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GENERALIZATION OF WOLFF'S IDEAL THEOREM ON $H^{\infty}_{K(B)}(\mathbb{D})$

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Abstract. We consider an open question proposed in [2] when generalizing Wolff's Ideal Theorem on uniformly closed subalgebras of $H^{\infty}(\mathbb{D})$. In this paper, we are able to resolve the open question; in addition, we look at some cases where Wolff's Ideal Theorem holds without the additional condition of $F_0 \neq 0$.

1. Introduction

The famous Corona Theorem (1962), by Carleson [3], characterizes when a finitely generated ideal of $H^{\infty}(\mathbb{D})$ is all of $H^{\infty}(\mathbb{D})$. Let $\mathscr{I}$ be the ideal generated by a finite set of functions $\{f_i\}_{i=1}^{n} \subset H^{\infty}(\mathbb{D})$. Then $\mathscr{I}$ is the entire space $H^{\infty}(\mathbb{D})$ provided that there exists $\delta > 0$ that satisfies

$$\left(\sum_{i=1}^{n} |f_i(z)|^2\right)^{\frac{1}{2}} \geqslant \delta \text{ for all } \quad z \in \mathbb{D}. \tag{1}$$

Wolff [18] later attempted to generalize the corona theorem by replacing 1 by any arbitrary H^{∞} function as follows: Does the condition

$$\left(\sum_{i=1}^{n} |f_i(z)|^2\right)^{\frac{1}{2}} \geqslant |h(z)| \text{ for all } \quad z \in \mathbb{D} \text{ and } \; f_j, h \in H^{\infty}(\mathbb{D}) \tag{2}$$

imply $h^p \in \mathscr{I}$? Rao provided an example (see Garnett [5], p. 369, Ex-3) to show that the condition (2) is not sufficient for $p = 1$, and Treil showed that the condition (2) is not sufficient for $p = 2$ as well [16]. Wolff proved the following theorem for all $p \geqslant 3$.

THEOREM 1.1. (Wolff's Theorem) *If* $f_j \in H^{\infty}(\mathbb{D})$, $j = 1,2,\dots,n$, $h \in H^{\infty}(\mathbb{D})$ *and*

$$\left(\sum_{j=1}^{n} |f_j(z)|^2\right)^{\frac{1}{2}} \geqslant |h(z)| \quad \textit{for all} \quad z \in \mathbb{D},$$

then

$$h^p \in \mathscr{I}(\{f_j\}_{j=1}^{n}), \quad p \geqslant 3.$$

(See Garnett [5], *p. 319, Theorem 2.3.)*

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The Corona Theorem and Wolff's Theorem have been extended to various subalgebras of $H^\infty(\mathbb{D})$, for example the subalgebra $\mathbb{C}+BH^\infty(\mathbb{D})$ given by

$$\mathbb{C}+BH^\infty(\mathbb{D}) = \{\alpha + Bg : \alpha \in \mathbb{C}, g \in H^\infty(\mathbb{D})\},$$

where B is a fixed Blaschke product. In [7], Mortini, Sasane, and Wick proved the Corona Theorem for a finite number of generators, whereas the infinite version is due to Ryle and Trent [12, 13]. A similar result was obtained in [1] for the subalgebra $H^\infty_{\mathbb{I}}(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f = c+\phi,\ c \in \mathbb{C} \text{ and } \phi \in \mathbb{I}\}$, where $\mathbb{I}$ is a proper ideal of $H^\infty(\mathbb{D})$.

