begin{pmatrix} A_0 & 0 & 0\\ 0 & E_0 & 0\\ 0 & 0 & L_0\end{pmatrix}$ with domain $\mathscr{D}(\mathscr{A}_0):=\mathscr{D}(A_0)\times\mathscr{D}(E_0)\times\mathscr{D}(L_0)$.

It is now our aim, on the Banach space $X\times Y\times Z$, to write a 'virtual' operator matrix, that is to establish the relation between $\mathscr{A}-\lambda$ and $\mathscr{A}_0-\lambda$.

LEMMA 2.2. [26, Definition 2.3, p.4]
(i) Let $(C_0,\mathscr{D}(C_0))$ be a restriction of a closed operator $(C_m,\mathscr{D}(C_m))$ on some Banach space. For every $\lambda\in\rho(C_0)$, the following decomposition holds:

$$\mathscr{D}(C_m)=\mathscr{D}(C_0)\oplus\ker(C_m-\lambda)$$

(ii) For $\lambda\in\rho(A_0)$, $\lambda\in\rho(E_0)$ and $\lambda\in\rho(L_0)$, we have that the operators

$$\phi_{1\lambda}:=\phi_1|_{\ker(A_m-\lambda)},\quad \phi_{2\lambda}:=\phi_2|_{\ker(E_m-\lambda)}\ \text{ and }\ \phi_{3\lambda}:=\phi_3|_{\ker(L_m-\lambda)}.$$

are continuous bijections from $\ker(A_m-\lambda)$ onto X_1 and from $\ker(E_m-\lambda)$ onto Y_1 and from $\ker(L_m-\lambda)$ onto Z_1 respectively.

As a direct consequence of the above Lemma, for $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$, we define the following operators

$$\begin{cases} L_\lambda:\mathscr{D}(A_m)\longrightarrow\mathscr{D}(E_m),\\ \qquad f\longmapsto L_\lambda(f)=\phi_{2\lambda}^{-1}\circ\psi_1(f)\end{cases}\qquad \begin{cases} N_\lambda:\mathscr{D}(A_m)\longrightarrow\mathscr{D}(L_m),\\ \qquad f\longmapsto N_\lambda(f)=\phi_{3\lambda}^{-1}\circ\psi_4(f)\end{cases}$$

$$\begin{cases} K_\lambda:\mathscr{D}(E_m)\longrightarrow\mathscr{D}(A_m),\\ \qquad g\longmapsto K_\lambda(g)=\phi_{1\lambda}^{-1}\circ\psi_2(g)\end{cases}\qquad \begin{cases} P_\lambda:\mathscr{D}(E_m)\longrightarrow\mathscr{D}(L_m),\\ \qquad g\longmapsto P_\lambda(g)=\phi_{3\lambda}^{-1}\circ\psi_5(g)\end{cases}$$

$$\begin{cases} M_\lambda:\mathscr{D}(L_m)\longrightarrow\mathscr{D}(A_m),\\ \qquad h\longmapsto M_\lambda(h)=\phi_{1\lambda}^{-1}\circ\psi_3(h)\end{cases}\qquad \begin{cases} Q_\lambda:\mathscr{D}(L_m)\longrightarrow\mathscr{D}(E_m),\\ \qquad h\longmapsto Q_\lambda(h)=\phi_{2\lambda}^{-1}\circ\psi_6(h).\end{cases}$$

Then, for all $f \in \mathscr{D}(A_m)$, $g \in \mathscr{D}(E_m)$ and $h \in \mathscr{D}(L_m)$, the operators K_λ, L_λ, M_λ, N_λ, P_λ and Q_λ are bounded and the following results are evident:

$$\phi_1(\mathrm{K}_\lambda g) = \psi_2(g), \quad \phi_1(\mathrm{M}_\lambda h) = \psi_3(h) \tag{2.2}$$

$$\phi_2(L_\lambda f) = \psi_1(f), \quad \phi_2(Q_\lambda h) = \psi_6(h) \tag{2.3}$$

$$\phi_3(N_\lambda f) = \psi_4(f), \quad \phi_3(P_\lambda g) = \psi_5(g). \tag{2.4}$$

REMARK 2.3. For $\lambda \in \rho(A_0) \cap \rho(E_0) \cap \rho(L_0)$, we have $\{Im\mathrm{K}_\lambda, Im\mathrm{M}_\lambda\} \subset \ker(A_m - \lambda)$, $\{Im\mathrm{L}_\lambda, Im\mathrm{Q}_\lambda\} \subset \ker(E_m - \lambda)$ and $\{Im\mathrm{N}_\lambda, Im\mathrm{P}_\lambda\} \subset \ker(L_m - \lambda)$.

LEMMA 2.4. *Let $\lambda \in \rho(A_0) \cap \rho(E_0) \cap \rho(L_0)$. Then, the 'virtual' operator matrix $(\mathscr{A} - \lambda)$ is write by the decomposition as follows*

$$(\mathscr{A} - \lambda) = (\mathscr{A}_0 - \lambda)\mathbb{Q}_\lambda \quad on \ \mathscr{D}(\mathscr{A}). \tag{2.5}$$

where

$$\mathbb{Q}_\lambda := \begin{pmatrix} Id & -K_\lambda + (A_0 - \lambda)^{-1}B & -M_\lambda + (A_0 - \lambda)^{-1}C \\ -L_\lambda + (E_0 - \lambda)^{-1}D & Id & -Q_\lambda + (E_0 - \lambda)^{-1}F \\ -N_\lambda + (L_0 - \lambda)^{-1}G & -P_\lambda + (L_0 - \lambda)^{-1}H & Id \end{pmatrix}$$

Proof. We decompose $\mathbb{Q}_\lambda$ in the form

$$\mathbb{Q}_\lambda = \mathbb{B}_\lambda + \mathbb{C}_\lambda$$

with

$$\mathbb{B}_\lambda := \begin{pmatrix} Id & -K_\lambda & -M_\lambda \\ -L_\lambda & Id & -Q_\lambda \\ -N_\lambda & -P_\lambda & Id \end{pmatrix}$$

and

$$\mathbb{C}_\lambda := \begin{pmatrix} 0 & (A_0 - \lambda)^{-1}B & (A_0 - \lambda)^{-1}C \\ (E_0 - \lambda)^{-1}D & 0 & (E_0 - \lambda)^{-1}F \\ (L_0 - \lambda)^{-1}G & (L_0 - \lambda)^{-1}H & 0 \end{pmatrix}$$

which the both defines a bounded operators on $\mathscr{D}(A_m) \times \mathscr{D}(E_m) \times \mathscr{D}(L_m)$. Furthermore

$$\mathscr{D}(\mathbb{Q}_\lambda) = \mathscr{D}(A_m) \times \mathscr{D}(D_m) \times \mathscr{D}(L_m).$$

