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(n,k)–QUASI CLASS Q AND (n,k)–QUASI CLASS Q^* WEIGHTED COMPOSITION OPERATORS

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Abstract. Let T be a bounded linear operator on a complex Hilbert space H. An operator T is called (n,k)-quasi class Q if it satisfies

$$\|T(T^k x)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{1+n}(T^k x)\|^2 + n\|T^k x\|^2\right),$$

and (n,k)-quasi class Q^* if it satisfies

$$\|T^*(T^k x)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{1+n}(T^k x)\|^2 + n\|T^k x\|^2\right),$$

for all $x \in H$ and for some nonnegative integers n and k.

In this paper, we will be studying the conditions under which composition operators and weighted composition operators on $L^2(\mu)$ spaces become (n,k)-quasi class Q operators and (n,k)-quasi class Q^* operators have been obtained in terms of Radon-Nikodym derivative h_m. Some necessary and sufficient conditions for a composition operator C_ϕ on Fock Spaces to be a (n,k)-quasi class Q operators and (n,k)-quasi class Q^* operators have also been explored.

1. Introduction

Throughout this paper, let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $L(H)$ denote the C^* algebra of all bounded operators on H. For $T \in L(H)$, we denote by $\ker(T)$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint.

We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all positive integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$. An operator $T \in L(H)$ is a positive operator, $T \geqslant O$, if $\langle Tx, x\rangle \geqslant 0$ for all $x \in H$.

An operator $T \in L(H)$, is said to be paranormal [9], if $\|Tx\|^2 \leqslant \|T^2x\|$ for any unit vector x in H. An operator $T \in L(H)$, is said to be $*$-paranormal [2], if $\|T^*x\|^2 \leqslant \|T^2x\|$ for any unit vector x in H.

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Mecheri, [18] introduced a new class of operators called k-quasi paranormal operators. An operator T is called k-quasi paranormal if

$$\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where k is a nonnegative integer number. Hoxha and Braha, [10] introduced a new class of operators called k-quasi-$*$-paranormal operators. An operator T is called k-quasi-$*$-paranormal if

$$\|T^*T^kx\|^2 \leqslant \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where k is a nonnegative integer number.

DEFINITION 1.1. [11] An operator T is said to be of the (n,k)-quasi class Q if

$$\|T(T^kx)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{1+n}(T^kx)\|^2 + n\|T^kx\|^2\right),$$

for all $x \in H$ and for some nonnegative integers n and k.

A $(1,k)$-quasi class Q operator is a k-quasi class Q operator, [13]:

$$\|T^{k+1}x\|^2 \leqslant \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^kx\|^2\right);$$

$(1,1)$-quasi class Q operator is a quasi class Q operator: $\|T^2x\|^2 \leqslant \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$; $(1,0)$-quasi class Q operator is a class Q operator, [7]: $\|Tx\|^2 \leqslant \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$; $(n,0)$-quasi class Q operator is a n-class Q operator, [20]:

$$\|Tx\|^2 \leqslant \frac{1}{n+1}(\|T^{1+n}x\|^2 + n\|x\|^2).$$

THEOREM 1.2. [11] *An operator $T \in L(H)$ is of the (n,k)-quasi class Q, if and only if*

$$T^{*k}\left(T^{*(n+1)}T^{n+1} - (n+1)T^*T + nI\right)T^k \geqslant O,$$

where k and n are nonnegative integer numbers.

An operator $T \in L(H)$ is said to be (n,k)-quasi paranormal operators if

$$\|T(T^kx)\| \leqslant \|T^{1+n}(T^kx)\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{n+1}},$$

for all $x \in H$, [21].

In [11] the authors have proved the following Lemma:

LEMMA 1.3. *If T is an (n,k)-quasi paranormal operator, then T is an (n,k)-quasi class Q operator.*

DEFINITION 1.4. [12] An operator $T \in L(H)$ is said to be (n,k)-quasi class Q^* if

$$\|T^*(T^k x)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{n+1}(T^k x)\|^2 + n\|T^k x\|^2\right),$$

for all $x \in H$ and for some nonnegative integer numbers n and k.

A $(1,k)$-quasi class Q^* operator is a k-quasi class Q^* operator, [14]:

$$\|T^*(T^k x)\|^2 \leqslant \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^k x\|^2\right);$$

a $(1,1)$-quasi class Q^* operator is a quasi class Q^* operator: $\|T^*(Tx)\|^2 \leqslant \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$; a $(1,0)$-quasi class Q^* operator is a class Q^* operator: $\|T^*x\|^2 \leqslant \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$; an $(n,0)$-quasi class Q^* operator is an n-class Q^* operator, [20]:

$$\|T^*x\|^2 \leqslant \frac{1}{n+1}(\|T^{n+1}x\|^2 + n\|x\|^2).$$

THEOREM 1.5. [12] *An operator $T \in L(H)$ is of the (n,k)-quasi class Q^*, if and only if*

$$T^{*k}\left(T^{*(n+1)}T^{n+1} - (n+1)TT^* + nI\right)T^k \geqslant O,$$

where k and n are nonnegative integer numbers.

An operator $T \in L(H)$ is said to be (n,k)-quasi-$*$-paranormal operators if

$$\|T^*(T^k x)\| \leqslant \|T^{n+1}(T^k x)\|^{\frac{1}{n+1}} \|T^k x\|^{\frac{n}{n+1}},$$

for all $x \in H$ and for some nonnegative integers n and k, [22].