We will use f and f_i to represent complex-valued scalar functions, and F to denote a vector-valued function. For $\{f_j\}_{j=1}^\infty \subset H^\infty(\mathbb{D})$, if we let $F(z) = (f_1(z), f_2(z), \ldots)$, we will use $F(z)^*$ to denote the adjoint of $F(z)$. We use $H^\infty_{l^2}(\mathbb{D})$ to denote the Hilbert space of bounded analytic functions that map $\mathbb{D}$ to l^2. That is, an element $F \in H^\infty_{l^2}(\mathbb{D})$ is an infinite-dimensional row vector whose entries consist of functions $f_i \in H^\infty(\mathbb{D})$ such that

$$\|F\|_\infty^2 = \sum_{i=1}^\infty \sup_{z\in\mathbb{D}} |f_i(z)|^2 < \infty.$$

Similarly, we use the notation $H^\infty_n(\mathbb{D})$ for

$$H^\infty_n(\mathbb{D}) = \{\{f_j\}_{j=1}^n : f_j \in H^\infty(\mathbb{D}) \text{ for } j = 1,2,\ldots,n\}.$$

The subalgebra, $H^\infty_{K(B)}(\mathbb{D})$, we consider for this paper is as follows: for a (finite) subset $K = \{k_1, k_2, \ldots, k_p\}$ of $\mathbb{Z}_+$, define

$$H^\infty_K(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f^{(j)}(0) = 0 \text{ for all } j \in K\},$$

where $f^{(j)}$ is the j^{th} derivative of f.

We consider those sets K for which $H^\infty_K(\mathbb{D})$ is a subalgebra of $H^\infty(\mathbb{D})$ under the usual product of functions.

Next, fix a Blashke product B and define

$$H^\infty_{K(B)}(\mathbb{D}) = \left\{ \sum_{j\notin K,\, 0\leqslant j<k_p} a_j B^j + B^{k_p+1} g : g \in H^\infty(\mathbb{D}) \text{ and } a_j \in \mathbb{C} \right\}.$$

Later on we will see that $H^\infty_{K(B)}(\mathbb{D})$ is an algebra when $H^\infty_K(\mathbb{D})$ is.

We define algebras comprised of vectors with entries in $H^\infty_{K(B)}(\mathbb{D})$ as follows:

$$\mathscr{H}^\infty_{K(B),n}(\mathbb{D}) = \{\{f_j\}_{j=1}^n : f_j \in H^\infty_{K(B)}(\mathbb{D}) \text{ for } j = 1,2,\ldots,n \text{ and } \sup_{z\in\mathbb{D}} \sum_{j=1}^n |f_j(z)|^2 < \infty\}.$$

Multiplication here is entrywise, and n can be either a positive integer or ∞. We write the elements of $\mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ as row vectors F and denoted as

$$F(z) = \sum_{\substack{j\notin K \\ 0\leqslant j<k_p}} B^j(z) F_j + B^{k_p+1}(z) F_{k_p+1}(z),$$

where $F_j's$, $0 \leqslant j < k_p$, are constants and $F_{k_p+1}(z) \in H_n^\infty(\mathbb{D})$. Notice that the constant term is no longer $F(0)$. In fact, the constant term is $F_0 = F(c)$, where c is a zero of the Blashke product B. These algebras were introduced and function problems were considered in J. Solazzo [14], M. Ragupathi [9], and Davidson, Paulsen, Ragupathi, and Singh [4]. Note that the algebra $\mathbb{C}+BH^\infty(\mathbb{D})$ is a special case of the type of algebra $H^\infty_{K(B)}(\mathbb{D})$, where the set K is just the empty set.

Just as in the H^∞ case, condition (2) is not sufficient to guarantee ideal membership of the function h in these algebras, as can be shown by simple modification of Rao's counterexample [11]. While extending Wolff's Theorem to various subalgebras of $H^\infty(\mathbb{D})$, the authors in [2] have proved the following theorem with an additional condition that $F_0 \neq \mathbf{0}$ as follows:

THEOREM 1.2. (Banjade-Holloway-Trent) *Suppose* $F(z) = (f_1(z), f_2(z), \ldots) \in \mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ *and let* $h(z) \in H^\infty_{K(B)}(\mathbb{D})$, *with* $1 \geqslant [F(z)F(z)^*]^{\frac{1}{2}} \geqslant |h(z)|\ \forall\, z \in \mathbb{D}$. *Suppose also that* $F_0 \neq \mathbf{0}$. *Then there exists* $V(z) = (v_1(z), v_2(z), \ldots) \in \mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ *such that*

$$F(z)V(z)^T = h^3(z)\ \forall\, z \in \mathbb{D}.$$

The extra condition $F_0 \neq \mathbf{0}$ was required in Theorem 1.2 whereas this condition was not necessary for $H^\infty(\mathbb{D})$ or even for $\mathbb{C}+BH^\infty(\mathbb{D})$. In [2], the authors proposed the question if Wolff's Theorem can be fully extended to the subalgebra $H^\infty_{K(B)}(\mathbb{D})$ without the additional assumption that $F_0 \neq \mathbf{0}$. The main purpose of this paper is to provide the answer to that question. We will prove that the answer is negative. In addition, we will also discuss subalgebras on which the assumption $F_0 \neq \mathbf{0}$ may be removed and still establish Wolff's Theorem. Before presenting our main result, we first provide some basic properties of $H^\infty_{K(B)}(\mathbb{D})$ in the following section.

2. The subalgebra $H^\infty_{K(B)}(\mathbb{D})$

It is clear that $H^\infty_K(\mathbb{D})$ is not an algebra for every subset K of $\mathbb{Z}_+$; for example when $K = \{2\}$, the set $H^\infty_K(\mathbb{D})$ is not an algebra.

Though there is not a complete characterization of the set K for which $H^\infty_K(\mathbb{D})$ is an algebra, Ryle and Trent [12] have given certain criteria that the set K must meet.

LEMMA 2.1. (Ryle-Trent) *Let* $K \subseteq \mathbb{N}$ *such that* $H^\infty_K(\mathbb{D})$ *is an algebra. Then*

(i) $k_0 \notin K$ *if and only if* $z^{k_0} \in H^\infty_K(\mathbb{D})$.

(ii) If $j, k \notin K$, *then* $j+k \notin K$.

(iii) Suppose $k_0 \in K$. *If* $1 < j < k_0$ *satisfies* $j \notin K$, *then* $k_0 - j \in K$.

LEMMA 2.2. *Let $K \subseteq \mathbb{Z}_+$. If $H_K^\infty(\mathbb{D})$ is an algebra, then $H_{K(B)}^\infty(\mathbb{D})$ is also an algebra.*

Proof. Let f and g be functions in $H_{K(B)}^\infty(\mathbb{D})$.

Then $f = \sum_{j \notin K,\, 0 \leqslant j < k_p} a_j B^j + B^{k_p+1} f_1$ and $g = \sum_{j \notin K,\, 0 \leqslant j < k_p} b_j B^j + B^{k_p+1} g_1$, for some $f_1, g_1 \in H^\infty(\mathbb{D})$. So

$$\begin{aligned} fg = & \sum_{i,j \notin K,\, 0 \leqslant i,j < k_p} \sum a_i b_j B^i B^j + \sum_{j \notin K,\, 0 \leqslant j < k_p} a_j B^j B^{k_p+1} g_1 \\ & + \sum_{j \notin K,\, 0 \leqslant j < k_p} b_j B^j B^{k_p+1} f_1 + B^{k_p+1} B^{k_p+1} f_1 g_1 \\ = & \sum_{i,j \notin K,\, 0 \leqslant i+j < k_p} \sum a_i b_j B^{i+j} + \sum_{i,j \notin K,\, i+j \geqslant k_p} \sum a_i b_j B^{i+j} \\ & + \sum_{j \notin K,\, 0 \leqslant j < k_p} a_j B^j B^{k_p+1} g_1 + \sum_{j \notin K,\, 0 \leqslant j < k_p} b_j B^j B^{k_p+1} f_1 + B^{k_p+1} B^{k_p+1} f_1 g_1. \end{aligned}$$

