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DISJOINT HYPERCYCLIC POWERS OF WEIGHTED TRANSLATIONS ON LOCALLY COMPACT HAUSDORFF SPACES

YA WANG AND HONG-GANG ZENG*

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Abstract. In this article, we study the disjoint hypercyclic powers of weighted translations on the weighted space $L^p(G,\omega)$ in two cases, where G is a locally compact second countable Hausdorff space with a positive regular Borel measure and ω is a weight on G. In addition, some examples are given to illustrate our results.

1. Introduction

The notion of disjointness in linear dynamics was introduced by Bernal-González [3] and by Bès and Peris [8] in 2007, respectively. After that, the disjoint hypercyclicity was studied intensely by many scholars ([4, 5, 6, 7, 18, 20, 21, 22, 23]). For instance, Shkarin studied the existence of disjoint hypercyclic operators on separable infinite dimensional topological vector space in [22]. The disjoint hypercyclicity of bilateral and unilateral weighted backward shifts were characterized by Bès, Martin and Sanders [5] and by Bès and Peris [8]. In addition, Bès, Martin and Peris in [6] and Martin in [18] investigated the disjoint hypercyclicity of composition operators.

The notion of disjoint hypercyclicity comes from the much older notion of hypercyclicity in linear dynamics. Let X be a separable, infinite dimensional Banach space over the complex scalar field $\mathbb{C}$, and $L(X)$ be the algebra of bounded linear operators on X. An operator $T \in L(X)$ is said to be *hypercyclic* if there is a vector $x \in X$ for which its orbit $\mathrm{Orb}(T,x) = \{T^n x : n \in \mathbb{N}\}$ (where $\mathbb{N}$ denotes the set of non-negative integers)is dense in X. In linear dynamics, it is well known that an operator T is hypercyclic if and only if it is topologically transitive. An operator T is said to be topologically transitive if for any nonempty open sets V_0, V_1 in X, there is a positive integer m for which $V_0 \cap T^{-m}(V_1) \neq \emptyset$. This classical conclusion was put forward by Birkhoff in [9]. The excellent monographs [2], [15] and [17] provide a great deal of basic information about hypercyclicity.

Given $N \geqslant 2$, hypercyclic operators $T_1, T_2, \ldots, T_N$ acting on the same space X are said to be *disjoint hypercyclic* (in short, *d-hypercyclic*) if their direct sum $\oplus_{m=1}^{N} T_m$

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* Corresponding author.

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has a hypercyclic vector of the form $(x,x,\cdots,x)$ in X^N. Such a vector x is called a *d-hypercyclic vector* for $T_1,T_2,\dots,T_N$. If the set of d-hypercyclic vectors is dense in X, we say $T_1,T_2,\dots,T_N$ are *densely d-hypercyclic*. We say that $T_1,T_2,\dots,T_N$ in $L(X)$ with $N\geqslant 2$ are *disjoint topologically transitive* (in short, *d-topologically transitive*) if for any non-empty open subsets $V_0,V_1,\dots,V_N$ in X, there exists a positive integer m such that $V_0\cap T_1^{-m}(V_1)\cap T_2^{-m}(V_2)\cap\cdots\cap T_N^{-m}(V_N)\neq\emptyset$. Similarly, in the disjoint setting, the disjoint topological transitivity is equivalent to densely disjoint hypercyclicity [8]. A disjoint Hypercyclicity Criterion (in short, d-Hypercyclicity Criterion) is provided in [8]. The criterion offers a sufficient condition for densely d-hypercyclicity and it has the following equivalence relation with disjoint topological transitivity.

THEOREM 1.1. [8, Theorem 2.7] *Let $T_1,T_2,\dots,T_N$ be operators in $L(X)$ with $N\geqslant 2$. The following statements are equivalent:*

(a) The operators $T_1,T_2,\dots,T_N$ satisfy the d-Hypercyclicity Criterion.

(b) For each integer $r\geqslant 1$, the direct sum operators $\overbrace{T_1\oplus\cdots\oplus T_1}^{r}$, $\dots$, $\overbrace{T_N\oplus\cdots\oplus T_N}^{r}$ are d-topologically transitive on X^r.

Recently, hypercyclic and disjoint hypercyclic weighted translations on locally compact groups were studied in [10, 11, 12, 13, 16, 24], which generalized the characterizations of hypercyclic and disjoint hypercyclic bilateral weighted backward shifts offered in [5], [8] and [19].

Now, we introduce a more general definition of weighted translation, which generated by a continuous injective map on a locally compact Hausdorff space.

Let G be a locally compact Hausdorff space and λ be a positive regular Borel measure on G. Let $\varphi: G\to G$ be a continuous injective map such that λ is invariant under φ (that is, $\lambda(A)=\lambda(\varphi(A))$ for each A in the Borel $\sigma-$algebra $\mathscr{B}(G)$). Let $\omega: G\to\mathbb{R}$ be a positive continuous function such that $\sup\limits_{x\in G}\frac{\omega(x)}{\omega(\varphi(x))}<\infty$. For $1\leqslant p<\infty$, we consider the weighted space $L^p(G,\omega)=\{f:\int_G|f(x)\omega(x)|^p d\lambda(x)<\infty\}$ of complex-valued functions on G. $L^p(G,\omega)$ is a Banach space with the norm $\|f\|_{p,\omega}=(\int_G|f(x)\omega(x)|^p d\lambda(x))^{\frac{1}{p}}$. Since a complex Banach space admits a hypercyclic operator if and only if it is separable and infinite-dimensional [1], we also assume that G is second countable so that the question of hypercyclicity is meaningful for the space $L^p(G,\omega)$. A bounded continuous function $u: G\to\mathbb{C}\backslash\{0\}$ is called a *weight* on G. Now we define a *weighted translation* $T_{u,\varphi}: L^p(G,\omega)\to L^p(G,\omega)$ by

$$T_{u,\varphi}f(x)=u(x)f(\varphi(x)),\ \ f\in L^p(G,\omega),\ \ x\in G. \tag{1.1}$$

We call $T_{u,\varphi}$ is a *weighted translation generated by* φ and u. Since $\sup\limits_{x\in G}\frac{\omega(x)}{\omega(\varphi(x))}<\infty$ and u is bounded, it is easy to see that $T_{u,\varphi}$ is a bounded operator on $L^p(G,\omega)$.

For each integer n with $n>1$,

$$T_{u,\varphi}^n f(x)=\prod_{s=0}^{n-1}u(\varphi^s(x))f(\varphi^n(x)),\ \ f\in L^p(G,\omega),\ \ x\in G, \tag{1.2}$$

where $\varphi^n(x) = (\varphi \circ \varphi \circ \cdots \circ \varphi)(x) \quad (n-\text{fold})$.

We also define a self-map $S_{u,\varphi}$ on the subspace $L^p_c(G,\omega)$, which consists of functions in $L^p(G,\omega)$ with compact support, by

$$S_{u,\varphi} f(x) = \begin{cases} \frac{1}{u(y)} f(y) & \text{if there exists an } y \in G \text{ such that } x = \varphi(y), \\ 0 & \text{if } x \in G \backslash \varphi(G). \end{cases} \tag{1.3}$$

Then for any integer n with $n > 1$ we have

$$S^n_{u,\varphi} f(x) = \begin{cases} \frac{1}{\prod_{s=0}^{n-1} u(\varphi^s(y))} f(y) & \text{if there exists an } y \in G \text{ such that } x = \varphi^n(y), \\ 0 & \text{if } x \in G \backslash \varphi^n(G). \end{cases}$$

Since for any $f \in L^p_c(G,\omega)$ and $x \in G$,

$$\begin{aligned} (T_{u,\varphi} S_{u,\varphi} f)(x) &= T_{u,\varphi}(S_{u,\varphi} f)(x) \\ &= u(x)(S_{u,\varphi} f(\varphi(x))) \\ &= u(x) \frac{1}{u(x)} f(x) \\ &= f(x), \end{aligned}$$

we have

$$T_{u,\varphi} S_{u,\varphi}(f) = f \qquad \text{for } f \in L^p_c(G,\omega).$$

REMARKS 1.2. (1) The mapping φ is continuous injective and Proposition 7.1.5 in [14] imply that for each A in $\mathscr{B}(G)$, $\varphi(A) \in \mathscr{B}(G)$.