Denote by $\Theta_\lambda = (\mathscr{A}_0 - \lambda)\mathbb{Q}_\lambda$. So

$$\begin{aligned} \mathscr{D}(\Theta_\lambda) &= \left\{ \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(\mathbb{Q}_\lambda) \text{ such that } Q_\lambda \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(\mathscr{A}_0) \right\} \\ &= \left\{ \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(A_m) \times \mathscr{D}(E_m) \times \mathscr{D}(L_m)\,; \ \begin{matrix} \phi_1(f) - \phi_1(K_\lambda g) - \phi_1(M_\lambda h) = 0, \\ \phi_2(g) - \phi_2(L_\lambda f) - \phi_2(Q_\lambda h) = 0, \\ \phi_3(h) - \phi_3(N_\lambda f) - \phi_3(P_\lambda g) = 0 \end{matrix} \right\} \end{aligned}$$

using (2.2), (2.3) and (2.4)

$$\mathscr{D}(\Theta_\lambda) = \left\{ \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(A_m)\times\mathscr{D}(E_m)\times\mathscr{D}(L_m) \text{ such that } \begin{array}{l} \phi_1(f)=\psi_2(g)=\psi_3(h), \\ \phi_2(g)=\psi_1(f)=\psi_6(h), \\ \phi_3(h)=\psi_4(f)=\psi_5(g), \end{array} \right\}$$
$$= \mathscr{D}(\mathscr{A}).$$

We show that the expression (2.5) is verified. As a first step, for $\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(\mathscr{A})$, if we take $B=C=D=F=G=H=0$, we get the following property by analogy with the case of a 2×2 operator matrix (see [26, Lemma 2.6]), we introduce the following result:

$$(\mathscr{A}_0-\lambda)\mathbb{B}_\lambda \begin{pmatrix} f \\ g \\ h \end{pmatrix} = (\mathscr{A}-\lambda)\begin{pmatrix} f \\ g \\ h \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathscr{D}(\mathscr{A}). \tag{2.6}$$

Therefore, we can prove with this equalities

$$(\mathscr{A}_0-\lambda)\mathbb{Q}_\lambda \begin{pmatrix} f \\ g \\ h \end{pmatrix} = (\mathscr{A}_0-\lambda)\mathbb{B}_\lambda \begin{pmatrix} f \\ g \\ h \end{pmatrix} + (\mathscr{A}_0-\lambda)\mathbb{C}_\lambda \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

and according to the decomposition (2.6) and with a simple calculus, that

$$(\mathscr{A}_0-\lambda)\mathbb{Q}_\lambda \begin{pmatrix} f \\ g \\ h \end{pmatrix} = (\mathscr{A}-\lambda)\begin{pmatrix} f \\ g \\ h \end{pmatrix}. \quad \square$$

In addition, the following lemma consists of showing what are the conditions that we impose on the operators entries of $\mathscr{A}-\lambda$ which make it to become invertible. For this purpose, we need firstly to show the invertibility of the matrix operator $\mathbb{Q}_\lambda$. So, by analogy with the case of a 2×2 operator matrix (see [2, 31]), we start with the following Frobenius Schur type factorization of $\mathbb{Q}_\lambda$:

$$\mathbb{Q}_\lambda = \begin{pmatrix} Id & 0 & 0 \\ F_1(\lambda) & Id & 0 \\ F_2(\lambda) & F_3(\lambda) & Id \end{pmatrix} \begin{pmatrix} Id & 0 & 0 \\ 0 & S_1(\lambda) & 0 \\ 0 & 0 & S_2(\lambda) \end{pmatrix} \begin{pmatrix} Id & G_1(\lambda) & G_2(\lambda) \\ 0 & Id & G_3(\lambda) \\ 0 & 0 & Id \end{pmatrix} \tag{2.7}$$

where

$$\begin{aligned}
F_1(\lambda) &= -\mathrm{L}_\lambda + (E_0-\lambda)^{-1}D, \\
F_2(\lambda) &= -\mathrm{N}_\lambda + (L_0-\lambda)^{-1}G, \\
G_1(\lambda) &= -\mathrm{K}_\lambda + (A_0-\lambda)^{-1}B, \\
G_2(\lambda) &= -\mathrm{M}_\lambda + (A_0-\lambda)^{-1}C, \\
S_1(\lambda) &= Id - F_1(\lambda)G_1(\lambda), \\
F_3(\lambda) &= [\theta_1(\lambda) - F_2(\lambda)G_1(\lambda)]S_1^{-1}(\lambda), \text{ with } \theta_1(\lambda) = -\mathrm{P}_\lambda + (L_0-\lambda)^{-1}H. \\
G_3(\lambda) &= S_1^{-1}(\lambda)[\theta_2(\lambda) - F_1(\lambda)G_2(\lambda)], \text{ with } \theta_2(\lambda) = -\mathrm{Q}_\lambda + (E_0-\lambda)^{-1}F. \\
S_2(\lambda) &= Id - F_2(\lambda)G_2(\lambda) - F_3(\lambda)S_1(\lambda)G_3(\lambda).
\end{aligned}$$

Then, we can easily derive the following result:

LEMMA 2.5. *Let* $\lambda \in \rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$,

(i) $\mathbb{Q}_\lambda$ *is invertible in* $\mathscr{L}(\mathscr{D}(A_m)\times\mathscr{D}(E_m)\times\mathscr{D}(Z_m))$ *if and only if* $S_1(\lambda)$ *and* $S_2(\lambda)$ *are invertible respectively in* $\mathscr{L}(\mathscr{D}(E_m))$ *and* $\mathscr{L}(\mathscr{D}(L_m))$.