However, since $k_p \in K$, we have $i + j \neq k_p$ whenever $i, j \notin K$. Thus, this gives us:

$$\begin{aligned} fg = & \sum_{i,j \notin K,\ 0 \leqslant i+j < k_p} \sum a_i b_j B^{i+j} + \sum_{i,j \notin K,\ i+j \geqslant k_p+1} \sum a_i b_j B^{i+j} \\ & + \sum_{j \notin K,\, 0 \leqslant j < k_p} a_j B^j B^{k_p+1} g_1 + \sum_{j \notin K,\, 0 \leqslant j < k_p} b_j B^j B^{k_p+1} f_1 + B^{k_p+1} B^{k_p+1} f_1 g_1 \\ = & \sum_{i,j \notin K,\ 0 \leqslant i+j < k_p} \sum a_i b_j B^{i+j} + B^{k_p+1} \left[\sum_{i,j \notin K,\ i+j \geqslant k_p} \sum a_i b_j B^{i+j-(k_p+1)} \right] \\ & + B^{k_p+1} \left[\sum_{j \notin K,\, 0 \leqslant j < k_p} a_j B^j g_1 + \sum_{j \notin K,\, 0 \leqslant j < k_p} b_j B^j f_1 + B^{k_p+1} f_1 g_1 \right] \\ = & \sum_{i,j \notin K,\ 0 \leqslant i+j < k_p} \sum a_i b_j B^{i+j} + B^{k_p+1} h, \end{aligned}$$

where

$$\begin{aligned} h = & \sum_{i,j \notin K,\ i+j \geqslant k_p+1} \sum a_i b_j B^{i+j-(k_p+1)} + \sum_{j \notin K,\, 0 \leqslant j < k_p} a_j B^j g_1 \\ & + \sum_{j \notin K,\, 0 \leqslant j < k_p} b_j B^j f_1 + B^{k_p+1} f_1 g_1 \in H^\infty(\mathbb{D}). \end{aligned}$$

Moreover, $i + j \notin K$ whenever $i, j \notin K$. Therefore, $fg \in H_{K(B)}^\infty(\mathbb{D})$. $\square$

Now, we are ready to present our main results.

3. Main results

PROPOSITION 3.1. *Let* $F=(f_1,f_2,\ldots)\in\mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ *and* $h\in H^\infty_{K(B)}(\mathbb{D})$ *with* $1\geqslant\sqrt{F(z)F(z)^*}\geqslant|h(z)|\ \forall\, z\in\mathbb{D}$. *If* $F_0=\mathbf{0}$ *then the existence of* $V=(v_1,v_2,\ldots)\in\mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ *such that*

$$F(z)V(z)^T=h^3(z)\ \forall\, z\in\mathbb{D}$$

cannot be guaranteed.

Proof. We proceed by counterexample. If we consider the set $K=\{1,2,3,6,7,11\}$, then there exists $f,h\in H^\infty_{K(B)}(\mathbb{D})$ such that

$$|h(z)|\leqslant|f(z)| \text{ for all } z\in\mathbb{D}.$$

But, as we see below there is not necessarily a g in $H^\infty_{K(B)}(\mathbb{D})$ such that $h^3=fg$.

We first show that $H^\infty_{K(B)}(\mathbb{D})$ is an algebra. For this, by Lemma 2.2, it is enough to show that $H^\infty_K(\mathbb{D})$ is an algebra. We know that

$$(fg)^{(k)}(0)=\sum_{j=0}^{k}\binom{k}{j}f^{(j)}(0)g^{(k-j)}(0).$$

For any $f,g\in H^\infty_K(\mathbb{D})$, if $j\in K$, then $f^{(j)}(0)=0$. Otherwise, we can see in the set K that $k-j\in K$ and so $g^{(k-j)}(0)=0$. Hence, $fg\in H^\infty_K(\mathbb{D})$. That means, $H^\infty_K(\mathbb{D})$ is an algebra and so is $H^\infty_{K(B)}(\mathbb{D})$.