In [12] the authors have proved the following Lemma:

LEMMA 1.6. *If T is an (n,k)-quasi-$*$-paranormal operator, then T is an (n,k)-quasi class Q^* operator.*

Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space. The space $L^2(X,\mathscr{A},\mu) := L^2(\mu)$ is defined as

$$L^2(\mu) = \{f : X \to \mathbb{C} : f \text{ is a measurable function and } \int_X |f|^2 d\mu < \infty\}.$$

A transformation $T : X \to X$ is said to be measurable if $T^{-1}(B) \in \mathscr{A}$ for $B \in \mathscr{A}$. If T is a measurable transformation, then T^m is also a measurable transformation for all natural numbers m. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(B)) = 0$ whenever $\mu(B) = 0$ for every $B \in \mathscr{A}$. Let T be a measurable transformation on X. The composition operator $C_T : L^2(\mu) \to L^2(\mu)$ is given by $C_T f = f \circ T$ for $f \in L^2(\mu)$. For $m \in \mathbb{N}$ we have

$$C_T^m f = f \circ T^m \text{ for } f \in L^2(\mu).$$

If T is non-singular, then we say that $\mu \circ T^{-1}$ is absolutely continuous with respect to μ. Hence by the Radon-Nikodym theorem exists a unique non-negative essentially bounded measurable function h such that

$$\mu \circ T^{-1}(B) = \int_B h d\mu \text{ for } B \in \mathscr{A}.$$

The function h is called the Radon-Nikodym derivative and we have $h = h_1 := \frac{d\mu \circ T^{-1}}{d\mu}$. In addition, we assume that h is almost everywhere finite valued or equivalently $T^{-1}(\mathscr{A}) \subset \mathscr{A}$ is a sub-sigma finite algebra.

The Radom-Nikodym derivative of the measure $\mu \circ (T^{-1})^m$ with respect to μ is denoted by h_m and we have

$$\mu \circ (T^{-1})^m(B) = \int_B h_m d\mu \text{ for } B \in \mathscr{A} \text{ and } h_m := \frac{d\mu \circ (T^{-1})^m}{d\mu}.$$

It can be seen that

$$h_m = h \cdot h \circ T^{-1} \cdot h \circ T^{-2} \cdot \ldots \cdot h \circ T^{-(m-1)} \text{ and } h_m = h_{m-1} \cdot h \circ T^{-(m-1)}.$$

From [4, 16, Lemma 1] we have

$$C_T^* C_T f = hf \text{ and } C_T C_T^* f = (h \circ T)Pf \text{ for all } f \in L^2(\mu), \tag{1.1}$$

where P is the projection from $L^2(\mu)$ onto the closure of the range of the composition operator C_T,

$$\overline{C_T(L^2(\mu))} = \{f \in L^2(\mu) : f \text{ is } T^{-1}(\mathscr{A}) \text{ measurable}\}.$$

If $T^{-1}(\mathscr{A}) \subset \mathscr{A}$, there exists an operator $E : L^p(\mathscr{A}) \to L^p(T^{-1}(\mathscr{A}))$ which is called conditional expectation operator. The conditional expectation operator $E(f|T^{-1}(\mathscr{A})) = E(f)$ is defined for each nonnegative function f in $L^p(1 \leqslant p < \infty)$ and is uniquely determined by the following set of conditions: $E(f)$ is $T^{-1}(\mathscr{A})$ measurable and if B is any $T^{-1}(\mathscr{A})$ measurable set for which $\int_B f d\mu$ converges, then we have

$$\int_B f d\mu = \int_B E(f) d\mu.$$

For $m \geqslant 2$ let $E_m = E(f|T^{-m}(\mathscr{A}))$. The conditional expectation operator E has the following properties:

1. $E(g) = g$ if and only if g is $T^{-1}(\mathscr{A})$ measurable, [17].
2. If g is $T^{-1}(\mathscr{A})$ measurable, then $E(fg) = E(f)g$.
3. $E(f \cdot g \circ T) = (E(f))(g \circ T)$ and $E(E(f)g) = E(f)E(g)$ for $f, g \in L^2(\mu)$.
4. If $f \leqslant g$ a.e., then $E(f) \leqslant E(g)$ a.e., for $f, g \in L^2(\mu)$.
5. $E(1) = 1$, and E is the identity operator in $L^2(\mu)$ if and only if $T^{-1}(\mathscr{A}) = \mathscr{A}$.

6. $E(f)$ has the form $E(f) = g \circ T$ for exactly one $\mathscr{A}$-measurable function g provided that the support of g lies in the support of h which is given by $\sigma(h) = \{x : h(x) \neq 0\}$.

7. E is the projection operator from $L^2(\mu)$ onto $\overline{C_T(L^2(\mu))}$. So, as an operator on $L^2(\mu)$, E is the projection P used in relation (1.1), [4].

A detailed discussion and verification of most of these properties may be found in [19].

The adjoint C_T^* of C_T is given by $C_T^* f = hE(f) \circ T^{-1}$, [4]. For $m \in \mathbb{N}$ and for $f \in L^2(\mu)$, we have

$$C_T^{*m} f = h_m E(f) \circ T^{-m}.$$

2. On (n,k)-quasi class Q and (n,k)-quasi class Q^* composition operators on $L^2(\mu)$ space

THEOREM 2.1. *Let C_T be the composition operator induced by T on $L^2(\mu)$. Then, the following statements are equivalent:*

1. The operator C_T is of (n,k)-quasi class Q

2.
$$h_{n+k+1} - (n+1)h_{k+1} + nh_k \geqslant 0. \tag{2.1}$$

Proof. Let C_T be the composition operator induced by T on $L^2(\mu)$. By Theorem 1.2, the operator C_T is of (n,k)-quasi class Q if and only if

$$\left\langle C_T^{*(n+k+1)} C_T^{n+k+1} f - (n+1) C_T^{*(k+1)} C_T^{k+1} f + n C_T^{*k} C_T^k f, f \right\rangle \geqslant O. \tag{2.2}$$

For every $f \in L^2(\mu)$, we have

$$C_T^{*(n+k+1)} C_T^{n+k+1} f = C_T^{*(n+k+1)} (f \circ T^{n+k+1})$$

$$= h_{n+k+1} E(f \circ T^{n+k+1}) \circ T^{-(n+k+1)} = h_{n+k+1} f. \tag{2.3}$$

From above relation and relation (2.2) we have

$$\langle h_{n+k+1} f - (n+1) h_{k+1} f + n h_k f, f \rangle \geqslant O.$$

Hence, relation (2.1) is proved. □

THEOREM 2.2. *Let C_T be the composition operator induced by T on $L^2(\mu)$. Then, the following statement are uquivalent:*

1. Operator C_T^* is of (n,k)-quasi class Q

2.
$$h_{n+k+1}\circ T^{(n+k+1)}-(n+1)h_{k+1}\circ T^{(k+1)}+nh_k\circ T^k\geqslant 0. \tag{2.4}$$