(ii) If $(Id-S_1(\lambda))\in\mathscr{PK}(\mathscr{D}(E_m))$ *and if* $(Id-S_2(\lambda))\in\mathscr{PK}(\mathscr{D}(L_m))$, *then*

$$\lambda\in\rho(\mathscr{A})\Leftrightarrow 0\in\rho(S_1(\lambda)) \text{ and } 0\in\rho(S_2(\lambda)).$$

Proof. (i) follows from (2.7). In order to show (ii), for $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$, using item (i) and if $S_1(\lambda)$ and $S_2(\lambda)$ are invertible respectively in $\mathscr{L}(\mathscr{D}(E_m))$ and $\mathscr{L}(\mathscr{D}(L_m))$ then $\mathscr{A}-\lambda$ is invertible in $\mathscr{L}(X\times Y\times Z)$.

Conversely, assume that $\mathscr{A}-\lambda$ is invertible, then $\mathscr{A}-\lambda$ is injective. Therefore, decomposition (2.5) revels that $S_1(\lambda)$ and $S_2(\lambda)$ are injective. Using Theorem (2.2) in [19], under the assumption that $(Id-S_1(\lambda))\in\mathscr{PK}(\mathscr{D}(E_m))$ and $(Id-S_2(\lambda))\in\mathscr{PK}(\mathscr{D}(L_m))$, amounts that $S_1(\lambda)$ and $S_2(\lambda)$ are invertible. □

THEOREM 2.6. *For* $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$ *such that* $0\in\rho(S_1(\lambda))$ *and* $0\in\rho(S_2(\lambda))$, *the resolvent of* $\mathscr{A}$ *is formally given by:*

$$\mathscr{R}_\lambda(\mathscr{A})=\begin{pmatrix} R_1 & R_2 & R_3\\ R_4 & R_5 & R_6\\ R_7 & R_8 & R_9\end{pmatrix}$$

where:

$$\begin{aligned}
R_1 &= [I+G_1(\lambda)S_1^{-1}(\lambda)F_1(\lambda)+[G_1(\lambda)G_3(\lambda)-G_2(\lambda)]S_2^{-1}(\lambda)[F_1(\lambda)F_3(\lambda)-F_2(\lambda)]]\\
&\quad\times\mathscr{R}_\lambda(A_0),\\
R_2 &= [-G_1(\lambda)S_1^{-1}(\lambda)-[G_1(\lambda)G_3(\lambda)-G_2(\lambda)]S_2^{-1}(\lambda)F_3(\lambda)]\mathscr{R}_\lambda(E_0),\\
R_3 &= [G_1(\lambda)G_3(\lambda)-G_2(\lambda)]S_2^{-1}(\lambda)\mathscr{R}_\lambda(L_0),\\
R_4 &= [-S_1^{-1}(\lambda)F_1(\lambda)-G_3(\lambda)[S_2^{-1}(\lambda)(F_1(\lambda)F_3(\lambda)-F_2(\lambda))]]\mathscr{R}_\lambda(A_0),\\
R_5 &= [S_1^{-1}(\lambda)+G_3(\lambda)S_2^{-1}(\lambda)F_3(\lambda)]\mathscr{R}_\lambda(E_0),\\
R_6 &= -G_3(\lambda)S_2^{-1}(\lambda)\mathscr{R}_\lambda(L_0),\\
R_7 &= [S_2^{-1}(\lambda)(F_1(\lambda)F_3(\lambda)-F_2(\lambda))]\mathscr{R}_\lambda(A_0),\\
R_8 &= -S_2^{-1}(\lambda)F_3(\lambda)\mathscr{R}_\lambda(E_0),\\
R_9 &= S_2^{-1}(\lambda)\mathscr{R}_\lambda(L_0).
\end{aligned}$$

REMARK 2.7. For $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$ such that $0\in\rho(S_1(\lambda))$ and $0\in\rho(S_2(\lambda))$, we obtain

$$\begin{aligned}
S_1^{-1}(\lambda) &= I+S_1^{-1}(\lambda)F_1(\lambda)G_1(\lambda)\\
S_2^{-1}(\lambda) &= I+S_2^{-1}(\lambda)[F_2(\lambda)G_2(\lambda)\\
&\qquad -[\theta_1(\lambda)-F_2(\lambda)G_1(\lambda)]S_1^{-1}(\lambda)[\theta_2(\lambda)-F_1(\lambda)G_2(\lambda)]].
\end{aligned}$$

2.2. Jeribi, Rakočević, Schechter, Schmoeger and Wolf essential spectra of $\mathscr{A}$ with non diagonal domain

Having obtained the resolvent $\mathscr{R}_\lambda(\mathscr{A})$ of the operator $\mathscr{A}$, in this subsection we discuss its essential spectra. As a first step we prove the first result.

THEOREM 2.8. *Suppose that* $0\in\rho(S_1(\lambda))\cap\in\rho(S_2(\lambda))$ *are satisfied. If for* $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$, *we have* $\mathscr{R}_\lambda(\mathscr{A})-\mathscr{R}_\lambda(\mathscr{A}_0)\in\mathscr{F}(X\times Y\times Z)$, *then the following results are satisfied:*

$$\sigma_{e4}(\mathscr{A})=\sigma_{e4}(A_0)\cup\sigma_{e4}(E_0)\cup\sigma_{e4}(L_0), \tag{2.8}$$

$$\sigma_{e5}(\mathscr{A})\subseteq\sigma_{e5}(A_0)\cup\sigma_{e5}(E_0)\cup\sigma_{e5}(L_0). \tag{2.9}$$

Moreover, if the sets $\mathbb{C}\setminus\sigma_{e4}(A_0)$ *and* $\mathbb{C}\setminus\sigma_{e4}(E_0)$ *are connected, then*

$$\sigma_{e5}(\mathscr{A})=\sigma_{e5}(A_0)\cup\sigma_{e5}(E_0)\cup\sigma_{e5}(L_0). \tag{2.10}$$

In addition if X *has no reflexive infinite dimensional subspaces, then we have*

$$\sigma_J(\mathscr{A})=\sigma_J(A_0)\cup\sigma_J(E_0)\cup\sigma_J(L_0). \tag{2.11}$$