Also, any element f in $H^\infty_{K(B)}(\mathbb{D})$ is of the form $f(z)=c_0+c_4B^4+c_5B^5+c_8B^8+c_9B^9+c_{10}B^{10}+B^{12}\phi$, where $\phi\in H^\infty(\mathbb{D})$ and $c_i\in\mathbb{C}$.

If we take $h(z)=B^4$ and $f(z)=2B^4+B^5$, then we see that $h,f\in H^\infty_{K(B)}(\mathbb{D})$.

Also,

$$|2B^4+B^5|=|B^4||2+B|\geqslant|B^4|(2-|B|)\geqslant|B^4|.$$

That means, $|h(z)|\leqslant|f(z)|$ for all $z\in\mathbb{D}$ [notice here that $c_0=0$].

If there was a $g\in H^\infty_{K(B)}(\mathbb{D})$ such that $h^3=fg$, then we would have

$$B^{12}=(2B^4+B^5)g=B^4(2+B)g.$$

This implies that g must satisfy $(2+B)g=B^8$. Also, we know that g is of the form $g(z)=d_0+d_4B^4+d_5B^5+d_8B^8+d_9B^9+d_{10}B^{10}+B^{12}\psi$, where $\psi\in H^\infty(\mathbb{D})$ and $d_i\in\mathbb{C}$.

That means,

$$\begin{aligned}B^8=(2+B)(&d_0+d_4B^4+d_5B^5+d_8B^8+d_9B^9\\&+d_{10}B^{10}+B^{12}\psi).\end{aligned}$$

Multiplying and collecting terms we get:

$$\begin{aligned}B^8=\;&2d_0+d_0B+2d_4B^4+(2d_5+d_4)B^5+d_5B^6+2d_8B^8\\&+(2d_9+d_8)B^9+(2d_{10}+d_9)B^{10}+d_{10}B^{11}+(2B^{12}+B^{13})\psi.\end{aligned}$$

This is true for all $z \in \mathbb{D}$. Hence, each coefficient of the right side should equal zero except for the coefficient of B^8 which equals 1. That is, $0 = d_0 = d_4 = 2d_5 + d_4 = d_5 = \ldots$. What are relevant for our discussion are that $2d_8 = 1$, i.e., $d_8 = 1/2$, and $2d_9 + d_8 = 2d_{10} + d_9 = d_{10} = 0$ which implies that $d_{10} = 0 \Rightarrow d_9 = 0 \Rightarrow d_8 = 0$. This creates a contradiction. Thus there is no g in $H^\infty_{K(B)}$ such that $h^3 = fg$. $\square$

This proves that the answer to the question asked in [2] is no. However, there are some algebras for which Theorem 1.2 holds true without the additional condition that $F_0 \neq \mathbf{0}$.

We saw in the proposition that Wolff's theorem does not necessarily hold if the constant term F_0 is equal to $\mathbf{0}$ in $F(z) = \sum_{\substack{j \notin K \\ 0 \leqslant j < k_p}} B^j(z) F_j + B^{k_p+1}(z) F_{k_p+1}(z)$. However, in [2] (see the second part of the proof of Theorem 1.1), they made the observation that if $F_j = \mathbf{0}$ for all $j \in K$, then the result holds true. We can demonstrate that with the following example.

EXAMPLE 3.2. Let $K = \{1,2,3\}$. We see that $H^\infty_K(\mathbb{D})$ is an algebra and so is $H^\infty_{K(B)}(\mathbb{D})$. Take

$$f(z) = B^4 g,\ g \in H^\infty(\mathbb{D}).$$

It's clear that $f \in H^\infty_{K(B)}(\mathbb{D})$. Also, $f_j = 0$ for all j. Let $h \in H^\infty_{K(B)}(\mathbb{D})$ such that $|h(z)| \leqslant |f(z)|$. We know that the general form of h looks like $h(z) = h_0 + B^4\, h_B(z)$, $h_B \in H^\infty(\mathbb{D})$. However, $|h(z)| \leqslant |f(z)|$ implies that $h_0 = 0$. This implies $|h_B(z)| \leqslant |g(z)|$. Hence, by Wolff's theorem there exists $u \in H^\infty(\mathbb{D})$ such that $g(z)u(z) = h_B^3(z)$. Thus,

$$h^3(z) = (B^4 h_B)^3 = (B^4 g(z))(B^8 u(z)) = f(z)v(z),$$

where $v(z) = B^4(B^4 u(z)) \in H^\infty_{K(B)}(\mathbb{D})$.