Proof. Let C_T be the composition operator induced by T on $L^2(\mu)$. By Theorem 1.2, the operator C_T^* is of (n,k)-quasi class Q if and only if

$$\left\langle C_T^{n+k+1}C_T^{*(n+k+1)}f-(n+1)C_T^{k+1}C_T^{*(k+1)}f+nC_T^kC_T^{*k}f,f\right\rangle\geqslant O. \tag{2.5}$$

For every $f\in L^2(\mu)$, we have

$$\begin{aligned}
C_T^{n+k+1}C_T^{*(n+k+1)}f &= C_T^{n+k+1}\left(h_{n+k+1}E(f)\circ T^{-(n+k+1)}\right) && (2.6)\\
&= \left(h_{n+k+1}\circ T^{(n+k+1)}E(f)\circ T^{-(n+k+1)}\right)\circ T^{n+k+1} && (2.7)\\
&= h_{n+k+1}\cdot\circ T^{(n+k+1)}E(f). && (2.8)
\end{aligned}$$

Let $u_{h,T}=h_{n+k+1}\circ T^{n+k+1}-(n+1)h_{k+1}\circ T^{k+1}+nh_k\circ T^k$. Since $u_{h,T}$ is a $T^{-1}(\Sigma)$-measurable function, then E commutes with $M_{u_{h,T}}$. Also, we know that E is a positive operator. By these observations and the relation (2.5) we have

$$\left\langle u_{h,T}.f,f\right\rangle\geqslant O,$$

and we have proved relation (2.4). □

THEOREM 2.3. *Let C_T be the composition operator induced by T on $L^2(\mu)$. Then, the following statement are equivalent:*

1. *Operator C_T is of (n,k)-quasi class Q^**

2.
$$h_{n+k+1}-(n+1)h_k\cdot h\circ T^{1-k}+nh_k\geqslant 0. \tag{2.9}$$

Proof. By Theorem 1.5, the operator C_T is of (n,k)-quasi class Q^* if and only if

$$\left\langle C_T^{*(n+k+1)}C_T^{n+k+1}f-(n+1)C_T^{*k}(C_TC_T^*)C_T^kf+nC_T^{*k}C_T^kf,f\right\rangle\geqslant O. \tag{2.10}$$

For every $f\in L^2(\mu)$, we have

$$\begin{aligned}
C_T^{*k}(C_TC_T^*)C_T^kf &= C_T^{*k}(C_TC_T^*)(f\circ T^k)=C_T^{*k}(h\circ T)E(f\circ T^k)\\
&= h_kE\left((h\circ T)E(f\circ T^k)\right)\circ T^{-k}\\
&= h_kE(h\circ T)\circ T^{-k}f\\
&= h_k\cdot h\circ T^{1-k}\cdot f.
\end{aligned}$$

From above relation and relation (2.3) we get

$$\left\langle h_{n+k+1}f-(n+1)h_k\cdot h\circ T^{1-k}\cdot f+n\cdot h_kf,f\right\rangle \geqslant O.$$

This proves relation (2.9). □

THEOREM 2.4. *Let C_T be the composition operator induced by T on $L^2(\mu)$. Then, the following statement are equivalent:*

1. *Operator C_T^* is of (n,k)-quasi class Q^**

2. $$h_{n+k+1}\circ T^{n+k+1}-(n+1)h\circ T^k\cdot h_k\circ T^k+nh_k\circ T^k\geqslant 0. \tag{2.11}$$

Proof. The operator C_T^* is of (n,k)-quasi class Q^* if and only if

$$\left\langle C_T^{n+k+1}C_T^{*(n+k+1)}f-(n+1)C_T^k\left(C_T^*C_T\right)C_T^{*k}f+nC_T^kC_T^{*k}f,f\right\rangle \geqslant O. \tag{2.12}$$

For every $f\in L^2(\mu)$, we have

$$\begin{aligned}C_T^k\left(C_T^*C_T\right)C_T^{*k}f &= C_T^k(C_T^*C_T)\left(h_kE(f)\circ T^{-k}\right)\\ &= C_T^k(h\cdot h_kE(f)\circ T^{-k})\\ &= h\circ T^k\cdot h_k\circ T^k\cdot E(f).\end{aligned}$$

By the same reasons that we mentioned in the proof of Theorem 2.2 and the relation (2.8) we obtain

$$\left\langle h_{n+k+1}\circ T^{n+k+1}\cdot f-(n+1)h\circ T^k\cdot h_k\circ T^k\cdot f+nh_k\circ T^k\cdot f,f\right\rangle \geqslant O. \quad \square$$

3. On (n,k)-quasi class Q and (n,k)-quasi class Q^* weighted composition operators on $L^2(\mu)$ space

Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space and let u be a complex-valued measurable function. Then the weighted composition operator $W_{u,T}$ on the space $L^2(\mu)$ induced by u and a measurable transformation T is given by

$$W_{u,T}f=Wf=u\cdot f\circ T \text{ for } f\in L^2(\mu),$$

and we have $W^mf=u_m\cdot f\circ T^m$ where m is any number natural and

$$u_m=u\cdot u\circ T\cdot u\circ T^2\cdot\ldots\cdot u\circ T^{(m-1)} \text{ and } u_m=u_{m-1}\cdot u\circ T^{(m-1)}.$$

The adjoint W^* of W is given by $W^*f=h\cdot E(\bar{u}\cdot f)\circ T^{-1}$, and $f\in L^2(\mu)$, we have

1. $W^{*m}f = h_m E(\bar{u_m} f) \circ T^{-m}$;
2. $(W^*W)f = W^*(Wf) = W^*(uf \circ T) = hE(\bar{u}u \cdot f \circ T) \circ T^{-1} = hE(|u|^2) \circ T^{-1} f$ $= Jf$;
3. $(WW^*)f = W(hE(\bar{u} \cdot f) \circ T^{-1}) = \left(uh \cdot E(\bar{u}f) \circ T^{-1}\right) \circ T = u(h \circ T)E(\bar{u}f)$;
4. $W^{*m}W^m f = h_m E(|u_m|^2) \circ T^{-m} f$;
5. $W^m W^{*m} f = u_m(h_m \circ T^m)E(\bar{u_m} f)$;

where $J_m = h_m E(|u_m|^2) \circ T^{-m}$ $(J_1 = J)$ and m is a natural number, (see [5], [8], [15]).