(2) The assumption that the mapping φ is injective ensures that $S_{u,\varphi}$ is well defined.

(3) Every unilateral or bilateral weighted backward shift on $\ell^p(\mathbb{N})$ or $\ell^p(\mathbb{Z})$ is a weighted translation with $G = \mathbb{N}$ or $\mathbb{Z}, \varphi(i) = i+1$ $(i \in G)$ and $\omega \equiv 1$ on G.

(4) If we let G be a locally compact group with a right invariant Haar measure λ and choose $a \in G$. Define the continuous injective map φ and positive continuous function ω on G by

$$\varphi(x) = xa^{-1} \quad \text{for } x \in G, \quad \omega \equiv 1 \text{ on } G.$$

Let u be a weight on G and let $T_{u,\varphi}$ be the weighted translation on $L^p(G,\omega)$ generated by φ and u. That is

$$T_{u,\varphi} f(x) = u(x) f(\varphi(x)) = u(x) f(xa^{-1}) \text{ for } f \in L^p(G,\omega) = L^p(G).$$

In this special case, $T_{u,\varphi}$ becomes the weighted convolution operator $T_{a,u}$ studied in [10, 11, 12, 13, 24].

Note that, every unilateral weighted backward shift on $\ell^p(\mathbb{N})$ satisfy that, for each $i \in \mathbb{N}, i \notin \varphi^n(\mathbb{N})$ when n sufficiently large. Since each compact subset of $\mathbb{N}$ is a finite set, the above assertion is equivalent with that for each nonempty compact subset K of $G = \mathbb{N}$, $K \cap \varphi^n(\mathbb{N}) = \emptyset$ when n sufficiently large.

For the weighted convolution operators $T_{a,u_1}, T_{a,u_2}, \cdots, T_{a,u_N}$ $(N \geqslant 2)$ acting on the space $L^p(G)$ of a locally compact group G, Chen showed that if a is a torsion element (an element $a \in G$ is called a torsion element if it is of finite order) then $T_{a,u_1}, T_{a,u_2}, \cdots, T_{a,u_N}$ are not disjoint hypercyclic (see [13, Lemma 2.1]). Thus, Chen in [13] and Zhang, Lu, Fu and Zhou in [24] were focus on the aperiodic group element $a \in G$(an element $a \in G$ is called aperiodic if the closed subgroup $G(a)$ generated by a is not compact). For aperiodic elements, Chen and Chu [11] showed that an element $a \in G$ is aperiodic if and only if for any compact set $K \subset G$, there exists some positive integer N such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > N$.

Inspired by the above statement, in this paper, we characterize the disjoint hypercyclic powers of weighted translations on $L^p(G, \omega)$ in the following two cases.

Case 1: Each compact subset $K \subset G$ lies outside $\varphi^n(G)$ for all sufficiently large n, that is, for each compact subset $K \subset G$, there exists a positive integer N_K such that $K \cap \varphi^n(G) = \emptyset$ for all $n > N_K$.

Case 2: The mapping φ is onto and $(\varphi^n)_{n \geqslant 1}$ is run away. We call $(\varphi^n)_{n \geqslant 1}$ is *run away*, if for each compact subset $K \subset G$, there exists a positive integer N_K such that $K \cap \varphi^n(K) = \emptyset$ for all $n > N_K$.

The result in the first case generalizes [8, Theorem 4.1], and the result in the second case generalizes [13, Theorem 2.2] and [24, Theorem 2.1], respectively.

2. Disjoint hypercyclic powers of weighted translations

In this section, let G be a locally compact second countable Hausdorff space with a positive regular Borel measure λ, where λ is invariant under a continuous injective mapping $\varphi : G \to G$. We characterize the d-hypercyclic powers of finite weighted translations generated by φ in two cases. Before stating the main theorems, we give a preliminary result.

LEMMA 2.1. *Let $f \in L^p(G, \omega)$ $(1 \leqslant p < \infty)$ with $\|f\|_{p,\omega}^p < \varepsilon^{p+1}$ for some $\varepsilon > 0$. Then for any compact subset $K \subset G$ with $\lambda(K) > 0$ and any non-negative integer n, there is a subset $E \subset K$ such that $\lambda(K \backslash E) < \varepsilon$ and $\sup_{x \in E} |f(\varphi^n x)\omega(\varphi^n x)| \leqslant \varepsilon$. If φ is onto, we can get the same result for any compact set $K \subset G$ $(\lambda(K) > 0)$ and $n \in \mathbb{Z}$.*

Proof. Let K be any compact subset of G with $\lambda(K) > 0$ and $n \in \mathbb{N}$. Let $E = \{x \in K : |\omega(\varphi^n x) f(\varphi^n x)| < \varepsilon\}$, then $\varepsilon^{p+1} > \int_G |f(x)\omega(x)|^p d\lambda x \geqslant \int_{\varphi^n(K)} |f(x)\omega(x)|^p d\lambda x \geqslant \int_{K \backslash E} |f(\varphi^n x)\omega(\varphi^n x)|^p d\lambda x \geqslant \varepsilon^p \lambda(K \backslash E)$. Thus,

$$\lambda(K \backslash E) < \varepsilon \text{ and } \sup_{x \in E} |f(\varphi^n x)\omega(\varphi^n x)| \leqslant \varepsilon.$$

If φ is onto, then φ becomes a bijection by assumption that φ is injective. For each compact set $K \subset G$ with $\lambda(K) > 0$ and each integer $n \in \mathbb{Z}$ the above argument is also valid. Thus, the same result follows. $\square$

Now we are ready to state the main results.

THEOREM 2.2. *Let $1 \leqslant p < \infty$ and integers $1 \leqslant r_1 < r_2 < \cdots < r_N$ be given, where $N \geqslant 2$. For each integer $1 \leqslant l \leqslant N$, let $T_{u_l,\varphi}$ be a weighted translation on $L^p(G,\omega)$ generated by φ and the weight u_l. If each compact subset $K \subset G$ lies outside $\varphi^n(G)$ for all sufficiently large n, then the following conditions are equivalent:*

(1) $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$ are d-hypercyclic.

(2) For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets $(E_k)_{k=1}^{\infty}$ in K such that $\lambda(K) = \lim\limits_{k\to\infty} \lambda(E_k)$ and a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for each $1 \leqslant l \leqslant N$ we have

$$\lim_{k\to\infty} \left\| \left. \frac{\omega \circ \varphi^{r_l n_k}}{\prod\limits_{t=0}^{r_l n_k - 1} u_l \circ \varphi^t} \right|_{E_k} \right\|_{\infty} = 0 \ , \tag{2.1}$$

and for $1 \leqslant s < l \leqslant N$ we have

$$\lim_{k\to\infty} \left\| \left. \frac{\left(\omega \circ \varphi^{(r_l - r_s)n_k}\right) \cdot \left(\prod\limits_{t=1}^{r_s n_k} u_s \circ \varphi^{r_l n_k - t}\right)}{\prod\limits_{t=0}^{r_l n_k - 1} u_l \circ \varphi^t} \right|_{E_k} \right\|_{\infty} = 0 \ . \tag{2.2}$$

(3) $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$ satisfy the d-Hypercyclicity Criterion.

(4) $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$ are densely d-hypercyclic.