Note that, for the example above, one may take any set K of the first n natural numbers.

We can actually make an improvement on this result. If there exists $k_p/2 < j_1 < k_p$, $j_1 \notin K$ such that $F_j = 0$ for all $j < j_1$ then Wolff's theorem holds.

THEOREM 3.3. *Let* $F(z) = (f_1(z), f_2(z), \ldots) \in \mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ *and* $h(z) \in H^\infty_{K(B)}(\mathbb{D})$, *with* $1 \geqslant \sqrt{(F(z)F(z)^*)} \geqslant |h(z)|$ *for all* $z \in \mathbb{D}$. *Suppose there exists* $j_1 \notin K$, $k_p/2 < j_1 < k_p$ *such that* $F_j = 0$ *whenever* $j < j_1$. *Then there exists* $V(z) = (v_1(z), v_2(z), \ldots) \in \mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ *such that*

$$F(z)V(z)^T = h^3(z) \quad \textit{for all } z \in \mathbb{D}$$

and

$$\|V\|_\infty \leqslant \left(1 + \frac{1}{\|F(\alpha)\|_{l^2}}\right)\left(1 + 4\sqrt{e} + 8\sqrt{2}e + 72e^{\frac{3}{2}}\right),$$

where α *is a zero of* $B(z)$.

Proof. Let $h(z) = \sum_{\substack{j \notin K \\ 0 \leqslant j < k_p}} B^j(z) h_j + B^{k_p+1}(z) h_{k_p+1}(z)$, $h_j \in \mathbb{C}$, and $h_{k_p+1}(z) \in H^\infty(\mathbb{D})$.

Since $F_j = \mathbf{0}$ for each $j < j_1$,

$$\begin{aligned}
F(z) &= \sum_{\substack{j \notin K \\ j_1 \leqslant j < k_p}} B^j(z) F_j + B^{k_p+1}(z) F_{k_p+1}(z) \\
&= B^{j_1}(z) \left[\sum_{\substack{j \notin K \\ j_1 \leqslant j < k_p}} B^{j-j_1}(z) F_j + B^{k_p+1-j_1}(z) F_{k_p+1}(z) \right] \\
&= B^{j_1}(z) \left[F_{j_1} + B(z) \left(\sum_{\substack{j \notin K \\ j_1 < j < k_p}} B^{j-j_1-1}(z) F_j + B^{k_p-j_1}(z) F_{k_p+1}(z) \right) \right],
\end{aligned}$$

$F_j \in \mathbb{C}^n$, $F_{k_p+1}(z) \in H_n^\infty(\mathbb{D})$.

That is,

$$F(z) = B^{j_1}(z) \left[F_{j_1} + B(z) \Phi(z) \right],$$

where $\Phi(z) = \sum_{\substack{j \notin K \\ j_1 < j < k_p}} B^{j-j_1-1}(z) F_j + B^{k_p-j_1}(z) F_{k_p+1}(z) \in H_n^\infty(\mathbb{D})$.

Thus we have

$$\sqrt{\left(B^{j_1}(z) \left[F_{j_1} + B(z)\Phi(z)\right]\right) \left(B^{j_1}(z) \left[F_{j_1} + B(z)\Phi(z)\right]\right)^\star} \geqslant |h(z)|$$

for all $z \in \mathbb{D}$.