THEOREM 3.1. *Let W be a weighted composition operator on $L^2(\mu)$. Then, W is an (n,k)-quasi class Q operator if and only if*

$$J_{n+k+1} \cdot f - (n+1)J_{k+1} \cdot f + n \cdot J_k \cdot f \geqslant O. \tag{3.1}$$

Proof. By Theorem 1.2, the operator W is of (n,k)-quasi class Q if and only if

$$\langle (W^{*(n+k+1)}W^{n+k+1} - (n+1)W^{*(k+1)}W^{k+1} + nW^{*k}W^k)f, f\rangle \geqslant O. \tag{3.2}$$

For every $f \in L^2(\mu)$ we have

$$W^{*(n+k+1)}W^{n+k+1} f = J_{n+k+1} \cdot f. \tag{3.3}$$

From relations (3.1) and (3.2) we get

$$\langle J_{n+k+1} \cdot f - (n+1)J_{k+1} \cdot f + n \cdot J_k \cdot f, f\rangle \geqslant O.$$

Hence, it is proved relation (3.1). $\square$

THEOREM 3.2. *Let W be a weighted composition operator on $L^2(\mu)$. Then, W^* is an (n,k)-quasi class Q operator if and only if*

$$\begin{aligned}\langle u_{n+k+1}h_{n+k+1} \circ T^{n+k+1}E(\bar{u_{n+k+1}} f) - (n+1)u_{k+1}h_{k+1} \circ T^{k+1}E(\bar{u_{k+1}} f)\\ +nu_k h_k \circ T^k E(\bar{u_k} f), f\rangle \geqslant O.\end{aligned} \tag{3.4}$$

Proof. By Theorem 1.2, the operator W^* is of (n,k)-quasi class Q if and only if

$$\langle (W^{n+k+1}W^{*(n+k+1)} - (n+1)W^{k+1}W^{*(k+1)} + nW^k W^{*k})f, f\rangle \geqslant O. \tag{3.5}$$

For every $f \in L^2(\mu)$ we have

$$W^{n+k+1}W^{*(n+k+1)} f = u_{n+k+1} \cdot h_{n+k+1} \circ T^{n+k+1} \cdot E(\bar{u_{n+k+1}} \cdot f) \tag{3.6}$$

From relations (3.6) and (3.5) we obtain

$$\begin{aligned}\langle u_{n+k+1}h_{n+k+1} \circ T^{n+k+1}E(\bar{u_{n+k+1}} f) - (n+1)u_{k+1}h_{k+1} \circ T^{k+1}E(\bar{u_{k+1}} f)\\ +nu_k h_k \circ T^k E(\bar{u_k} f), f\rangle \geqslant O,\end{aligned}$$

and this proves relation (3.2). $\square$

THEOREM 3.3. *Let W be a weighted composition operator on $L^2(\mu)$. Then, W is an (n,k)-quasi class Q^* operator if and only if*

$$\langle J_{n+k+1}\cdot f-(n+1)h_k\cdot|E(\bar{u}u_k)|^2\circ T^{-k}\cdot h\circ T^{1-k}f+n\cdot J_k\cdot f,f\rangle \geqslant O. \tag{3.7}$$

Proof. The operator W is of (n,k)-quasi class Q^* if and only if

$$\langle (W^{*(n+k+1)}W^{n+k+1}-(n+1)W^{*k}(WW^*)W^k+nW^{*k}W^k)f,f\rangle \geqslant O. \tag{3.8}$$

For every $f\in L^2(\mu)$ we have

$$\begin{aligned} W^{*k}(WW^*)W^k f &= W^{*k}(WW^*)(u_k\cdot f\circ T^k)\\ &= W^{*k}(u(h\circ T)E(\bar{u}\cdot u_k\cdot f\circ T^k))\\ &= h_k\cdot|E(\bar{u}u_k)|^2\circ T^{-k}\cdot h\circ T^{1-k}f. \end{aligned}\tag{3.9}$$

Now our result follows from relations (3.8) and relation (3.9). □

THEOREM 3.4. *Let W be a weighted composition operator on $L^2(\mu)$. Then, W^* is an (n,k)-quasi class Q^* operator if and only if*

$$\begin{aligned}\langle u_{n+k+1}\cdot h_{n+k+1}\circ T^{n+k+1}E(\bar{u}_{n+k+1}.f)-(n+1)u_kJ\circ T^k h_k\circ T^k E(\bar{u}_k.f)\\ +nu_k\cdot h_k\circ T^k E(\bar{u}_k.f),f\rangle \geqslant O.\end{aligned}\tag{3.10}$$

Proof. The operator W^* is of (n,k)-quasi class Q^* if and only if

$$\langle (W^{n+k+1}W^{*(n+k+1)}-(n+1)W^k(W^*W)W^{*k}+nW^kW^{*k})f,f\rangle \geqslant O. \tag{3.11}$$

For every $f\in L^2(\mu)$ we have

$$\begin{aligned} W^k(W^*W)W^{*k}f &= W^k(W^*W)\left(h_kE(u_kf)\circ T^{-k}\right)\\ &= W^k(Jh_kE(u_kf)\circ T^{-k})\\ &= u_kJ\circ T^k h_k\circ T^k E(\bar{u}_k.f). \end{aligned}\tag{3.12}$$

From relations (3.6) and relation (3.12) we get

$$\begin{aligned}\langle u_{n+k+1}\cdot h_{n+k+1}\circ T^{n+k+1}E(\bar{u}_{n+k+1}.f)-(n+1)u_kJ\circ T^k h_k\circ T^k E(\bar{u}_k.f)\\ +nu_k\cdot h_k\circ T^k E(\bar{u}_k.f),f\rangle \geqslant O. \quad \square\end{aligned}$$

In what follows we will give a necessary and sufficient condition for operator W to be an (n,k)-quasi class Q operator, using into consideration support of the measurable function u, which we denote by $\sigma(u)=\{x\in X : u(x)\neq 0\}$ and expectation . As it is known, for every $f\in L^2$, $\int_X|E(f)|^2d\mu \leqslant \int_X|f|^2d\mu$ and $\sigma(f)\subseteq\sigma(|f|)$.