Proof. (1) $\Rightarrow$ (2). Let $K \subset G$ be a compact set with $\lambda(K) > 0$ and $\chi_K \in L^p(G,\omega)$ denote the characteristic function of K. By assumption, there is a positive integer N_K such that $K \cap \varphi^n(K) \subset K \cap \varphi^n(G) = \emptyset$ for all $n > N_K$. Since ω is a positive continuous function, $c := \inf\limits_{x\in K} \omega(x) > 0$. Let k be any fixed positive integer, choose a real number δ_k such that $0 < \delta_k < \frac{1}{k}$, $0 < \frac{\delta_k}{c} < \frac{1}{k}$ and $\frac{\delta_k}{1-\frac{\delta_k}{c}} < \frac{1}{k}$. By the d-hypercyclicity of $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$, there is a d-hypercyclic vector f_k in $L^p(G,\omega)$ and positive integer $n_k > N_K$ (in fact the selection of n_k here can be sufficiently large) such that

$$\|f_k\|^p_{p,\omega} < \delta_k^{p+1} \tag{2.3}$$

and for each integer $1 \leqslant l \leqslant N$,

$$\left\| T^{r_l n_k}_{u_l,\varphi} f_k - \chi_K \right\|^p_{p,\omega} < \delta_k^{p+1}. \tag{2.4}$$

Applying Lemma 2.1 N times to (2.3), we can obtain a subset $E_k^1 \subset K$ with $\lambda(K \backslash E_k^1) < N\delta_k$ such that for each integer $1 \leqslant l \leqslant N$,

$$\sup_{x \in E_k^1} |f_k(\varphi^{r_l n_k}(x))\,\omega(\varphi^{r_l n_k}(x))| \leqslant \delta_k. \tag{2.5}$$

Applying Lemma 2.1 N times to (2.4), we can obtain a subset $E_k^2 \subset K$ with $\lambda(K \backslash E_k^2) < N\delta_k$ such that for each integer $1 \leqslant l \leqslant N$,

$$\begin{aligned} &\sup_{x \in E_k^2} \left| T_{u_l,\varphi}^{r_l n_k} f_k(x) - 1 \right| \omega(x) \\ &= \sup_{x \in E_k^2} \left| \left(\prod_{t=0}^{r_l n_k - 1} u_l(\varphi^t(x)) \right) f_k(\varphi^{r_l n_k}(x)) - 1 \right| \omega(x) \leqslant \delta_k, \end{aligned} \tag{2.6}$$

Applying Lemma 2.1 to (2.4) for any integers s, l with $1 \leqslant s < l \leqslant N$ (that is, applying Lemma 2.1 $N \cdot \frac{N-1}{2}$ times to (2.4)), we can obtain a subset $E_k^3 \subset K$ with $\lambda(K \backslash E_k^3) < (N \cdot \frac{N-1}{2})\delta_k$ such that for any $1 \leqslant s < l \leqslant N$,

$$\begin{aligned} &\sup_{x \in E_k^3} \left| T_{u_s,\varphi}^{r_s n_k} f_k(\varphi^{(r_l - r_s)n_k}(x)) - \chi_K(\varphi^{(r_l - r_s)n_k}(x)) \right| \omega(\varphi^{(r_l - r_s)n_k}(x)) \\ &= \sup_{x \in E_k^3} \left| \omega(\varphi^{(r_l - r_s)n_k}(x)) \left(\prod_{t=0}^{r_s n_k - 1} u_s(\varphi^t(\varphi^{(r_l - r_s)n_k}(x))) \right) f_k(\varphi^{r_l n_k}(x)) \right| \\ &\leqslant \delta_k. \end{aligned} \tag{2.7}$$

Let $E_k = E_k^1 \cap E_k^2 \cap E_k^3$, then $\lambda(K \backslash E_k) < (2N + N \cdot \frac{N-1}{2})\delta_k$. And by (2.5) and (2.6), for each $1 \leqslant l \leqslant N$ and any $x \in E_k$ we have

$$\frac{\omega(\varphi^{r_l n_k}(x))}{\left| \prod_{t=0}^{r_l n_k - 1} u_l(\varphi^t(x)) \right|} \leqslant \frac{\omega(\varphi^{r_l n_k}(x)) |f_k(\varphi^{r_l n_k}(x))|}{1 - \frac{\delta_k}{\omega(x)}} \leqslant \frac{\delta_k}{1 - \frac{\delta_k}{c}} < \frac{1}{k}. \tag{2.8}$$

Also, for any $x \in E_k$ and $1 \leqslant s < l \leqslant N$, from (2.6) and (2.7) we can get

$$\begin{aligned} &\left| \frac{\omega\left(\varphi^{(r_l - r_s)n_k}(x)\right) \prod_{t=1}^{r_s n_k} u_s\left(\varphi^{r_l n_k - t}(x)\right)}{\prod_{t=0}^{r_l n_k - 1} u_l\left(\varphi^t(x)\right)} \right| \\ &= \frac{\left| \omega\left(\varphi^{(r_l - r_s)n_k}(x)\right) \left(\prod_{t=0}^{r_s n_k - 1} u_s\left(\varphi^t\left(\varphi^{(r_l - r_s)n_k}(x)\right)\right) \right) f_k(\varphi^{r_l n_k}(x)) \right|}{\left| \prod_{t=0}^{r_l n_k - 1} u_l\left(\varphi^t(x)\right) f_k(\varphi^{r_l n_k}(x)) \right|} \\ &\leqslant \frac{\delta_k}{1 - \frac{\delta_k}{c}} < \frac{1}{k}. \end{aligned} \tag{2.9}$$

Now condition (2) can be proved by (2.8) and (2.9). Indeed, we just need to take $k = 1,2,3,\dots,$ and then find the sequences $(E_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ by induction.

(2) $\Rightarrow$ (3). By Theorem 1.1, we show that for any positive integer $r \geqslant 1$ the direct sum operators $\overbrace{T_{u_1,\varphi}^{r_1} \oplus \cdots \oplus T_{u_1,\varphi}^{r_1}}^{r}$, $\dots,$ $\overbrace{T_{u_N,\varphi}^{r_N} \oplus \cdots \oplus T_{u_N,\varphi}^{r_N}}^{r}$ are d-topologically transitive. Fix $r \in \mathbb{N}$ with $r \geqslant 1$, let

$$V_{0,j}, V_{1,j}, \cdots, V_{N,j}\,(j = 1,\dots,r)$$

be non-empty open subsets of $L^p(G,\omega)$. Our aim is to find a positive integer n such that

$$V_{0,j} \cap T_{u_1,\varphi}^{-r_1 n}(V_{1,j}) \cap \cdots \cap T_{u_N,\varphi}^{-r_N n}(V_{N,j}) \neq \emptyset \text{ for each } 1 \leqslant j \leqslant r.$$

Since the space $C_c(G)$ of continuous functions on G with compact support is dense in $L^p(G,\omega)$, for each integer $1 \leqslant j \leqslant r$ we can pick $f_{0,j}$, $g_{1,j}, \cdots, g_{N,j}$ in $C_c(G)$ such that $f_{0,j} \in V_{0,j}$, $g_{1,j} \in V_{1,j}, \cdots, g_{N,j} \in V_{N,j}$. Let K be the union of the compact supports of $f_{0,j}$, $g_{1,j}, \cdots, g_{N,j}$ $(j = 1,\cdots,r)$ and set $C := \sup\limits_{x\in K} \omega(x) < \infty$. Suppose $(E_k)_{k\geqslant 1}$ and $(n_k)_{k\geqslant 1}$ be the sequences satisfying condition (2). Let N_K be the positive integer such that

$$K \cap \varphi^n(G) = \emptyset \text{ for all } n > N_K. \tag{2.10}$$

For each $1 \leqslant l \leqslant N$, we consider the self-map $S_{u_l,\varphi}$ defined as (1.3) on the subspace $L_c^p(G,\omega)$.