Proof. As a consequence of Remark 2.7, we can easily drive a same expression of the resolvent of operator matrices $\mathscr{A}$ will play a prominent role. If $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$, the resolvent of $\mathscr{A}$ is given by:

$$\mathscr{R}_\lambda(\mathscr{A})=\mathscr{R}_\lambda(\mathscr{A}_0)+\mathbb{M}(\lambda),$$

with

$$\mathbb{M}(\lambda)=\begin{pmatrix} I_1 & R_2 & R_3\\ R_4 & I_2 & R_6\\ R_7 & R_8 & I_3\end{pmatrix}$$

where:

$$\begin{aligned} I_1 &= [G_1(\lambda)S_1^{-1}(\lambda)F_1(\lambda)+[G_1(\lambda)G_3(\lambda)-G_2(\lambda)]S_2^{-1}(\lambda)[F_1(\lambda)F_3(\lambda)-F_2(\lambda)]]\\ &\quad\times\mathscr{R}_\lambda(A_0),\\ I_2 &= [S_1^{-1}(\lambda)F_1(\lambda)G_1(\lambda)+G_3(\lambda)S_2^{-1}(\lambda)F_3(\lambda)]\mathscr{R}_\lambda(E_0),\\ I_3 &= [S_2^{-1}(\lambda)[F_2(\lambda)G_2(\lambda)-[\theta_1(\lambda)-F_2(\lambda)G_1(\lambda)]S_1^{-1}(\lambda)(\theta_2(\lambda)-F_1(\lambda)G_2(\lambda))]]\\ &\quad\times\mathscr{R}_\lambda(L_0).\end{aligned}$$

Since $\mathbb{M}(\lambda)$ is Fredholm perturbation. Hence, according to Theorem (3.2) (i) in [23], one gets $\sigma_{e4}(\mathscr{A})=\sigma_{e4}(\mathscr{A}_0)$. Which shows (2.8). The second result stems from

$$i(\mathscr{A}-\lambda)=i(A_0-\lambda)+i(E_0-\lambda)+i(L_0-\lambda). \tag{2.12}$$

If $i(\mathscr{A}-\lambda)\neq 0$, then one of the terms in (2.12) is non-zero, hence

$$\sigma_{e5}(\mathscr{A})=\sigma_{e5}(\mathscr{A}_0)\subseteq\sigma_{e5}(A_0)\cup\sigma_{e5}(E_0)\cup\sigma_{e5}(L_0).$$

According to the second result (2.9), it is sufficient to verify the opposite inclusion. Since $\mathbb{C}\setminus\sigma_{e4}(A_0)$ is connected, by Theorem 2.1 in [1], $\sigma_{e4}(A_0)=\sigma_{e5}(A_0)$. Using the same argument that $\mathbb{C}\setminus\sigma_{e4}(E_0)$ is connected, then $\sigma_{e4}(E_0)=\sigma_{e5}(E_0)$ and $i(E_0-\lambda)=0$ for each $\lambda\in\mathbb{C}\setminus\sigma_{e4}(E_0)$. If $\lambda\in\mathbb{C}\setminus\sigma_{e5}(\mathscr{A})$, then $\lambda\in\mathbb{C}\setminus\sigma_{e4}(A_0)$, $\lambda\in\mathbb{C}\setminus\sigma_{e4}(E_0)$ and $\lambda\in\mathbb{C}\setminus\sigma_{e4}(L_0)$. Further, $i(\mathscr{A}-\lambda)=i(L_0-\lambda)$, hence $\lambda\in\mathbb{C}\setminus\sigma_{e5}(L_0)$ and (2.10) is proved.

From Remark 7.2.1 in [13], the Jeribi essential spectrum always satisfying the inclusion $\sigma_J(A_0)\subseteq\sigma_{e5}(A_0)$, $\sigma_J(E_0)\subseteq\sigma_{e5}(E_0)$ and $\sigma_J(L_0)\subseteq\sigma_{e5}(L_0)$, furthermore if the Banach space X has no reflexive infinite dimensional subspaces and according to Theorem 3.3 in [3] we have $\sigma_{e4}(A_0)\subset\sigma_J(A_0)$, then by Eq (2.10) we obtain

$$\sigma_{e5}(A_0)=\sigma_{e4}(A_0)\subset\sigma_J(A_0)\subseteq\sigma_{e5}(A_0)$$

Hence $\sigma_J(A_0)=\sigma_{e5}(A_0)$.

In the same way we have $\sigma_J(E_0)=\sigma_{e5}(E_0)$ and $\sigma_J(L_0)=\sigma_{e5}(L_0)$. So,

$$\sigma_J(\mathscr{A})=\sigma_J(A_0)\cup\sigma_J(E_0)\cup\sigma_J(L_0). \quad \square$$

In the next result, we discuss the Rakočević and Schmoeger essential spectra of unbounded operator matrix (1.1) with non diagonal domain.

THEOREM 2.9. *Suppose that* $0\in\rho(S_1(\lambda))\cap\rho(S_2(\lambda))$ *are satisfied.*
(i) *If for* $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$, *we have* $\mathbb{M}(\lambda)\in\mathscr{F}_+(X\times Y\times Z)$, *then*

$$\sigma_{eap}(\mathscr{A})\subseteq\sigma_{eap}(A_0)\cup\sigma_{eap}(E_0)\cup\sigma_{eap}(L_0).$$

Moreover, if the sets $\mathbb{C}\setminus\sigma_{e4}(A_0)$, $\mathbb{C}\setminus\sigma_{e4}(E_0)$, $\mathbb{C}\setminus\sigma_{e4}(L_0)$ *and* $\mathbb{C}\setminus\sigma_{e4}(\mathscr{A})$ *are connected, then*

$$\sigma_{eap}(\mathscr{A})=\sigma_{eap}(A_0)\cup\sigma_{eap}(E_0)\cup\sigma_{eap}(L_0).$$

(ii) *If for* $\lambda\in\rho(A_0)\cap\rho(E_0)\cap\rho(L_0)$, *we have* $\mathbb{M}(\lambda)\in\mathscr{F}_-(X\times Y\times Z)$, *then*

$$\sigma_{e\delta}(\mathscr{A})\subseteq\sigma_{e\delta}(A_0)\cup\sigma_{e\delta}(E_0)\cup\sigma_{e\delta}(L_0).$$

Moreover, if the sets $\mathbb{C}\setminus\sigma_{e4}(A_0)$, $\mathbb{C}\setminus\sigma_{e4}(E_0)$, $\mathbb{C}\setminus\sigma_{e4}(L_0)$ *and* $\mathbb{C}\setminus\sigma_{e4}(\mathscr{A})$ *are connected, then*

$$\sigma_{e\delta}(\mathscr{A})=\sigma_{e\delta}(A_0)\cup\sigma_{e\delta}(E_0)\cup\sigma_{e\delta}(L_0).$$

Proof. (i) We infer by Lemma 2.5 that $\lambda\in\rho(\mathscr{A})\cap\rho(\mathscr{A}_0)$. This together with the fact that $\mathbb{M}(\lambda)\in\mathscr{F}_+(X,Y,Z)$, leads from Theorem 3.3 (i) in [20] to $\sigma_{eap}(\mathscr{A})=\sigma_{eap}(\mathscr{A}_0)$.