LEMMA 3.5. (see [3]) *Let α and β be non-negative functions. Then the following condition are equivalent:*

1. For every $f \in L^2(\mu)$,

$$\int\limits_X \alpha |f|^2 d\mu \geqslant \int\limits_X |E(\beta f)|^2 d\mu;$$

2. $\sigma(\beta) \subset \sigma(\alpha)$ *and* $E\left(\frac{\beta^2}{\alpha}\chi_{\sigma(\alpha)}\right) \leqslant 1$ *almost everywhere.*

THEOREM 3.6. *Let* W *be a weighted composition operator on* $L^2(\mu)$. *If* W *is an* (n,k)*-quasi class* Q *operator, then*

$$\sigma\left(J_{k+1}^{\frac{1}{2}}\right) \subset \sigma\left(J_{n+k+1} + nJ_k\right)$$

and

$$E\left(\frac{J_{k+1}}{J_{n+k+1} + nJ_k}\chi_{J_{n+k+1}+nJ_k}\right) \leqslant 1.$$

Proof. For every $f \in L^2(\mu)$, W is an (n,k)-quasi class Q operator if

$$\langle (W^{*(n+k+1)}W^{n+k+1} - (n+1)W^{*(k+1)}W^{k+1} + nW^{*k}W^k)f, f\rangle \geqslant O.$$

Respectively

$$\langle (W^{*(n+k+1)}W^{n+k+1} + nW^{*k}W^k)f, f\rangle \geqslant \langle (n+1)W^{*(k+1)}W^{k+1}f, f\rangle.$$

By definition we get

$$\langle W^{*(n+k+1)}W^{n+k+1}f, f\rangle = \int_X J_{n+k+1}|f|^2 d\mu. \tag{3.13}$$

On the other hand

$$\begin{aligned} \langle (n+1)W^{*(k+1)}W^{k+1}f, f\rangle &= (n+1)\int_X J_{k+1}|f|^2 d\mu \\ \geqslant \int_X J_{k+1}|E(f)|^2 d\mu &= \int_X |E(J_{k+1}^{\frac{1}{2}}f)|^2 d\mu. \end{aligned} \tag{3.14}$$

From relations (3.13), (3.14) and Lemma 3.5, we obtain that if W is an (n,k)-quasi class Q operator, then

$$\sigma\left(J_{k+1}^{\frac{1}{2}}\right) \subset \sigma\left(J_{n+k+1} + nJ_k\right)$$

and

$$E\left(\frac{J_{k+1}}{J_{n+k+1} + nJ_k}\chi_{J_{n+k+1}+nJ_k}\right) \leqslant 1. \quad \square$$

THEOREM 3.7. *Let W be a weighted composition operator on $L^2(\mu)$. If W is an (n,k)-quasi class Q^* operator, then*

$$\sigma\left(s_k^{\frac{1}{2}}h^{\frac{1}{2}}\circ T^{1-k}\right)\subset\sigma\left(J_{n+k+1}+nJ_k\right)$$

and $E\left(\frac{s_k h\circ T^{1-k}}{J_{n+k+1}+nJ_k}\chi_{J_{n+k+1}+nJ_k}\right)\leqslant 1.$

Proof. For every $f\in L^2(\mu)$, W is an (n,k)-quasi class Q^* operator if and only if

$$\langle(W^{*(n+k+1)}W^{n+k+1}-(n+1)W^{*k}(WW^*)W^k+nW^{*k}W^k)f,f\rangle\geqslant O. \tag{3.15}$$

Respectively

$$\langle(W^{*(n+k+1)}W^{n+k+1}+nW^{*k}W^k)f,f\rangle\geqslant\langle(n+1)W^{*k}(WW^*)W^kf,f\rangle.$$

By definition we get

$$\langle W^{*(n+k+1)}W^{n+k+1}f,f\rangle=\int_X h_{n+k+1}\cdot u_{n+k+1}^2|f|^2d\mu. \tag{3.16}$$

On the other hand

$$\begin{aligned}\langle(n+1)W^{*k}(WW^*)W^kf,f\rangle&=(n+1)\int_X h_k\cdot|E(\bar{u}u_k)|^2\circ T^{-k}\cdot h\circ T^{1-k}f\bar{f}d\mu\\&\geqslant(n+1)\int_X|E(s_k^{\frac{1}{2}}h^{\frac{1}{2}}\circ T^{1-k}f)|^2d\mu\\&\geqslant\int_X|E(s_k^{\frac{1}{2}}h^{\frac{1}{2}}\circ T^{1-k}f)|^2d\mu,\end{aligned}$$

in which $s_k=h_k\cdot|E(\bar{u}u_k)|^2\circ T^{-k}$. By these observations we get that if W is an (n,k)-quasi class Q^* operator, then

$$\sigma\left(s_k^{\frac{1}{2}}h^{\frac{1}{2}}\circ T^{1-k}\right)\subset\sigma\left(J_{n+k+1}+nJ_k\right)$$

and $E\left(\frac{s_k h\circ T^{1-k}}{J_{n+k+1}+nJ_k}\chi_{J_{n+k+1}+nJ_k}\right)\leqslant 1.$ □

Recall that the Althuge transformation of operator $A\in L(h)$, is the operator $\widetilde{A}$ defined as follows: $\widetilde{A}=|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. And for $0<r\leqslant 1$, it is defined $\widetilde{A}_r=|A|^rU|A|^{1-r}$. In the next result we describe the (n,k)-quasi class Q operators via Althuge transformation.