By (2.10), it is easy to calculate that for any integer $n_k \in (n_k)_{k\geqslant 1}$ with $n_k > N_K$,

$$T_{u_l,\varphi}^{r_l n_k}\left(f_{0,j}\chi_{E_k}\right) \equiv 0 \quad \text{on } G \quad (1 \leqslant l \leqslant N, 1 \leqslant j \leqslant r) \tag{2.11}$$

and

$$T_{u_l,\varphi}^{r_l n_k} S_{u_s,\varphi}^{r_s n_k}\left(g_{s,j}\chi_{E_k}\right) \equiv 0 \quad \text{on } G \quad (1 \leqslant s < l \leqslant N, 1 \leqslant j \leqslant r). \tag{2.12}$$

From (2.1), for integers j,l with $1 \leqslant j \leqslant r$ and $1 \leqslant l \leqslant N$ we have

$$\begin{aligned}
&\lim_{k\to\infty} \left\| S_{u_l,\varphi}^{r_l n_k}\left(g_{l,j}\chi_{E_k}\right)\right\|_{p,\omega} \\
=\ & \lim_{k\to\infty} \left(\int_{\varphi^{r_l n_k}(G)} \left| S_{u_l,\varphi}^{r_l n_k}\left(g_{l,j}\chi_{E_k}\right)(x)\omega(x)\right|^p d\lambda(x)\right)^{\frac{1}{p}} \\
\overset{x=\varphi^{r_l n_k}(y)}{=}\ & \lim_{k\to\infty} \left(\int_G \left| S_{u_l,\varphi}^{r_l n_k}\left(g_{l,j}\chi_{E_k}\right)(\varphi^{r_l n_k}(y))\omega(\varphi^{r_l n_k}(y))\right|^p d\lambda(y)\right)^{\frac{1}{p}} \\
=\ & \lim_{k\to\infty} \left(\int_G \left| \frac{\omega(\varphi^{r_l n_k}(y))}{\prod\limits_{t=0}^{r_l n_k - 1} u_l(\varphi^t(y))} g_{l,j}\chi_{E_k}(y)\right|^p d\lambda(y)\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\left(\int_{E_k}\left|\frac{\omega(\varphi^{r_l n_k}(y))}{\prod_{t=0}^{r_l n_k-1} u_l(\varphi^t(y))} g_{l,j}(y)\right|^p d\lambda(y)\right)^{\frac{1}{p}} \\
&\leqslant \lim_{k\to\infty}\left\|\left.\frac{\omega\circ\varphi^{r_l n_k}}{\prod_{t=0}^{r_l n_k-1} u_l\circ\varphi^t}\right|_{E_k}\right\|_\infty \left\|g_{l,j}\right\|_\infty \lambda(K)^{\frac{1}{p}} \\
&= 0. \qquad (2.13)
\end{aligned}
$$

And for $1\leqslant s<l\leqslant N$, $1\leqslant j\leqslant r$, by (2.2) we have

$$
\begin{aligned}
&\lim_{k\to\infty}\left\|T_{u_s,\varphi}^{r_s n_k} S_{u_l,\varphi}^{r_l n_k}\left(g_{l,j}\chi_{E_k}\right)\right\|_{p,\omega} \\
&= \lim_{k\to\infty}\left(\int_{E_k}\left|\frac{\omega\left(\varphi^{(r_l-r_s)n_k}(x)\right)\prod_{t=1}^{r_s n_k} u_s\left(\varphi^{r_l n_k-t}(x)\right)}{\prod_{t=0}^{r_l n_k-1} u_l\left(\varphi^t(x)\right)} g_{l,j}(x)\right|^p d\lambda(x)\right)^{\frac{1}{p}} \\
&= 0. \qquad (2.14)
\end{aligned}
$$

Now for any integers j,k with $k\geqslant 1$ and $1\leqslant j\leqslant r$ let

$$
v_{j,k} = f_{0,j}\chi_{E_k} + \sum_{i=1}^{N} S_{u_i,\varphi}^{r_i n_k}\left(g_{i,j}\chi_{E_k}\right) \in L^p(G,\omega).
$$

Since for each $1\leqslant j\leqslant r$ and any $x\in G$,

$$
\begin{aligned}
&\left|f_{0,j}\chi_{E_k}(x) - f_{0,j}(x) + \sum_{i=1}^{N} S_{u_i,\varphi}^{r_i n_k}\left(g_{i,j}\chi_{E_k}\right)(x)\right|^p \\
&\leqslant (N+1)^p\left(\left|f_{0,j}\chi_{E_k}(x) - f_{0,j}(x)\right|^p + \sum_{i=1}^{N}\left|S_{u_i,\varphi}^{r_i n_k}\left(g_{i,j}\chi_{E_k}\right)(x)\right|^p\right),
\end{aligned}
$$

thus

$$
\left\|v_{j,k}-f_{0,j}\right\|_{p,\omega}^p \leqslant (N+1)^p C^p \left\|f_{0,j}\right\|_\infty^p \lambda(K\backslash E_k) + (N+1)^p \sum_{i=1}^{N}\left\|S_{u_i,\varphi}^{r_i n_k}\left(g_{i,j}\chi_{E_k}\right)\right\|_{p,\omega}^p.
$$

Using a similar argument, for any $1\leqslant l\leqslant N$ and $1\leqslant j\leqslant r$ we have

$$
\begin{aligned}
\left\|T_{u_l,\varphi}^{r_l n_k} v_{j,k} - g_{l,j}\right\|_{p,\omega}^p \leqslant{}& (N+1)^p\left\|T_{u_l,\varphi}^{r_l n_k}\left(f_{0,j}\chi_{E_k}\right)\right\|_{p,\omega}^p + (N+1)^p C^p\left\|g_{l,j}\right\|_\infty^p \lambda(K\backslash E_k) \\
&+ (N+1)^p \sum_{i\neq l}^{N}\left\|T_{u_l,\varphi}^{r_l n_k} S_{u_i,\varphi}^{r_i n_k}\left(g_{i,j}\chi_{E_k}\right)\right\|_{p,\omega}^p.
\end{aligned}
$$

Hence by (2.11), (2.12), (2.13) and (2.14) for each $1 \leqslant l \leqslant N$ and $1 \leqslant j \leqslant r$,

$$\lim_{k\to\infty} v_{j,k} = f_{0,j} \quad \text{and} \quad \lim_{k\to\infty} T_{u_l,\varphi}^{r_l n_k} v_{j,k} = g_{l,j}.$$

Which implies that there is some $n_{k_0} \in (n_k)_{k=1}^{\infty}$ such that

$$V_{0,j} \cap T_{u_1,\varphi}^{-r_1 n_{k_0}}\left(V_{1,j}\right) \cap \cdots \cap T_{u_N,\varphi}^{-r_N n_{k_0}}\left(V_{N,j}\right) \neq \emptyset \quad (1 \leqslant j \leqslant r).$$

The implications $(3) \Rightarrow (4)$ and $(4) \Rightarrow (1)$ are obvious. $\square$

REMARK 1. If G is discrete, condition (2) in above theorem can be replaced by the following:

(2') There is a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for any $x \in G$, if $1 \leqslant l \leqslant N$,

$$\lim_{k\to\infty} \frac{\omega(\varphi^{r_l n_k}(x))}{\prod_{t=0}^{r_l n_k - 1} |u_l(\varphi^t(x))|} = 0, \tag{2.15}$$

and if $1 \leqslant s < l \leqslant N$,

$$\lim_{k\to\infty} \left| \frac{\omega(\varphi^{(r_l - r_s)n_k}(x)) \cdot \prod_{t=1}^{r_s n_k} u_s(\varphi^{r_l n_k - t}(x))}{\prod_{t=0}^{r_l n_k - 1} u_l(\varphi^t(x))} \right| = 0. \tag{2.16}$$

Indeed, if G is discrete, then each compact subset of G is a finite set, thus $(2') \Rightarrow (2)$ is obvious. To prove $(2) \Rightarrow (2')$, since G is discrete, we fix $G := \{i_1, i_2, \cdots, i_k, \cdots\}$ and set $G_k := \{i_1, i_2, \cdots, i_k\}$ for each integer $k \geqslant 1$. By condition (2), for each G_k $(k \geqslant 1)$, there is a strictly increasing sequence $(n_m^{(k)})_{m=1}^{\infty}$ of positive integers such that

$$\lim_{m\to\infty} \left\| \left. \frac{\omega \circ \varphi^{r_l n_m^{(k)}}}{\prod_{t=0}^{r_l n_m^{(k)} - 1} u_l \circ \varphi^t} \right|_{G_k} \right\|_{\infty} = 0 \ \ (1 \leqslant l \leqslant N)$$

and

$$\lim_{m\to\infty} \left\| \left. \frac{\left(\omega \circ \varphi^{(r_l - r_s)n_m^{(k)}}\right) \cdot \left(\prod_{t=1}^{r_s n_m^{(k)}} u_s \circ \varphi^{r_l n_m^{(k)} - t}\right)}{\prod_{t=0}^{r_l n_m^{(k)} - 1} u_l \circ \varphi^t} \right|_{G_k} \right\|_{\infty} = 0 \ \ (1 \leqslant s < l \leqslant N).$$