That means,

$$|B(z)|^{j_1} \sqrt{\left(F_{j_1} + B(z)\Phi(z)\right) \left(F_{j_1} + B(z)\Phi(z)\right)^\star} \geqslant |h(z)|.$$

Implying $h_j = 0$ for all $j < j_1$, and this gives us

$$\begin{aligned}
&|B(z)|^{j_1} \sqrt{\left(F_{j_1} + B(z)\Phi(z)\right) \left(F_{j_1} + B(z)\Phi(z)\right)^\star} \\
&\geqslant \left| \sum_{\substack{j \notin K \\ j_1 \leqslant j < k_p}} B^j(z) h_j + B^{k_p+1}(z) h_{k_p+1}(z) \right| \\
&= \left| B^{j_1}(z) \right| \left| \sum_{\substack{j \notin K \\ j_1 \leqslant j < k_p}} B^{j-j_1}(z) h_j + B^{k_p+1-j_1}(z) h_{k_p+1}(z) \right|
\end{aligned}$$

$$= \left|B^{j_1}(z)\right| \left| h_{j_1} + B(z) \left[\sum_{\substack{j \notin K \\ j_1 < j < k_p}} B^{j-j_1-1}(z) h_j + B^{k_p - j_1}(z) h_{k_p+1}(z) \right] \right|.$$

This implies

$$\sqrt{\left(F_{j_1} + B(z)\Phi(z)\right)\left(F_{j_1} + B(z)\Phi(z)\right)^{\star}}$$
$$\geqslant \left| h_{j_1} + B(z) \left[\sum_{\substack{j \notin K \\ j_1 < j < k_p}} B^{j-j_1-1}(z) h_j + B^{k_p - j_1}(z) h_{k_p+1}(z) \right] \right|.$$

However, $h_{j_1} + B(z) \left[\sum_{\substack{j \notin K \\ j_1 < j < k_p}} B^{j-j_1-1}(z) h_j + B^{k_p - j_1}(z) h_{k_p+1}(z) \right] = h_{j_1} + B(z)\psi(z)$ is an element of $\mathbb{C} + BH^{\infty}(\mathbb{D})$, where $\psi(z) = \sum_{\substack{j \notin K \\ j_1 < j < k_p}} B^{j-j_1-1}(z) h_j + B^{k_p - j_1}(z) h_{k_p+1}(z)$. Also $F_{j_1} + B(z)\Phi(z) \in \mathbb{C}^n + BH_n^{\infty}(\mathbb{D})$.

By Wolff's Theorem for $\mathbb{C} + BH^{\infty}(\mathbb{D})$, there exists G in $\mathbb{C}^n + BH_n^{\infty}(\mathbb{D})$ such that

$$(F_{j_1} + B(z)\Phi(z))G(z)^T = (h_{j_1} + B(z)\psi(z))^3.$$

Also, from Theorem 1.1 of [2], we can see that

$$\|G\|_{\infty} \leqslant \left(1 + \frac{1}{\|F(\alpha)\|_{l^2}}\right)\left(1 + 4\sqrt{e} + 8\sqrt{2}e + 72e^{\frac{3}{2}}\right),$$

where α is a zero of $B(z)$.

Multiplying both sides by B^{3j_1} we get

$$B^{3j_1}(z)(h_{j_1} + B(z)\psi(z))^3 = B^{j_1}(z)(F_{j_1} + B(z)\Phi(z))(B^{2j_1}(z)G(z))^T.$$

Hence,

$$h^3(z) = \left[B^{j_1}(z)(h_{j_1} + B(z)\psi(z))\right]^3 = F(z)V(z)^T,$$

where $V(z) = B^{2j_1}(z)G(z) \in \mathscr{H}^{\infty}_{K(B),n}(\mathbb{D})$ as it can be expressed as

$$V(z) = B^{k_p+1}(z)B^{2j_1 - k_p - 1}(z)G(z) \in H^{\infty}_{K(B),n}(\mathbb{D}),$$

this is because $2j_1 > n_p$. Moreover, since $|B(z)| \leqslant 1$ on $\mathbb{D}$,

$$\|V\|_{\infty} \leqslant \|G\|_{\infty} \leqslant \left(1 + \frac{1}{\|F(\alpha)\|_{l^2}}\right)\left(1 + 4\sqrt{e} + 8\sqrt{2}e + 72e^{\frac{3}{2}}\right),$$