THEOREM 3.8. *Let $T=U|T|$ be the polar decomposition of the operator T, and $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. If $|\widetilde{T}|\geqslant|T|$, then T is an (n,k)-quasi class Q operator.*

Proof. From given conditions and Theorem 2.2 in [1] it follows that T is an paranormal operator. Every paranormal operator is n-paranormal operator and it is (n,k)-quasiparanormal operator ([21]). From Lemma 1.3 it follows that T is an (n,k)-quasi class Q operator. $\square$

THEOREM 3.9. *Let $T=U|T|$ be the polar decomposition of the operator T, and $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. If $\|\widetilde{T}^{\frac{1}{2}}|T|^{\frac{1}{2}}x\| \geqslant |||T|x||^2\cdot||T^*x||^2$, then T is an (n,k)-quasi class Q^* operator.*

Proof. First we prove that under given condition operator T is $*$-paranormal. For this we start from

$$
\begin{aligned}
||T^2x|| &= |||T|Tx|| \geqslant |||T|^{\frac{1}{2}}Tx||^2\cdot||Tx||^{-1}; \quad \text{from Lemma 2.1 in [1]}\\
&= ||\widetilde{T}|T|^{\frac{1}{2}}x||^2\cdot||Tx||^{-1}\\
&= |||\widetilde{T}||T|^{\frac{1}{2}}x||^2\cdot||Tx||^{-1}\\
&\geqslant |||\widetilde{T}|^{\frac{1}{2}}|T|^{\frac{1}{2}}x||^2\cdot|||T|x||^{-1}\cdot||Tx||^{-1}\\
&\geqslant |||T|x||^2\cdot||T^*x||^2\cdot|||T|x||^{-1}\cdot||Tx||^{-1}\\
&= ||T^*x||^2.
\end{aligned}
$$

Every $*$-paranormal operator is n-$*$-paranormal operator and it is (n,k)-$*$-quasi-paranormal operator ([22]). From Lemma 1.6 it follows that T is an (n,k)-quasi class Q^* operator. $\square$

In [3], are described the properties of the composition operators via Althuge transformation.

LEMMA 3.10. *For a weighted composition operator W we have the following entities:*

$$W_rf=\omega_r\cdot f\circ T, |W_r|f=\sqrt{h[E(\omega_r^2)]\circ T^{-1}f}$$

and

$$|W_r^*|f=P_{v_r}f=v_rE(v_rf),$$

where $\omega_r=u\left(\frac{J\chi_{\sigma(E(u))}}{h\circ TE(u^2)}\right)^{\frac{r}{2}}$, *and* $v_r=\frac{\omega_r\sqrt{h\circ T}}{\sqrt[4]{E([\omega_r\sqrt{h\circ T}]^2)}}$.

The next results characterized that W_r is an (n,k)-quasi class Q operator, via Althuge transformation.

THEOREM 3.11. *Let W be a weighted composition operator in $L^2(\mu)$. Then $|W_r|\geqslant|W|$ if and only if $E(\omega_r^2)\geqslant E(u^2)$.*

Proof. Proof of the Theorem follows directly from the above facts. $\square$

OPEN PROBLEM. The authors didn't know how to describe the weighted composition operator W_r, to be an (n,k)-quasi class Q^* operator, via Althuge transformation.

4. Examples

EXAMPLE 4.1. Let $X=[0,\pi]$, $d\mu=dx$ and Σ be the Lebesgue sets. Define the non-singular transformation $\varphi : X \to X$ by

$$\varphi(x)=\begin{cases} 2x & x\in[0,\frac{\pi}{2}];\\ 2x-1 & x\in(\frac{\pi}{2},1],\end{cases}$$

Easily we get that $h(x)=1$ and so $h_m(x)=1$, for each $m>0$. Thus C_φ is a bounded composition operator on $L^2(\Sigma)$. And so by Theorems 2.1, 2.2, 2.3 and 2.4 we get that C_φ and C_φ^* are of (n,k)-quasi class Q and (n,k)-quasi class $Q*$, for all $n,k\in\mathbb{N}$.

Moreover, for each $0\leqslant a<b\leqslant 1$ and $f\in L^2(\Sigma)$ we have

$$\begin{aligned}\int_{\varphi_1^{-1}(a,b)} f(x)dx &= \int_{\frac{a}{2}}^{\frac{b}{2}} f(x)dx+\int_{\frac{a+1}{2}}^{\frac{b+1}{2}} f(x)dx\\ &= \int_{(a,b)} \frac{1}{2}\left\{f\left(\frac{x}{2}\right)+f\left(\frac{1+x}{2}\right)\right\}dx.\end{aligned}$$

Hence

$$(E(f)\circ\varphi_1^{-1})(x)=\frac{1}{2}\left\{f\left(\frac{x}{2}\right)+f\left(\frac{1+x}{2}\right)\right\},$$

It follows that

$$E(f)(x)=\frac{1}{2}\left\{f(x)+f\left(\frac{1+2x}{2}\right)\right\}\chi_{[0,\frac{\pi}{2}]}+\frac{1}{2}\left\{f\left(\frac{2x-1}{2}\right)+f(x)\right\}\chi_{(\frac{\pi}{2},1]}.$$

Let

$$u(x)=\begin{cases} \sin(x) & x\in[0,\frac{\pi}{2}];\\ \sin(x-\frac{\pi}{2}) & x\in(\frac{\pi}{2},1],\end{cases}.$$

Direct computations show that u is $\varphi^{-1}(\Sigma)$-measurable. Since $h=1$, then we have $h_m=1$. Consequently, we get that $J_m(x)=u_m^2\circ T^{-m}$. Therefore by Theorem 3.1, we have that $W=uC_\varphi$ is an (n,k)-quasi class Q operator if and only if

$$u_{n+k+1}^2\circ T^{-(n+k+1)}\cdot f-(n+1)u_{k+1}^2\circ T^{-(k+1)}\cdot f+n\cdot u_k^2\circ T^{-k}\cdot f\geqslant O.$$

Also, by Theorem 3.3 we get that W is an (n,k)-quasi class Q^* operator if and only if

$$\langle u_{n+k+1}^2\circ T^{-(n+k+1)}\cdot f-(n+1)\cdot|u|^2\circ T^{-k}.|u_k|^2\circ T^{-k}.f+n\cdot u_k^2\circ T^{-k}\cdot f,f\rangle\geqslant O.$$

EXAMPLE 4.2. Let $X=[0,1]$, $d\mu=dx$ and Σ be the Lebesgue sets. Define the non-singular transformation $\varphi : X\to X$ by

$$\varphi(x)=\begin{cases} 1-2x & x\in[0,\frac{1}{2}];\\ 2x-1 & x\in(\frac{1}{2},1].\end{cases}$$

It is easy to see that $h(x) = 1$ and similar to example 4.1 we get that

$$\int_{\varphi^{-1}(a,b)} f(x)dx = \int_{\frac{1-b}{2}}^{\frac{1-a}{2}} f(x)dx + \int_{\frac{a+1}{2}}^{\frac{b+1}{2}} f(x)dx = \int_{(a,b)} \frac{1}{2}\left\{f\left(\frac{1-x}{2}\right) + f\left(\frac{1+x}{2}\right)\right\} dx.$$