Then for each integer $k \geqslant 1$, there is a positive integer $n_{m_0}^{(k)} \in (n_m^{(k)})_{m=1}^{\infty}$ such that, for any $x \in G_k$ and integer $m \geqslant m_0$ we have

$$\frac{\omega(\varphi^{r_l n_m^{(k)}}(x))}{\prod_{t=0}^{r_l n_m^{(k)}-1} |u_l(\varphi^t(x))|} < \frac{1}{k} \quad \text{for } 1 \leqslant l \leqslant N, \tag{2.17}$$

and

$$\left| \frac{\omega(\varphi^{(r_l - r_s) n_m^{(k)}}(x)) \cdot \prod_{t=1}^{r_s n_m^{(k)}} u_s(\varphi^{r_l n_m^{(k)} - t}(x))}{\prod_{t=0}^{r_l n_m^{(k)}-1} u_l(\varphi^t(x))} \right| < \frac{1}{k} \quad \text{for } 1 \leqslant s < l \leqslant N. \tag{2.18}$$

If we take $k = 1, 2, 3, \ldots$ in above argument and denote $n_{m_0}^{(k)}$ by n_k, then by induction we can find a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that (2.15) and (2.16) hold.

The following two examples are provided to illustrate Theorem 2.2, where G is discrete in Example 2.3 and G is not discrete in Example 2.4.

EXAMPLE 2.3. Let $1 \leqslant p < \infty$, $N \geqslant 2$. For each $1 \leqslant l \leqslant N$, let T_l be a unilateral backward weighted shift on $\ell^p(\mathbb{N})$ with positive weight sequence $(a_{l,j})_{j \geqslant 1}$, that is, $T_l e_0 = 0$ and $T_l e_j = a_{l,j} e_{j-1}$ for $j \geqslant 1$, where $(e_j)_{j \in \mathbb{N}}$ is the canonical basis of $\ell^p(\mathbb{N})$. If we let $G = \mathbb{N}$ and define the injective map φ on G by $\varphi(j) = j + 1$ for $j \in \mathbb{N}$. For each $1 \leqslant l \leqslant N$, let u_l be a weight on G defined by $u_l(j) = a_{l,j+1}$ $(j \in \mathbb{N})$. Then each unilateral backward weighted shift T_l is the weighted translation $T_{u_l,\varphi}$ on $\ell^p(\mathbb{N})$ given by

$$T_{u_l,\varphi} f(j) = u_l(j) f(j+1) \quad (f \in \ell^p(\mathbb{N})).$$

Hence, by Remark 1, for any integers $1 \leqslant r_1 < r_2 < \cdots < r_N$, the operators $T_1^{r_1}, \ldots, T_N^{r_N}$ are d-hypercyclic if and only if there is a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for any $j \in \mathbb{N}$,

$$\lim_{k \to \infty} \frac{1}{\prod_{t=1}^{r_l n_k} a_{l,j+t}} = 0 \quad \text{for } 1 \leqslant l \leqslant N, \tag{2.19}$$

and

$$\lim_{k \to \infty} \frac{\prod_{t=0}^{r_s n_k - 1} a_{s,j+r_l n_k - t}}{\prod_{t=1}^{r_l n_k} a_{l,j+t}} = 0 \quad \text{for } 1 \leqslant s < l \leqslant N. \tag{2.20}$$

Which are the same with [8, Theorem 4.1].

EXAMPLE 2.4. Let $G=\{re^{i\theta}:0<r<\infty,-\frac{\pi}{4}\leqslant\theta\leqslant\frac{\pi}{4}\}$. And define the continuous injective mapping φ and positive continuous function ω on G by

$$\varphi(x)=x+e^{\frac{\pi}{4}i}\quad\text{for } x\in G,\quad \omega(re^{i\theta})=(\sqrt{3/2})^r\quad\text{for } re^{i\theta}\in G.$$

We define a weight u on G by

$$u(re^{i\theta})=\begin{cases}\sqrt{2} & \text{if } \frac{1}{2}\leqslant\theta\leqslant\frac{\pi}{4},\\ 2^{\theta} & \text{if } -\frac{1}{2}<\theta<\frac{1}{2},\\ \frac{\sqrt{2}}{2} & \text{if } -\frac{\pi}{4}\leqslant\theta\leqslant-\frac{1}{2}.\end{cases}$$

Let $u_1=u_2=u$, $r_1=1$, $r_2=2$ and let $T_{u_i,\varphi}$ $(i=1,2)$ be the weighted translations on $L^p(G,\omega)$ $(1\leqslant p<\infty)$ induced by φ and u_i $(i=1,2)$. That is, for $i=1,2$,

$$T_{u_i,\varphi}f(x)=u_i(x)f(x+e^{\frac{\pi}{4}i})=u(x)f(x+e^{\frac{\pi}{4}i})\quad(f\in L^p(G,\omega)).$$

Then $T^{r_1}_{u_1,\varphi},T^{r_2}_{u_2,\varphi}$ satisfy condition (2) in Theorem 2.2. Indeed, if we let K be any compact subset of G with $\lambda(K)>0$, then there exists a positive integer N_K such that for any integer $n>N_K$, if $re^{i\theta}\in K+ne^{\frac{\pi}{4}i}$ then $\frac{1}{2}<\theta\leqslant\frac{\pi}{4}$. Since K is compact, there is a positive integer r_0 such that, $\sup\limits_{x\in K}|x|<r_0$. Thus for any integers l,n with $1\leqslant l\leqslant 2$ and $n>N_K+1$ we have

$$\left\|\left.\frac{\omega\circ\varphi^{r_l n}}{\prod\limits_{t=0}^{r_l n-1}u_l\circ\varphi^t}\right|_K\right\|_\infty=\sup_{x\in K}\frac{\omega(x+lne^{\frac{\pi}{4}i})}{u(x)u(x+e^{\frac{\pi}{4}i})\cdots u(x+(ln-1)e^{\frac{\pi}{4}i})}$$
$$\leqslant\frac{(\sqrt{3/2})^{r_0+ln}}{(\frac{\sqrt{2}}{2})^{N_K+1}(\sqrt{2})^{ln-1-N_K}}\to 0\quad\text{as } n\to\infty,$$

and for $s=1$, $l=2$,

$$\left\|\left.\frac{\left(\omega\circ\varphi^{(r_l-r_s)n}\right)\cdot\left(\prod\limits_{t=1}^{r_s n}u_s\circ\varphi^{r_l n-t}\right)}{\prod\limits_{t=0}^{r_l n-1}u_l\circ\varphi^t}\right|_K\right\|_\infty$$
$$=\sup_{x\in K}\frac{\omega(x+ne^{\frac{\pi}{4}i})}{u(x)u(x+e^{\frac{\pi}{4}i})\cdots u(x+(n-1)e^{\frac{\pi}{4}i})}$$
$$\leqslant\frac{(\sqrt{3/2})^{r_0+n}}{(\frac{\sqrt{2}}{2})^{N_K+1}(\sqrt{2})^{n-1-N_K}}\to 0\quad\text{as } n\to\infty.$$

Next we consider the second case.

THEOREM 2.5. *Let $1 \leqslant p < \infty$ and $1 \leqslant r_1 < r_2 < \cdots < r_N$, where $N \geqslant 2$, $r_i \in \mathbb{N}$, $i = 1, \cdots, N$. For each $1 \leqslant l \leqslant N$, let $T_{u_l,\varphi}$ be a weighted translation on $L^p(G,\omega)$ generated by φ and the weight u_l. If φ is onto and $(\varphi^n)_{n\geqslant 1}$ is run away, then the following conditions are equivalent:*

(1) $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$ are densely d-hypercyclic.