As $\mathscr{A}_0$ is a diagonal operator matrices, and

$$i(\mathscr{A}-\lambda)=i(A_0-\lambda)+i(E_0-\lambda)+i(L_0-\lambda), \tag{2.13}$$

this shows that $\sigma_{eap}(\mathscr{A}_0)=\sigma_{eap}(A_0)\cup\sigma_{eap}(E_0)\cup\sigma_{eap}(L_0)$.

Since $\mathbb{C}\setminus\sigma_{e4}(A_0)$, $\mathbb{C}\setminus\sigma_{e4}(E_0)$, $\mathbb{C}\setminus\sigma_{e4}(L_0)$ and $\mathbb{C}\setminus\sigma_{e4}(\mathscr{A})$ are connected, the result follows [7, Proposition 2.3] together with [25, Theorem 3.2].

A same reasoning allows us to reach the result of item (ii). $\square$

3. Application to a three-group transport theory

The work presented in this section concerns the application of Theorem 2.8 and Theorem 2.9 to a three-group transport operator in an L_1-space. Let $a>0$ and

$$X_1 := L_1((-a,a)\times(-1,1);dxdv), \qquad X=Y=Z:=X_1.$$

We consider the operator matrix

$$\mathscr{A}=\mathscr{T}+\mathscr{K}:=\begin{pmatrix} T_{m_1} & K_{12} & K_{13}\\ K_{21} & T_{m_2} & K_{23}\\ K_{31} & K_{32} & T_{m_3}\end{pmatrix}$$

where $\mathscr{T}$ is defined by

$$\mathscr{T}\begin{pmatrix} f\\ g\\ h\end{pmatrix} = \begin{pmatrix} -v\frac{\partial f}{\partial x}-\sigma_1(v)f & 0 & 0\\ 0 & -v\frac{\partial g}{\partial x}-\sigma_2(v)g & 0\\ 0 & 0 & -v\frac{\partial h}{\partial x}-\sigma_3(v)h\end{pmatrix}$$
$$:= \begin{pmatrix} T_{m_1} & 0 & 0\\ 0 & T_{m_2} & 0\\ 0 & 0 & T_{m_3}\end{pmatrix}\begin{pmatrix} f\\ g\\ h\end{pmatrix}.$$

For each operator T_{m_i}, $i=1,2,3$, is called streaming operator in X_1, defined by

$$T_{m_i}\varphi(x,v)=-v\frac{\partial\varphi}{\partial x}(x,v)-\sigma_i(v)\varphi(x,v),\ \varphi\in\mathscr{W}_1,$$

with $\mathscr{W}_1$ is the partial Sobolev space $\mathscr{W}_1:=\{\varphi\in X_1$ such that $v\frac{\partial\varphi}{\partial x}\in X_1\}$, and K is defined by

$$\mathscr{K}=\begin{pmatrix} 0 & K_{12} & K_{13}\\ K_{21} & 0 & K_{23}\\ K_{31} & K_{32} & 0\end{pmatrix}$$

where K_{ij}, $i,j=1,2,3$ $i\neq j$, are bounded linear operators in X_1, defined by

$$K_{ij}u(x,v)=\int_{-1}^{1}k_{ij}(x,v,v')u(x,v')dv',\ \ u\in X_1; \tag{3.1}$$

the kernels $k_{ij}:(a,a)\times(-1,1)\times(-1,1)\to\mathbb{R}$ are assumed to be measurable.

We consider the boundary spaces

$$\begin{aligned} X_1 &= L_1([-a,a]\times[-1,1];dxdv), a>0,\\ X_1^i &= L_1(\{-a\}\times[-1,0];|v|dv)\times L_1(\{a\}\times[0,1];|v|dv).\end{aligned}$$

Let $\lambda_j^* := \inf_{v\in[-1,1]}\sigma_j(v),\ j=1,2,3.$ We define the non diagonal domain of $\mathscr{A}$ by

$$\mathscr{D}(\mathscr{A})=\left\{\begin{pmatrix} f\\ g\\ h\end{pmatrix}\in\mathscr{W}_1\times\mathscr{W}_1 \text{ such that } \begin{pmatrix} f\\ g\\ h\end{pmatrix}^i = H\begin{pmatrix} f\\ g\\ h\end{pmatrix}^o\right\},$$

where $\begin{pmatrix} f \\ g \\ h \end{pmatrix}^o$ and $\begin{pmatrix} f \\ g \\ h \end{pmatrix}^i$ represent respectively the outgoing and the incoming fluxes related by the boundary operator $H = \begin{pmatrix} 0 & H_{12} & H_{13} \\ H_{21} & 0 & H_{23} \\ H_{31} & H_{32} & 0 \end{pmatrix}$.