Hence we have

$$(E(f)\circ\varphi_2^{-1})(x) = \frac{1}{2}\left\{f\left(\frac{1-x}{2}\right) + f\left(\frac{1+x}{2}\right)\right\}.$$

And so

$$E(f)(x) = \frac{1}{2}\{f(x)+f(1-x)\}\chi_{[0,\frac{1}{2}]} + \frac{1}{2}\{f(-x)+f(x)\}\chi_{(\frac{1}{2},1]}.$$

If we put $u(x) = 4(x+\frac{1}{2})$, then we have

$$J(x) = 2\{(2+x)^2 + (2-x)^2\}.$$

The operator W^*, by Theorem 3.2 is an (n,k)-quasi class Q operator if and only if

$$\langle u_{n+k+1}h_{n+k+1}\circ T^{n+k+1}E(u_{n+\bar{k}+1}f) - (n+1)u_{k+1}h_{k+1}\circ T^{k+1}E(u_{\bar{k}+1}f) \\ +nu_kh_k\circ T^kE(\bar{u_k}f), f\rangle \geqslant O.$$

And W^*, by Theorem 3.4 is an (n,k)-quasi class Q^* operator if and only if

$$\langle u_{n+k+1}\cdot h_{n+k+1}\circ T^{n+k+1}E(\bar{u}_{n+k+1}.f) - (n+1)u_kJ\circ T^kh_k\circ T^kE(\bar{u}_k.f) \\ +nu_k\cdot h_k\circ T^kE(\bar{u}_k.f), f\rangle \geqslant O.$$

EXAMPLE 4.3. Let $X = (0,a]$, $d\mu = dx$ and Σ be the Lebesgue sets. Define the non-singular transformation $\varphi : X\to X$ by $\varphi(x) = e^x$. Since $\varphi^{-1}(x) = \ln(x)$, then $h(x) = \frac{1}{x}$. Hence $\varphi^{-m}(x) = \ln\circ\ln\circ\ln\ldots.\circ\ln(x)$, m times, $h_2(x) = \frac{1}{x\ln(x)}$ and $h_3(x) = \frac{1}{x\ln(x).\ln\circ\ln(x)} = \frac{1}{x.\ln(x)\varphi^{-2}(x)}$. And so by induction we get that $h_m(x) = \frac{1}{x\ln(x).\varphi^{-m+1}(x)}$. Moreover, $\varphi^{-1}(\Sigma) = \Sigma$ and therefore $E = I$(identity operator). This implies that

$$J_m(x) = \frac{1}{x.\ln(x).\varphi^{-m+1}(x)}|u_m|\circ\varphi^{-m}(x).$$

Therefore by Theorem 3.1, we have that $W = uC_\varphi$ is an (n,k)-quasi class Q operator if and only if

$$u_{n+k+1}^2\circ T^{-(n+k+1)}\cdot f - (n+1)u_{k+1}^2\circ T^{-(k+1)}\cdot f + n\cdot u_k^2\circ T^{-k}\cdot f \geqslant O.$$

Also, by Theorem 3.3 we get that W is an (n,k)-quasi class Q^* operator if and only if

$$\langle u_{n+k+1}^2\circ T^{-(n+k+1)}\cdot f - (n+1)\cdot|u|^2\circ T^{-k}.|u_k|^2\circ T^{-k}.f + n\cdot u_k^2\circ T^{-k}\cdot f, f\rangle \geqslant O.$$

The operator W^*, by Theorem 3.2 is an (n,k)-quasi class Q operator if and only if

$$\langle u_{n+k+1}h_{n+k+1}\circ T^{n+k+1}E(u_{n+\bar{k}+1}f)-(n+1)u_{k+1}h_{k+1}\circ T^{k+1}E(u_{\bar{k}+1}f) \\ +nu_kh_k\circ T^kE(\bar{u_k}f),f\rangle \geqslant O.$$

And W^*, by Theorem 3.4 is an (n,k)-quasi class Q^* operator if and only if

$$\langle u_{n+k+1}\cdot h_{n+k+1}\circ T^{n+k+1}E(\bar{u}_{n+k+1}.f)-(n+1)u_kJ\circ T^kh_k\circ T^kE(\bar{u}_k.f) \\ +nu_k\cdot h_k\circ T^kE(\bar{u}_k.f),f\rangle \geqslant O.$$

5. On (n,k)-quasi class Q and (n,k)-quasi class Q^* composition on Fock-spaces

Let $z=(z_1,z_2,\dots,z_m)$ and $w=(w_1,w_2,\dots,w_m)$ be point in $\mathbb{C}^m$, $\langle z,w\rangle=\sum_{k=1}^m z_k\overline{w_k}$ and $|z|=\sqrt{\langle z,z\rangle}$. The Fock space $\mathscr{F}_m^2$ is the Hilbert space of all holomorphic functions on $\mathbb{C}^m$ (entire functions) with inner product

$$\langle f,g\rangle=\frac{1}{(2\pi)^m}\int_{\mathbb{C}^m} f(z)\overline{g(z)}e^{-\frac{1}{2}|z|^2}dA(z),$$

here $dA(z)$ denotes Lebesgue measure on $\mathbb{C}^m$, and $\frac{1}{(2\pi)^m}e^{-\frac{1}{2}|z|^2}dA(z)$ is called Gaussian measure on $\mathbb{C}^m$. The sequence $\{e_m=\sqrt{\frac{1}{m!}}z^m\}_{m\in\mathbb{N}}$ forms an orthonormal basis for $\mathscr{F}_m^2$.

Since each point evaluation is a bounded linear functional on $\mathscr{F}_m^2$, for each $w\in\mathbb{C}^m$ there exists a unique function $u_w\in\mathscr{F}_m^2$ such that $\langle f,u_w\rangle=f(w)$ for all $f\in\mathscr{F}_m^2$. The reproducing kernel functions for the Fock space are given by $u_w(z)=e^{\frac{\langle z,w\rangle}{2}}$ and $\|u_w\|=e^{\frac{|w|^2}{4}}$.