(2) For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets $(E_k)_{k=1}^{\infty}$ in K such that $\lambda(K) = \lim\limits_{k\to\infty} \lambda(E_k)$ and a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for $1 \leqslant l \leqslant N$,

$$\lim_{k\to\infty} \left\| \left. \frac{\omega \circ \varphi^{r_l n_k}}{\prod\limits_{t=0}^{r_l n_k - 1} u_l \circ \varphi^t} \right|_{E_k} \right\|_{\infty} = \lim_{k\to\infty} \left\| \left(\omega \circ \varphi^{-r_l n_k}\right) \cdot \prod_{t=1}^{r_l n_k} u_l \circ \varphi^{-t} \Big|_{E_k} \right\|_{\infty} = 0, \qquad (2.21)$$

and for $1 \leqslant s < l \leqslant N$,

$$\lim_{k\to\infty} \left\| \left. \frac{\left(\omega \circ \varphi^{(r_l - r_s)n_k}\right) \cdot \left(\prod\limits_{t=1}^{r_s n_k} u_s \circ \varphi^{r_l n_k - t}\right)}{\prod\limits_{t=0}^{r_l n_k - 1} u_l \circ \varphi^t} \right|_{E_k} \right\|_{\infty} = 0, \qquad (2.22)$$

$$\lim_{k\to\infty} \left\| \left. \frac{\left(\omega \circ \varphi^{(r_s - r_l)n_k}\right) \cdot \left(\prod\limits_{t=1}^{r_l n_k} u_l \circ \varphi^{r_s n_k - t}\right)}{\prod\limits_{t=0}^{r_s n_k - 1} u_s \circ \varphi^t} \right|_{E_k} \right\|_{\infty} = 0. \qquad (2.23)$$

(3) $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$ satisfy the d-Hypercyclicity Criterion.

Proof. (1) $\Rightarrow$ (2). Let $K \subset G$ be a compact set with $\lambda(K) > 0$ and $\chi_K \in L^p(G,\omega)$ denote the characteristic function of K. By assumption there is a positive integer N_K such that

$$K \cap \varphi^n(K) = \emptyset \text{ for } n > N_K. \qquad (2.24)$$

Since ω is a positive continuous function, $c := \inf\limits_{x\in K} \omega(x) > 0$. Let k be any fixed positive integer, choose a real number δ_k such that $0 < \delta_k < \frac{1}{k}$, $0 < \frac{\delta_k}{c} < \frac{1}{k}$ and $\frac{\delta_k}{1 - \frac{\delta_k}{c}} < \frac{1}{k}$. By densely d-hypercyclicity of $T^{r_1}_{u_1,\varphi}, \ldots, T^{r_N}_{u_N,\varphi}$, there is a d-hypercyclic vector f_k in $L^p(G,\omega)$ and a positive integer $n_k > N_K$ (in fact, the selection of n_k can be sufficiently large) such that

$$\|f_k - \chi_K\|^p_{p,\omega} < \delta_k^{p+1} \qquad (2.25)$$

and

$$\left\|T_{u_l,\varphi}^{r_l n_k} f_k - \chi_K\right\|_{p,\omega}^{p} < \delta_k^{p+1} \quad (1 \leqslant l \leqslant N). \tag{2.26}$$

Applying Lemma 2.1 $N+1$ times for (2.25) and $2N+N(N-1)$ times for (2.26), we can obtain a subset $E_k \subset K$ with $\lambda(K \backslash E_k) < (3N+1+N(N-1))\delta_k$ such that for each $1 \leqslant l \leqslant N$,

$$\begin{aligned}
&\sup_{x\in E_k} |f_k(\varphi^{r_l n_k}(x)) - \chi_K(\varphi^{r_l n_k}(x))|\,\omega(\varphi^{r_l n_k}(x)) \\
&= \sup_{x\in E_k} |\omega(\varphi^{r_l n_k}(x)) f_k(\varphi^{r_l n_k}(x))| \leqslant \delta_k,
\end{aligned} \tag{2.27}$$

$$\sup_{x\in E_k} |f_k(x) - 1|\,\omega(x) \leqslant \delta_k, \tag{2.28}$$

$$\sup_{x\in E_k} \left|T_{u_l,\varphi}^{r_l n_k} f_k(x) - 1\right| \omega(x) \leqslant \delta_k, \tag{2.29}$$

$$\begin{aligned}
&\sup_{x\in E_k} \left|T_{u_l,\varphi}^{r_l n_k} f_k(\varphi^{-r_l n_k}(x))\right| \omega(\varphi^{-r_l n_k} x) \\
&= \sup_{x\in E_k} \left|\omega(\varphi^{-r_l n_k}(x)) \left(\prod_{t=1}^{r_l n_k} u_l(\varphi^{-t}(x))\right) f_k(x)\right| \leqslant \delta_k,
\end{aligned} \tag{2.30}$$

and for s,l with $1 \leqslant s < l \leqslant N$,

$$\begin{aligned}
&\sup_{x\in E_k} \left|T_{u_s,\varphi}^{r_s n_k} f_k(\varphi^{(r_l-r_s)n_k}(x)) - \chi_K(\varphi^{(r_l-r_s)n_k}(x))\right| \omega(\varphi^{(r_l-r_s)n_k}(x)) \\
&= \sup_{x\in E_k} \left|\omega(\varphi^{(r_l-r_s)n_k}(x)) \left(\prod_{t=0}^{r_s n_k-1} u_s(\varphi^{t}(\varphi^{(r_l-r_s)n_k}(x)))\right) f_k(\varphi^{r_s n_k}(\varphi^{(r_l-r_s)n_k}(x)))\right| \\
&\leqslant \delta_k,
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\sup_{x\in E_k} \left|T_{u_l,\varphi}^{r_l n_k} f_k(\varphi^{(r_s-r_l)n_k}(x)) - \chi_K(\varphi^{(r_s-r_l)n_k}(x))\right| \omega(\varphi^{(r_s-r_l)n_k}(x)) \\
&= \sup_{x\in E_k} \left|\omega(\varphi^{(r_s-r_l)n_k}(x)) \left(\prod_{t=0}^{r_l n_k-1} u_l(\varphi^{t}(\varphi^{(r_s-r_l)n_k}(x)))\right) f_k(\varphi^{r_l n_k}(\varphi^{(r_s-r_l)n_k}(x)))\right| \\
&\leqslant \delta_k.
\end{aligned} \tag{2.32}$$

Using a similar argument as in (2.8), by (2.27) and (2.29) one can deduce

$$\sup_{x\in E_k} \frac{\omega(\varphi^{r_l n_k}(x))}{\left|\prod_{t=0}^{r_l n_k-1} u_l(\varphi^{t}(x))\right|} < \frac{1}{k} \qquad \text{for } 1 \leqslant l \leqslant N. \tag{2.33}$$

From (2.28) and (2.30), an easy computation shows that

$$\sup_{x\in E_k}\left|\omega(\varphi^{-r_l n_k}(x))\prod_{t=1}^{r_l n_k} u_l(\varphi^{-t}(x))\right| \leqslant \frac{\delta_k}{1-\frac{\delta_k}{c}} < \frac{1}{k} \quad \text{for } 1\leqslant l\leqslant N. \tag{2.34}$$

From (2.29), (2.31) and (2.32), repeating a similar argument as in (2.9), we can obtain that for $1\leqslant s<l\leqslant N$,

$$\sup_{x\in E_k}\left|\frac{\omega\left(\varphi^{(r_l-r_s)n_k}(x)\right)\prod_{t=1}^{r_s n_k} u_s\left(\varphi^{r_l n_k-t}(x)\right)}{\prod_{t=0}^{r_l n_k-1} u_l\left(\varphi^{t}(x)\right)}\right| < \frac{1}{k} \tag{2.35}$$

and

$$\sup_{x\in E_k}\left|\frac{\omega\left(\varphi^{(r_s-r_l)n_k}(x)\right)\prod_{t=1}^{r_l n_k} u_l\left(\varphi^{r_s n_k-t}(x)\right)}{\prod_{t=0}^{r_s n_k-1} u_s\left(\varphi^{t}(x)\right)}\right| < \frac{1}{k}. \tag{2.36}$$

Now, the proof of condition (2) can be completed by (2.33), (2.34), (2.35) and (2.36).