where α is a zero of $B(z)$. $\square$

There are other cases where the conclusions of Wolff's theorem hold true even when $F_0 = \mathbf{0}$. To see one of them, let $h \in H^\infty_{K(B)}(\mathbb{D})$ such that $[F(z)F(z)^*]^{\frac{1}{2}} \geqslant |h(z)| \; \forall z \in \mathbb{D}$. Let r be the smallest element of K such that $F_r \neq \mathbf{0}$. That means, $F(z)$ and $h(z)$ can be written as $F(z) = B^r F_H(z)$ and $h(z) = B^r H(z), F_H(z)$ and $H(z) \in H^\infty_{\mathscr{K}(B)}(\mathbb{D})$, where

$$\mathscr{K} = \{k - r : \; k \in K \text{ and } k - r > 0\}.$$

The following theorem, due to [2] provides us a case where the Wolff's theorem holds true even when $F_0 = \mathbf{0}$.

THEOREM 3.4. [2, Theorem 1.4] *Let $F = (f_1, f_2, \ldots) \in \mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ and $h \in H^\infty_{K(B)}(\mathbb{D})$, with $1 \geqslant \sqrt{F(z)F(z)^*} \geqslant |h(z)| \; \forall \, z \in \mathbb{D}$. If $H^\infty_{\mathscr{K}}(\mathbb{D})$ is a subalgebra of $H^\infty(\mathbb{D})$, then there exists $V = (v_1, v_2, \ldots) \in \mathscr{H}^\infty_{K(B),n}(\mathbb{D})$ such that*

$$F(z)V(z)^T = h^3(z) \; \forall \, z \in \mathbb{D}$$

EXAMPLE 3.5. We can demonstrate this theorem with the following example. Let $K = \{1, 3, 5, \ldots, 2n-1\}$. Then, $H^\infty_{K(B)}(\mathbb{D})$ is an algebra and the elements of $H^\infty_{K(B)}(\mathbb{D})$ are of the form

$$f = C_0 + C_2 B^2 + C_4 B^4 \cdots + C_{2n-2} B^{2n-2} + B^{2n} \Phi$$

If $f_0 = 0$, then $f(z) = B^2(C_2 + C_4 B^2 + C_6 B^4 + \cdots + B^{2n-2})\Phi$.

So, the set $\mathscr{K}$ associated to K is again going to be the set containing the first n consecutive odd numbers, i. e., $\mathscr{K} = \{1, 3, 5, \cdots + 2n - 3\}$. That means, the result holds true for $H^\infty_{K(B)}(\mathbb{D})$ without the condition $f_0 \neq 0$.

We note that not all subsets $\mathscr{K}$ form a subalgebra $H^\infty_{\mathscr{K}(B)}(\mathbb{D})$. For example, if we take $K = \{1, 2, 5\}$, $H^\infty_{K(B)}(\mathbb{D})$ is an algebra whose elements are of the form $f(z) = c_0 + c_3 B^3 + c_4 B^4 + B^6 \phi, \; \phi \in H^\infty(\mathbb{D})$.

If $c_0 = 0$, then $f(z) = B^3(c_3 + c_4 B + B^3 \phi)$. So, the subset $\mathscr{K}$ corresponding to this algebra is $\tilde{f} = c_3 + c_4 B + B^3 \phi$ is $\mathscr{K} = \{2\}$. As we discussed above, $H^\infty_{\mathscr{K}}(\mathbb{D})$ is not an algebra for $\mathscr{K} = \{2\}$, neither is $H^\infty_{\mathscr{K}(B)}(\mathbb{D})$. That means Theorem 3.4 can not be applied on the subalgebra $H^\infty_{K(B)}(\mathbb{D})$. However, in this particular case, we observe that $j_1 = 3 > k_p/2$, therefore Theorem 3.3 can be applied to get the result we need.

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