Furthermore, by the identification with the last section, we introduce the boundary operators ϕ_1, ϕ_2 and ϕ_3

$$\begin{cases} \phi_1 : X_1 \longrightarrow X_1^i \\ \qquad f \longmapsto f^i = H_{12}g = H_{13}h \end{cases}, \qquad \begin{cases} \phi_2 : X_1 \longrightarrow X_1^i \\ \qquad g \longmapsto g^i = H_{21}f = H_{23}h \end{cases}$$

and

$$\begin{cases} \phi_3 : X_1 \longrightarrow X_1^i \\ \qquad h \longmapsto h^i = H_{31}f = H_{32}g \end{cases}$$

and its ψ_i, $i = 1,2,3,4,5,6$ as allows

$$\begin{cases} \psi_1 : X_1 \longrightarrow X_1^i \\ \qquad f \longmapsto H_{21}f^o \end{cases} \text{and,} \quad \begin{cases} \psi_2 : X_1 \longrightarrow X_1^i \\ \qquad g \longmapsto H_{12}g^o, \end{cases} \quad \begin{cases} \psi_3 : X_1 \longrightarrow X_1^i \\ \qquad h \longmapsto H_{13}h^o \end{cases}$$

$$\begin{cases} \psi_4 : X_1 \longrightarrow X_1^i \\ \qquad f \longmapsto H_{31}f^o, \end{cases} \quad \begin{cases} \psi_5 : X_1 \longrightarrow X_1^i \\ \qquad g \longmapsto H_{32}g^o \end{cases} \text{and} \quad \begin{cases} \psi_6 : X_1 \longrightarrow X_1^i \\ \qquad h \longmapsto H_{23}h^o. \end{cases}$$

Let T_i $i = 1,2,3$ be the closed operator defined by $\mathscr{D}(T_i) := \{u \in \mathscr{W}_1 \text{ such that } u^i = 0\}$ and $T_i := T_{m_i}|_{\ker \phi_i}$. Hence, we obtain the operator matrices $\mathscr{A}_0 = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}$ on $X_1 \times X_1 \times X_1$ with diagonal domain $\mathscr{D}(\mathscr{A}_0) = \mathscr{D}(T_1) \times \mathscr{D}(T_2) \times \mathscr{D}(T_3)$.

In order to express the operators K_λ, L_λ, M_λ, N_λ, P_λ and Q_λ, we will determine the solution of the equation

$$(T_{m_i} - \lambda)\varphi_i = 0, \ \text{ for } \varphi_i \in \mathscr{W}_p \text{ and } i = 1,2. \tag{3.2}$$

LEMMA 3.1. *Let $\lambda \in \rho(T_1) \cap \rho(T_2) \cap \rho(T_3)$, the solution of equation (3.2) is given by*

$$\varphi_i(x,v) = \chi_{[-1,0[}(v)\varphi_i(a,v)e^{-\frac{(\lambda+\sigma_i(v))}{|v|}|a-x|} + \chi_{]0,1]}(v)\varphi_i(-a,v)e^{-\frac{(\lambda+\sigma_i(v))}{|v|}|a+x|}, \ \textit{for } i = 1,2.$$

Furthermore, the expression of the bounded operators K_λ, L_λ, M_λ, N_λ, P_λ, Q_λ are:

$$\begin{cases} K_\lambda : \mathscr{W}_1 \longrightarrow \ker(T_{m_1} - \lambda) \\ \qquad g \longmapsto K_\lambda g(x,v) = \chi_{(-1,0)}(v)H_{12}g(-a,v)e^{-\frac{(\lambda+\sigma_1(v))}{|v|}|a-x|} \\ \qquad\qquad\qquad + \chi_{(0,1)}(v)H_{12}g(a,v)e^{-\frac{(\lambda+\sigma_1(v))}{|v|}|x+a|} \end{cases}$$

$$\begin{cases} L_\lambda : \mathscr{W}_1 \longrightarrow \ker(T_{m_2}-\lambda) \\ \qquad f \longmapsto L_\lambda f(x,v) = \chi_{(-1,0)}(v)H_{21}f(-a,v)e^{-\frac{(\lambda+\sigma_2(v))}{|v|}|a-x|} \\ \qquad\qquad + \chi_{(0,1)}(v)H_{21}f(a,v)e^{-\frac{(\lambda+\sigma_2(v))}{|v|}|x+a|} \end{cases}$$

$$\begin{cases} M_\lambda : \mathscr{W}_1 \longrightarrow \ker(T_{m_1}-\lambda) \\ \qquad h \longmapsto M_\lambda h(x,v) = \chi_{(-1,0)}(v)H_{13}h(-a,v)e^{-\frac{(\lambda+\sigma_1(v))}{|v|}|a-x|} \\ \qquad\qquad + \chi_{(0,1)}(v)H_{13}h(a,v)e^{-\frac{(\lambda+\sigma_1(v))}{|v|}|x+a|} \end{cases}$$

$$\begin{cases} N_\lambda : \mathscr{W}_1 \longrightarrow \ker(T_{m_3}-\lambda) \\ \qquad f \longmapsto N_\lambda f(x,v) = \chi_{(-1,0)}(v)H_{31}f(-a,v)e^{-\frac{(\lambda+\sigma_3(v))}{|v|}|a-x|} \\ \qquad\qquad + \chi_{(0,1)}(v)H_{31}f(a,v)e^{-\frac{(\lambda+\sigma_3(v))}{|v|}|x+a|} \end{cases}$$

$$\begin{cases} P_\lambda : \mathscr{W}_1 \longrightarrow \ker(T_{m_3}-\lambda) \\ \qquad g \longmapsto P_\lambda g(x,v) = \chi_{(-1,0)}(v)H_{32}g(-a,v)e^{-\frac{(\lambda+\sigma_3(v))}{|v|}|a-x|} \\ \qquad\qquad + \chi_{(0,1)}(v)H_{32}g(a,v)e^{-\frac{(\lambda+\sigma_3(v))}{|v|}|x+a|} \end{cases}$$

and

$$\begin{cases} Q_\lambda : \mathscr{W}_1 \longrightarrow \ker(T_{m_2}-\lambda) \\ \qquad h \longmapsto Q_\lambda h(x,v) = \chi_{(-1,0)}(v)H_{23}h(-a,v)e^{-\frac{(\lambda+\sigma_2(v))}{|v|}|a-x|} \\ \qquad\qquad + \chi_{(0,1)}(v)H_{23}h(a,v)e^{-\frac{(\lambda+\sigma_2(v))}{|v|}|x+a|}. \end{cases}$$

Proof. The proof of the above lemma is the same proof in case 2×2 in the article [6] □

In the sequel, our object is to determine the essential spectra of the transport operator $(\mathscr{A}-\lambda)$, it is enough to prove that the operator matrices obtain by the difference between the resolvent as defined in Theorem 2.8 is weakly compact on $X_1\times X_1\times X_1$.