For a given holomorphic mapping $\phi:\mathbb{C}^m\mapsto\mathbb{C}^m$, the composition operator $C_\phi:\mathscr{F}_m^2\mapsto\mathscr{F}_m^2$ is given by $C_\phi(f)=f\circ\phi$, $f\in\mathscr{F}_m^2$, so $(C_\phi f)(z)=f(\phi(z)$. The multiplication operator M_u induced by an entire function u on $\mathscr{F}_m^2$ is defined as $M_uf(z)=u(z)f(z)$ for an entire function f.

LEMMA 5.1. [6, Lemma 2] *If $f(z)=Az+B$, where A is an $m\times m$ matrix with $\|A\|\leqslant 1$ and B is an $m\times 1$ vector and if $\langle A\xi,B\rangle=0$ whenever $|A\xi|=|\xi|$ then $C_\phi^*=M_{u_b}C_\tau$, where $\tau(z)=A^*z$ and M_{u_b} is the multiplication by the kernel function u_b.*

THEOREM 5.2. *A composition operator C_ϕ is an (n,k)-quasi class Q operator on $\mathscr{F}_m^2$ if and only if*

$$M_{u_b\circ\tau^k}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}-(n+1)M_{u_b\circ\tau^k}C_{\phi^{k+1}\circ\tau^{k+1}}+nC_{\phi^k\circ\tau^k}\geqslant 0.$$

Proof. A composition operator C_ϕ is an (n,k)-quasi class Q operator on $\mathscr{F}_m^2$ if and only if

$$C_\phi^{*(n+k+1)}C_\phi^{n+k+1}-(n+1)C_\phi^{*(k+1)}C_\phi^{k+1}+nC_\phi^{*k}C_\phi^k \geqslant O. \tag{5.1}$$

By Lemma 5.1 we have

$$C_\phi^{*(n+k)}(C_\phi^*C_\phi)C_\phi^{n+k}=C_\phi^{*(n+k)}((M_{u_b}C_\tau)C_\phi)C_\phi^{n+k}.$$

Since $C_\phi C_\tau=C_{\tau\circ\phi}$ we have

$$C_\phi^{*(n+k)}(C_\phi^*C_\phi)C_\phi^{n+k}=C_\phi^{*(n+k)}(M_{u_b}C_{\phi\circ\tau})C_\phi^{n+k}=C_\phi^{*(n+k)}(M_{u_b}C_{\phi^{n+k+1}\circ\tau}).$$

Again by using into consideration Lemma 5.1, we obtain

$$C_\phi^{*(n+k)}(C_\phi^*C_\phi)C_\phi^{n+k}=C_\phi^{*(n+k-1)}M_{u_b}C_\tau(M_{u_b}C_{\phi^{n+k+1}\circ\tau}).$$

Since

$$C_\tau M_{u_b}=M_{u_b\circ\tau}C_\tau$$

then

$$C_\phi^{*(n+k)}(C_\phi^*C_\phi)C_\phi^{n+k}=C_\phi^{*(n+k-1)}M_{u_b}M_{u_b\circ\tau}C_{\phi^{n+k+1}\circ\tau^2}.$$

Continuing this way we obtain

$$C_\phi^{*(n+k+1)}C_\phi^{n+k+1}=M_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}. \tag{5.2}$$

From relations (5.1) and (5.2) we have: C_ϕ is an (n,k)-quasi class Q operator on $\mathscr{F}_m^2$ if and only if

$$M_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}$$
$$-(n+1)M_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^k}C_{\phi^{k+1}\circ\tau^{k+1}}+nM_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^{k-1}}C_{\phi^k\circ\tau^k}\geqslant 0,$$

hence

$$M_{u_b\circ\tau^k}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}-(n+1)M_{u_b\circ\tau^k}C_{\phi^{k+1}\circ\tau^{k+1}}+nC_{\phi^k\circ\tau^k}\geqslant 0. \quad \square$$

THEOREM 5.3. *A composition operator C_ϕ is an (n,k)-quasi class Q^* operator on $\mathscr{F}_m^2$ if and only if*

$$M_{u_b\circ\tau^k}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}-(n+1)M_{u_b\circ\tau^k}M_{u_b\circ\phi\circ\tau^k}C_{\phi^k\circ\tau\circ\phi\circ\tau^k}+nC_{\phi^k\circ\tau^k}\geqslant 0.$$

Proof. A composition operator C_ϕ is an (n,k)-quasi class Q^* operator on $\mathscr{F}_m^2$ if and only if

$$C_\phi^{*(n+k+1)}C_\phi^{n+k+1}-(n+1)C_\phi^{*k}(C_\phi C_\phi^*)C_\phi^k+nC_\phi^{*k}C_\phi^k\geqslant O. \tag{5.3}$$

By Lemma 5.1 and since $C_\tau M_{u_b} = M_{u_b\circ\tau}C_\tau$, $C_\phi C_\tau = C_{\tau\circ\phi}$ we have

$$\begin{aligned} C_\phi^{*k}(C_\phi C_\phi^*)C_\phi^k &= C_\phi^{*k}(C_\phi M_{u_b}C_\tau)C_\phi^k = C_\phi^{*k}M_{u_b\circ\phi}C_{\phi^k\circ\tau\circ\phi} \\ &= C_\phi^{*(k-1)}M_{u_b}M_{u_b\circ\phi\circ\tau}C_{\phi^k\circ\tau\circ\phi\circ\tau} \\ &= M_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^k}M_{u_b\circ\phi\circ\tau^k}C_{\phi^k\circ\tau\circ\phi\circ\tau^k}. \end{aligned}$$

From above relation and from relations (5.2) and(5.3) we have: C_ϕ is an (n,k)-quasi class Q operator on $\mathscr{F}_m^2$ if and only if

$$M_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}$$

$$-(n+1)M_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^k}M_{u_b\circ\phi\circ\tau^k}C_{\phi^k\circ\tau\circ\phi\circ\tau^k} + nM_{u_b}M_{u_b\circ\tau}\dots M_{u_b\circ\tau^{k-1}}C_{\phi^k\circ\tau^k} \geqslant 0,$$

hence

$$M_{u_b\circ\tau^k}\dots M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}} - (n+1)M_{u_b\circ\tau^k}M_{u_b\circ\phi\circ\tau^k}C_{\phi^k\circ\tau\circ\phi\circ\tau^k} + nC_{\phi^k\circ\tau^k} \geqslant 0. \quad \square$$

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