(2)$\Rightarrow$(3). The proof of this implication is similar to that in Theorem 2.2. We just need to replace (2.10) with

$$K\cap\varphi^n(K)=\emptyset \text{ for all } n>N_K,$$

replace (2.11) with

$$\begin{aligned}\lim_{k\to\infty}\left\|T_{u_l,\varphi}^{r_l n_k}\left(f_{0,j}\chi_{E_k}\right)\right\|_{p,\omega} &= \lim_{k\to\infty}\left(\int_{E_k}\left|\omega(\varphi^{-r_l n_k}(x))\prod_{t=1}^{r_l n_k} u_l(\varphi^{-t}x) f_{0,j}(x)\right|^p d\lambda(x)\right)^{\frac{1}{p}}\\ &=0\end{aligned}$$

for any $1\leqslant l\leqslant N, 1\leqslant j\leqslant r.$

And replace (2.12) with

$$\begin{aligned}&\lim_{k\to\infty}\left\|T_{u_l,\varphi}^{r_l n_k} S_{u_s,\varphi}^{r_s n_k}\left(g_{s,j}\chi_{E_k}\right)\right\|_{p,\omega}\\ &=\lim_{k\to\infty}\left(\int_{E_k}\left|\frac{\omega\left(\varphi^{(r_s-r_l)n_k}(x)\right)\prod_{t=1}^{r_l n_k} u_l\left(\varphi^{r_s n_k-t}(x)\right)}{\prod_{t=0}^{r_s n_k-1} u_s\left(\varphi^{t}(x)\right)} g_{s,j}(x)\right|^p d\lambda(x)\right)^{\frac{1}{p}}\\ &=0\end{aligned}$$

for any $1\leqslant s<l\leqslant N, 1\leqslant j\leqslant r.$

(3)$\Rightarrow$ (1). This implication is obvious. $\square$

REMARK 2. The same argument as used in Remark 1 gives that: if G is discrete, condition (2) in Theorem 2.5 can be replaced by

(2') There is a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for any $x \in G$, if $1 \leqslant l \leqslant N$,

$$\lim_{k\to\infty}\left|\frac{\omega(\varphi^{r_l n_k}(x))}{\prod_{t=0}^{r_l n_k-1} u_l(\varphi^t(x))}\right| = \lim_{k\to\infty}\left|\omega(\varphi^{-r_l n_k}(x))\cdot\prod_{t=1}^{r_l n_k} u_l(\varphi^{-t}(x))\right| = 0,$$

if $1 \leqslant s < l \leqslant N$,

$$\lim_{k\to\infty}\left|\frac{\omega(\varphi^{(r_l-r_s)n_k}(x))\cdot\prod_{t=1}^{r_s n_k} u_s(\varphi^{r_l n_k-t}(x))}{\prod_{t=0}^{r_l n_k-1} u_l(\varphi^t(x))}\right| = 0$$

and

$$\lim_{k\to\infty}\left|\frac{\omega(\varphi^{(r_s-r_l)n_k}(x))\cdot\prod_{t=1}^{r_l n_k} u_l(\varphi^{r_s n_k-t}(x))}{\prod_{t=0}^{r_s n_k-1} u_s(\varphi^t(x))}\right| = 0.$$

The next example illustrates that the result in case 2 generalizes the works on disjoint hypercyclicity by Chen in [13] and by Zhang, Lu, Fu and Zhou in [24].

EXAMPLE 2.6. Let G be a locally compact group with a right invariant Haar measure λ and let a be an aperiodic element in G. The continuous injective mapping φ and positive continuous function ω be defined by

$$\varphi(x) = xa^{-1} \quad \text{for } x \in G, \quad \omega \equiv 1 \text{ on } G.$$

Given $N \geqslant 2$, for $1 \leqslant l \leqslant N$, let u_l be a weight on G and $T_{u_l,\varphi}$ be the weighted translation on $L^p(G,\omega)$ generated by φ and u_l. In this case, each $T_{u_l,\varphi}$ is the weighted translation T_{a,u_l} studied in [24, Theorem 2.1] (or [13, Theorem 2.2]). By Theorem 2.5, for any integers $1 \leqslant r_1 < r_2 < \cdots < r_N$, $T_{a,u_1}^{r_1}, T_{a,u_2}^{r_2}, \ldots, T_{a,u_N}^{r_N}$ are disjoint hypercyclic if and only if for each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets $(E_k)_{k=1}^{\infty}$ in K such that $\lambda(K) = \lim_{k\to\infty}\lambda(E_k)$ and a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for $1 \leqslant l \leqslant N$,

$$\lim_{k\to\infty}\sup_{x\in E_k}\left|\frac{1}{\prod_{t=0}^{r_l n_k-1} u_l(xa^{-t})}\right| = \lim_{k\to\infty}\sup_{x\in E_k}\left|\prod_{t=1}^{r_l n_k} u_l(xa^{t})\right| = 0,$$

and for $1 \leqslant s < l \leqslant N$,

$$\limsup_{k\to\infty} \sup_{x\in E_k} \left| \frac{\prod_{t=1}^{r_s n_k} u_s(xa^{t-r_l n_k})}{\prod_{t=0}^{r_l n_k-1} u_l(xa^{-t})} \right| = 0,$$

$$\limsup_{k\to\infty} \sup_{x\in E_k} \left| \frac{\prod_{t=1}^{r_l n_k} u_l(xa^{t-r_s n_k})}{\prod_{t=0}^{r_s n_k-1} u_s(xa^{-t})} \right| = 0.$$

Which are the same with [24, Theorem 2.1] or [13, Theorem 2.2].

Now we offer two examples, which are particular cases of Theorem 2.5 but not particular cases of Theorem 2.1 in [24] or Theorem 2.2 in [13].

EXAMPLE 2.7. Let $G = \{1,2\} \times \mathbb{Z}$ with the discrete topology and define the injective map φ and the weight ω on G by

$$\varphi(i,j) = (i, j+1) \text{ for } (i,j) \in G, \quad \text{and } \omega \equiv 1 \text{ on } G.$$

Let u be a weight on G given by

$$u(i,j) = \begin{cases} 2 & \text{if } j > 0, \\ 1 & \text{if } j = 0, \\ \frac{1}{2} & \text{if } j < 0. \end{cases}$$

Let $u_1 = u_2 = u$ and let $T_{u_i,\varphi}$ $(i = 1,2)$ be the weighted translation on $L^p(G,\omega)$ $(1 \leqslant p < \infty)$ generated by φ and u_i $(i = 1,2)$. Then $T_{u_1,\varphi}, T^2_{u_2,\varphi}$ satisfy condition (2') in Remark 2. Indeed, for any $(i,j) \in G$ and integers n,l with $n > 2|j|+1$ and $l = 1,2$ we have

$$\begin{aligned} \left| \frac{1}{\prod_{t=0}^{ln-1} u_l(\varphi^t(i,j))} \right| &= \frac{1}{\prod_{t=0}^{ln-1} u(i,j+t)} \\ &\leqslant \frac{1}{(\frac{1}{2})^{|j|+1} \cdot 2^{ln-|j|-1}} \to 0 \text{ as } n \to \infty, \end{aligned}$$

$$\begin{aligned} \left| \prod_{t=1}^{ln} u_l(\varphi^{-t}(i,j)) \right| &= \prod_{t=1}^{ln} u(i,j-t) \\ &\leqslant 2^{|j|} \cdot \left(\frac{1}{2}\right)^{ln-|j|} \to 0 \text{ as } n \to \infty \end{aligned}$$

and for $s=1$, $l=2$ we have

$$\left|\frac{\prod_{t=1}^{sn} u_s(\varphi^{ln-t}(i,j))}{\prod_{t=0}^{ln-1} u_l(\varphi^{t}(i,j))}\right| = \frac{1}{\prod_{t=0}^{n-1} u(i,j+t)} \to 0 \text{ as } n\to\infty$$

and

$$\left|\frac{\prod_{t=1}^{ln} u_l(\varphi^{sn-t}(i,j))}{\prod_{t=0}^{sn-1} u_s(\varphi^{t}(i,j))}\right| = \prod_{t=1}^{n} u(i,j-t) \to 0 \text{ as } n\to\infty.$$

EXAMPLE 2.8. Let $G=\mathbb{C}$. The continuous injective mapping φ and positive continuous function ω on G are defined by

$$\varphi(re^{i\theta}) = re^{-i\theta}+1 \quad \text{for } re^{i\theta}\in\mathbb{C}, \quad \omega(re^{i\theta}) = \left(\frac{1}{2}\right)^{r} \quad \text{for } re^{i\theta}\in\mathbb{C}.$$