Moreover, by an identification

$$\begin{aligned}
\widetilde{F}_1(\lambda) &= -\mathrm{L}_\lambda + (T_2-\lambda)^{-1}K_{21}, \\
\widetilde{F}_2(\lambda) &= -\mathrm{N}_\lambda + (T_3-\lambda)^{-1}K_{31}, \\
\widetilde{G}_1(\lambda) &= -\mathrm{M}_\lambda + (T_1-\lambda)^{-1}K_{12}, \\
\widetilde{G}_2(\lambda) &= -\mathrm{K}_\lambda + (T_1-\lambda)^{-1}K_{13}, \\
\widetilde{S}_1(\lambda) &= Id - \widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda), \\
\widetilde{F}_3(\lambda) &= [\widetilde{\theta}_1(\lambda) - \widetilde{F}_2(\lambda)\widetilde{G}_1(\lambda)]\widetilde{S}_1^{-1}(\lambda), \text{ with } \widetilde{\theta}_1(\lambda) = -\mathrm{P}_\lambda + (T_3-\lambda)^{-1}K_{32}. \\
\widetilde{G}_3(\lambda) &= \widetilde{S}_1^{-1}(\lambda)[\widetilde{\theta}_2(\lambda) - \widetilde{F}_1(\lambda)\widetilde{G}_2(\lambda)], \text{ with } \widetilde{\theta}_2(\lambda) = -\mathrm{Q}_\lambda + (T_2-\lambda)^{-1}K_{23}. \\
\widetilde{S}_2(\lambda) &= I - \widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda) - \widetilde{F}_3(\lambda)\widetilde{S}_1(\lambda)\widetilde{G}_3(\lambda).
\end{aligned}$$

we obtain

$$\mathbb{M}(\lambda) = \begin{pmatrix} I_1 & R_2 & R_3 \\ R_4 & I_2 & R_6 \\ R_7 & R_8 & I_3 \end{pmatrix}$$

where

$$\begin{aligned}
I_1 &= [\widetilde{G}_1(\lambda)\widetilde{S}_1^{-1}(\lambda)\widetilde{F}_1(\lambda) + [\widetilde{G}_1(\lambda)\widetilde{G}_3(\lambda) - \widetilde{G}_2(\lambda)]\widetilde{S}_2^{-1}(\lambda)[\widetilde{F}_1(\lambda)\widetilde{F}_3(\lambda) - \widetilde{F}_2(\lambda)]] \\
&\quad \times \mathscr{R}_\lambda(T_1), \\
R_2 &= [-\widetilde{G}_1(\lambda)\widetilde{S}_1^{-1}(\lambda) - [\widetilde{G}_1(\lambda)\widetilde{G}_3(\lambda) - \widetilde{G}_2(\lambda)]\widetilde{S}_2^{-1}(\lambda)\widetilde{F}_3(\lambda)]\mathscr{R}_\lambda(T_2), \\
R_3 &= [\widetilde{G}_1(\lambda)\widetilde{G}_3(\lambda) - \widetilde{G}_2(\lambda)]\widetilde{S}_2^{-1}(\lambda)\mathscr{R}_\lambda(T_3), \\
R_4 &= [-\widetilde{S}_1^{-1}(\lambda)\widetilde{F}_1(\lambda) - \widetilde{G}_3(\lambda)[\widetilde{S}_2^{-1}(\lambda)(\widetilde{F}_1(\lambda)\widetilde{F}_3(\lambda) - \widetilde{F}_2(\lambda))]]\mathscr{R}_\lambda(T_1), \\
I_2 &= [\widetilde{S}_1^{-1}(\lambda)\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda) + \widetilde{G}_3(\lambda)\widetilde{S}_2^{-1}(\lambda)\widetilde{F}_3(\lambda)]\mathscr{R}_\lambda(T_2), \\
R_6 &= -\widetilde{G}_3(\lambda)\widetilde{S}_2^{-1}(\lambda)\mathscr{R}_\lambda(T_3), \\
R_7 &= [\widetilde{S}_2^{-1}(\lambda)(\widetilde{F}_1(\lambda)\widetilde{F}_3(\lambda) - \widetilde{F}_2(\lambda))]\mathscr{R}_\lambda(T_1), \\
R_8 &= -\widetilde{S}_2^{-1}(\lambda)\widetilde{F}_3(\lambda)\mathscr{R}_\lambda(T_2), \\
I_3 &= [\widetilde{S}_2^{-1}(\lambda)[\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda) - [\widetilde{\theta}_1(\lambda) - \widetilde{F}_2(\lambda)\widetilde{G}_1(\lambda)]\widetilde{S}_1^{-1}(\lambda)(\widetilde{\theta}_2(\lambda) - \widetilde{F}_1(\lambda)\widetilde{G}_2(\lambda))]] \\
&\quad \times \mathscr{R}_\lambda(T_3).
\end{aligned}$$

So, to prove that the operator $\mathbb{M}_\lambda$ is weakly compact on $X_1 \times X_1 \times X_1$, it is suffix to prove that the operators $\widetilde{F}_i(\lambda)$ and $\widetilde{G}_i(\lambda)$, $i = 1,2,3$ are weakly compact.

DEFINITION 3.2. [24] A collision operator K_{ij} in the form (3.1), is said to be regular if it satisfies the following conditions:

$$\begin{cases} -\text{the function } K_{ij}(.) \text{ is mesurable,} \\ -\text{there exists a compact subset } \mathscr{C} \subset \mathscr{L}(L_1([-1,1],dv)) \text{ such that :} \\ \;K_{ij}(x) \in \mathscr{C} \text{ a.e on } [-a,a], \\ -K_{ij}(x) \in \mathscr{K}(L_1([-1,1],dv)) \text{ a.e on } [-a,a] \end{cases}$$

where $\mathscr{K}(L_1([-1,1],dv))$ is the set of compact operators on $L_1([-1,1],dv)$.

It follows from [25, Lemma 4.3] the following result:

LEMMA 3.3. *Let* $\lambda \in \rho(T_1) \cap \rho(T_2) \cap \rho(T_3)$.

(*i*) *If* K_{12}, K_{13}, K_{21} *and* K_{31} *are regular operators, and if* H_{12}, H_{13}, H_{21}, $H_{31} \in \mathscr{W}(X_1)$, *then for any* $\lambda \in \mathbb{C}$ *satisfying* $Re\lambda > -\lambda_1^*$ *the operators* $\widetilde{G}_1(\lambda)$, $\widetilde{G}_2(\lambda)$, $\widetilde{F}_1(\lambda)$ *and* $\widetilde{F}_2(\lambda)$, *respectively, are weakly compact in* X_1.