Let u be a weight on G defined by

$$u(x) = \begin{cases} 2 & \text{if } Re\, x > 1, \\ 2^{Re\, x} & \text{if } -1 \leqslant Re\, x \leqslant 1, \\ \frac{1}{2} & \text{if } Re\, x < -1. \end{cases}$$

Let $u_1=u_2=u$ and $T_{u_i,\varphi}$ $(i=1,2)$ be weighted translations on $L^p(G,\omega)$ $(1\leqslant p<\infty)$ induced by φ and u_i $(i=1,2)$. Then $T_{u_1,\varphi}, T^2_{u_2,\varphi}$ satisfy condition (2) in Theorem 2.5. Indeed, let K be any compact subset of G with $\lambda(K)>0$. Then by the definition of φ, there is a positive integer N_K such that for any integer n with $n>N_K$ and any $re^{i\theta}\in K$ we have $Re\,(\varphi^n(re^{i\theta}))>1$ and $Re\,(\varphi^{-n}(re^{i\theta}))<-1$. Since K is compact, there is a positive integer r_0 such that, $\sup_{x\in K}|x|<r_0$. Hence, for any integers l,n with $1\leqslant l\leqslant 2$ and $n>N_K+1$ we have

$$\begin{aligned}\left\|\left|\frac{\omega\circ\varphi^{ln}}{\prod_{t=0}^{ln-1} u_l\circ\varphi^t}\right|_K\right\|_\infty &= \sup_{re^{i\theta}\in K}\left|\frac{\omega(re^{(-1)^{ln}i\theta}+ln)}{\prod_{t=0}^{ln-1} u(re^{(-1)^t i\theta}+t)}\right| \\ &\leqslant \frac{(\frac{1}{2})^{ln-r_0}}{(\frac{1}{2})^{N_K+1}\cdot 2^{ln-N_K-1}} \to 0 \quad \text{as } n\to\infty\end{aligned}$$

and

$$\begin{aligned}\left\|\left(\omega\circ\varphi^{-ln}\right)\cdot\prod_{t=1}^{ln} u_l\circ\varphi^{-t}\Big|_K\right\|_\infty &= \sup_{re^{i\theta}\in K}\left|\omega(re^{(-1)^{ln}i\theta}-ln)\prod_{t=1}^{ln} u(re^{(-1)^t i\theta}-t)\right| \\ &\leqslant \left(\frac{1}{2}\right)^{ln-r_0}\cdot 2^{N_K}\cdot\left(\frac{1}{2}\right)^{ln-N_K} \to 0 \text{ as } n\to\infty.\end{aligned}$$

And for $s = 1$, $l = 2$ we have

$$\left\| \left. \frac{\left(\omega \circ \varphi^{(l-s)n}\right) \cdot \left(\prod_{t=1}^{sn} u_s \circ \varphi^{ln-t}\right)}{\prod_{t=0}^{ln-1} u_l \circ \varphi^t} \right|_K \right\|_\infty = \sup_{re^{i\theta} \in K} \left| \frac{\omega(re^{(-1)^n i\theta} + n)}{\prod_{t=0}^{n-1} u(re^{(-1)^t i\theta} + t)} \right|$$
$$\to 0 \text{ as } n \to \infty$$

and

$$\left\| \left. \frac{\left(\omega \circ \varphi^{(s-l)n}\right) \cdot \left(\prod_{t=1}^{ln} u_l \circ \varphi^{sn-t}\right)}{\prod_{t=0}^{sn-1} u_s \circ \varphi^t} \right|_K \right\|_\infty$$
$$= \sup_{re^{i\theta} \in K} \left| \omega(re^{(-1)^n i\theta} - n) \prod_{t=1}^{n} u(re^{(-1)^t i\theta} - t) \right| \to 0 \text{ as } n \to \infty.$$

REFERENCES

[1] S. I. ANSARI, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal. **148** (2) (1997), 384–390.

[2] F. BAYART AND É. MATHERON, *Dynamics of Linear Operators*, Cambridge University Press, Cambridge, 2009.

[3] L. BERNAL-GONZÁLEZ, *Disjoint hypercyclic operators*, Studia Math. **182** (2) (2007), 113–131.

[4] J. BÈS AND Ö. MARTIN, *Compositional disjoint hypercyclicity equals disjoint supercyclicity*, Houston J. Math. **38** (4) (2012), 1149–1163.

[5] J. BÈS, Ö. MARTIN AND R. SANDERS, *Weighted shifts and disjoint hypercyclicity*, J. Oper. Theory, **72** (1) (2014), 15–40.

[6] J. BÈS, Ö. MARTIN AND A. PERIS, *Disjoint hypercyclic linear fractional composition operators*, J. Math. Anal. Appl. **381** (2011), 843–856.

[7] J. BÈS, Ö. MARTIN, A. PERIS AND S. SHKARIN, *Disjoint mixing operators*, J. Funct. Anal. **263** (5) (2012), 1283–1322.

[8] J. BÈS AND A. PERIS, *Disjointness in hypercyclicity*, J. Math. Anal. Appl. **336** (2007), 297–315.

[9] G. D. BIRKHOFF, *Surface transformations and their dynamical applications*, Acta Math. **43** (1) (1922), 1–119.

[10] C. C. CHEN AND C. H. CHU, *Hypercyclicity of weighted convolution operators on homogeneous spaces*, Proc. Amer. Math. Soc. **137** (2009), 2709–2718.

[11] C. C. CHEN AND C. H. CHU, *Hypercyclic weighted translations on groups*, Proc. Amer. Math. Soc. **139** (2011), 2839–2846.

[12] C. C. CHEN, *Hypercyclic weighted translations generated by non-torsion elements*, Arch. Math. **101** (2013), 135–141.

[13] C. C. CHEN, *Disjoint hypercyclic weighted translations on groups*, Banach J. Math. Anal. **11** (3) (2017), 459–476.

[14] C. DONALD L, *Measure Theory*, Birkhäuser Boston, 1980.

[15] K.-G. GROSSE-ERDMANN AND A. PERIS MANGUILLOT, *Linear Chaos*, Universitext, Springer, New York, 2011.

[16] S. A. HAN AND Y. X. LIANG, *Disjoint hypercyclic weighted translations generated by aperiodic elements*, Collect. Math. **67** (3) (2016), 347–356.

[17] C. KITAI, *Invariant closed sets for linear operators*, PhD thesis, Univ. of Toronto, 1982.

[18] Ö. MARTIN, *Disjoint hypercyclic and supercyclic composition operators*, PhD thesis, Bowling Green State University, 2010.

[19] H. SALAS, *Hypercyclic weighted shifts*, Trans. Amer. Math. Soc. **347** (1) (1995), 993–1004.
[20] H. SALAS, *Dual disjoint hypercyclic weighted shifts*, J. Math. Anal. Appl. **374** (1) (2011), 106–117.
[21] R. SANDERS AND S. SHKARIN, *Existence of disjoint weakly mixing operators that fail to satisfy the Disjoint Hypercyclicity Criterion*, J. Math. Anal. Appl. **417** (2) (2014), 834–855.
[22] S. SHKARIN, *A short proof of existence of disjoint hypercyclic operators*, J. Math. Anal. Appl. **367** (2) (2010), 713–715.
[23] Y. WANG AND Z. H. ZHOU, *Disjoint hypercyclic weighted pseudo-shifts on Banach sequence spaces*, Collectanea Mathematica, **69** (3) (2018), 437–449.
[24] L. ZHANG, H. Q. LU, X. M. FU AND Z. H. ZHOU, *Disjoint hypercyclic powers of weighted translations on groups*, Czechoslovak Mathematical Journal, **67** (3) (2017), 839–853.

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Ya Wang
Department of Mathematics
Tianjin university of finance and economics
Tianjin 300222, P. R. China
e-mail: wangyasjxsy0802@163.com

Hong-Gang Zeng
School of Mathematics
Tianjin University
Tianjin 300354, P. R. China
e-mail: zhgng@tju.edu.cn