(*ii*) *If* K_{32} *and* K_{23} *are regular operators, and if* H_{23} *and* H_{32} *are in* $\mathscr{W}(X_1)$, *then for any* $\lambda \in \mathbb{C}$ *satisfying* $Re\lambda > -\lambda_1^*$ *the operators* $\widetilde{\theta}_1(\lambda)$ *and* $\widetilde{\theta}_2(\lambda)$, *respectively, are weakly compact in* X_1. *And therefore* $\widetilde{F}_3(\lambda)$ *and* $\widetilde{G}_3(\lambda)$ *are weakly compact in* X_1.

Proof. $(i)+(ii)$ We deduce from $H_{ij}\in\mathscr{W}(X_1)$ for $i,j=1,2,3$ that the operator K_λ (resp. L_λ, M_λ, N_λ, P_λ, Q_λ) is weakly compact on X_1. Following Lemma 4.2 in [25], one has that $(T_i-\lambda)^{-1}K_{jk}$ with $i,j,k=1,2,3$ and $j\neq k$ is weakly compact on X_1.

So, the fact that the set $\mathscr{W}(X_1)$ is a closed two sided ideal of $\mathscr{L}(X_1)$, allows us to conclude the desired results. □

REMARK 3.4. For the remainder, we observe that if H_{13} is weakly compact on X_1 (resp. H_{13}), K_{12} defines a regular operator (resp. H_{12}), then $(Id-\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda))\in W(X_1)$ (resp. $(Id-\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda))\in\mathscr{W}(X_1)$). Hence, one has $[\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda)]^2\in\mathscr{W}(X_1)$ (resp. $[\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda)]^2\in\mathscr{W}(X_1)$), we deduce that $\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda)\in\mathscr{PK}(X)$ (resp. $\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda)\in\mathscr{PK}(X)$).

Taking account from the last item and Theorem 2.2 in [19] we infer that the following properties are equivalent:

(1) $1\in\rho(\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda))$.

(2) $Id-\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda)$ is invertible.

(3) $Id-\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda)$ is injective.

and

(a) $1\in\rho(\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda))$.

(b) $Id-\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda)$ is invertible.

(c) $Id-\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda)$ is injective.

The following proposition makes precise the injectivity properties.

PROPOSITION 3.5. *Let $\lambda\in\rho(T_1)\cap\rho(T_2)\cap\rho(T_3)$, then the operators*

$$Id-\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda)\quad \textit{and}\quad Id-\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda)$$

are injective.

Proof. Let $\lambda\in\rho(T_1)\cap\rho(T_2)\cap\rho(T_3)$, and $h\in\ker(Id-\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda))$. Then we will solve the following equation:

$$(Id-\widetilde{F}_1(\lambda)\widetilde{G}_1(\lambda))h=0.$$

The explicit expression of $\widetilde{F}_1(\lambda)$ and $\widetilde{G}_1(\lambda)$ and their properties yield that to solve the equation

$$\big(T_2-\lambda-k_{21}(T_1-\lambda)^{-1}k_{12}\big)h=0.$$

Since $\lambda\in\rho(T_1)\cap\rho(T_2)\cap\rho(T_3)$ and the use of Remark 3.1 in [21] assert that $\lambda\in\rho(T_1)\cap\rho(T_2)\cap\rho(T_3)\cap\rho\big(T_2-k_{21}(T_1-\lambda)^{-1}k_{12}\big)$. That is $T_2-\lambda-k_{21}(T_1-\lambda M_1)^{-1}k_{12}$ is injective and so $h=0$. Hence, this argument yields the injectivity of the desert operator.

A same reasoning allows us to reach the injectivity of $I-\widetilde{F}_2(\lambda)\widetilde{G}_2(\lambda)$. □

REMARK 3.6. By Remark 3.4 and Proposition 3.5 we show that $\widetilde{S}_1(\lambda)$ and $\widetilde{S}_2(\lambda)$ are invertible. Hence in addition with Lemma 2.5(ii), allows us to deduce that the matrix operator pencil $(\mathscr{A}_H - \lambda)$ is invertible with bounded inverse.

Now, we are in the position to establish the essential spectra of three-group transport operators matrix with non diagonal domain.

THEOREM 3.7. *For $\lambda \in \rho(T_1)\cap\rho(T_2)\cap\rho(T_3)$, if K_{ij} with $i,j=1,2,3$ and $i\neq j$ are non-negative regular operators on X_1, and H_{kv} with $v,k=1,2,3$ and $v\neq k$ strictly singular operators on X_1, then*

$$\begin{aligned}\sigma_J(\mathscr{A}) = \sigma_{e4}(\mathscr{A}) = \sigma_{e5}(\mathscr{A}) = \sigma_{eap}(\mathscr{A}) = \sigma_{e\delta}(\mathscr{A})\\ = \{\lambda \in \mathbb{C} \textit{ such that } Re\lambda \leqslant -\min(\lambda_1^*,\lambda_2^*,\lambda_3^*)\}.\end{aligned}$$

Proof. According to the previous Lemma, the hypothesis $\mathbb{M}_\lambda \in \mathscr{W}(X\times X\times X)$ is verified. The results of Theorem can be applied for the operator $\mathscr{A}$.

It is known (see. [25, Remark 4.3]), that the essential spectra of the operators T_{m_i}, $i=1,2,3$,

$$\sigma_{e4}(T_{m_i}) = \sigma_{e5}(T_{m_i}) = \sigma_{eap}(T_{m_i}) = \sigma_{e\delta}(T_{m_i}) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leqslant -\lambda_i^*\}.$$

Applying Theorems 2.8 and 2.9 we get

$$\sigma_{e4}(\mathscr{A}) = \sigma_{eap}(\mathscr{A}) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leqslant -\min(\lambda_1^*,\lambda_2^*,\lambda_3^*)\}.$$

The same reasoning implies the corresponding result for the essential spectra $\sigma_{e5}(\mathscr{A})$ and $\sigma_{e\delta}(\mathscr{A})$.

Using Theorem 7.2.1 in [13], it follows that

$$\sigma_J(\mathscr{A}) = \sigma_{e,5}(\mathscr{A}) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leqslant -\min(\lambda_1^*,\lambda_2^*,\lambda_3^*)\}. \quad \square$$

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