Operators
and
Matrices
Volume 16, Number 1 (2022), 189–195
doi:10.7153/oam-2022-16-16

GENERALIZED GHABRIES–ABBAS–MOURAD LOG–MAJORIZATION

ZESHENG FENG AND JIAN SHI*

(*Communicated by R. Vandebril*)

Abstract. In this paper, Ghabries-Abbas-Mourad log-majorization is generalized in two different cases.

1. Introduction

A capital letter, such as T, stands for an $n\times n$ complex matrix. $T \geqslant O$ means that T is positive semidefinite and $T > O$ means that T is positive definite, respectively.

Recall that for two matrices X and Y, whose eigenvalues are all positive numbers, the log-majorization $X \prec_{log} Y$ means that

$$\begin{cases} \prod_{i=1}^{k} \lambda_i(X) \leqslant \prod_{i=1}^{k} \lambda_i(Y), & k = 1,2,\cdots,n-1; \\ \prod_{i=1}^{k} \lambda_i(X) = \prod_{i=1}^{k} \lambda_i(Y), & k = n, \end{cases}$$

where $\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_n(X)$ are the eigenvalues of X in decreasing order counting multiplicities.

There are many log-majorizations shown in [5]. Recently, Ghabries, Abbas and Mourad obtained a perfect log-majorization in [3] as follows.

THEOREM 1.1. ([3], Ghabries-Abbas-Mourad log-majorization) *Let A and B be two positive definite matrices. Then for all $0 \leqslant t \leqslant 1$ and $k \geqslant 4t$, the following log-majorization holds,*

$$A^{\frac{k}{2}-t}B^t \prec_{log} A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^t A^{\frac{k}{4}}. \tag{1.1}$$

Theorem 1.1 was recently generalized and improved in [4]. In this paper, Ghabries-Abbas-Mourad log-majorization is generalized in two different cases. In order to prove these results, we introduce the following two Lemmas.

Mathematics subject classification (2020): 47A63, 47A64.
Keywords and phrases: Furuta inequality, generalized Furuta inequality, Ghabries-Abbas-Mourad log-majorization.

* Corresponding author.

LEMMA 1.1. ([1], Furuta inequality) *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$ and $p \geqslant 1$,*

$$A^{1+r} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

and

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant B^{1+r}$$

hold.

LEMMA 1.2. ([2], Generalized Furuta inequalities) *If $A \geqslant B \geqslant 0$ and $A > 0$, then for $0 \leqslant t \leqslant 1$ and $p \geqslant 1$,*

$$A^{1-t+r} \geqslant [A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $s \geqslant 1$ and $r \geqslant t$.

2. Generalized Ghabries-Abbas-Mourad log-majorization in the case of $4t \leqslant k \leqslant 2t+2$

In this section, we will show a generalization of Theorem 1.1 in the case of $4t \leqslant k \leqslant 2t+2$.

THEOREM 2.1. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t} B^t \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q}}{2}} [B^{\frac{1}{2}} (B^{\frac{-vt}{2}} A^{-1} B^{\frac{-vt}{2}})^s B^{\frac{1}{2}}]^{\tilde{p}} A^{\frac{k}{2}-t-\frac{\tilde{q}}{2}}\}^l \tag{2.1}$$

holds for $1 \leqslant (1-v+\frac{1}{t})\alpha \leqslant 2$, $4t \leqslant k \leqslant 2t+2$, $0 \leqslant t \leqslant 1$, $0 \leqslant v \leqslant 1$, $1 \leqslant s \leqslant \frac{1}{vt}$ and $0 \leqslant \alpha \leqslant 1$, where $\tilde{p} = \frac{\alpha(1-v+\frac{1}{t})}{(\frac{2}{k-2t}-v)s+\frac{1}{t}}$, $\tilde{q} = (\frac{k}{2}-t)[2-(1-v+\frac{1}{t})\alpha]$, and $l = \frac{(\frac{2}{k-2t}-v)st+1}{\alpha(1-v+\frac{1}{t})(1-vts)} > 0$.

Proof. According to Schur's complement, we have

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \geqslant 0, \tag{2.2}$$

where $M_1 = A^{\frac{\tilde{q}}{2}} B^t [B^{\frac{-1}{2}} (B^{\frac{vt}{2}} A B^{\frac{vt}{2}})^s B^{\frac{-1}{2}}]^{\tilde{p}} B^t A^{\frac{\tilde{q}}{2}}$,
$M_2 = A^{\frac{\tilde{q}}{2}} B^t A^{\frac{k}{2}-t-\frac{\tilde{q}}{2}}$, $M_3 = A^{\frac{k}{2}-t-\frac{\tilde{q}}{2}} B^t A^{\frac{\tilde{q}}{2}}$, $M_4 = A^{\frac{k}{2}-t-\frac{\tilde{q}}{2}} [B^{\frac{1}{2}} (B^{\frac{-vt}{2}} A^{-1} B^{\frac{-vt}{2}})^s B^{\frac{1}{2}}]^{\tilde{p}} A^{\frac{k}{2}-t-\frac{\tilde{q}}{2}}$.

Then we have

$$(\lambda_1(A^{\frac{k}{2}-t} B^t))^2 \leqslant \lambda_1(M_4)\lambda_1(M_1). \tag{2.3}$$

It follows that

$$(\lambda_1(A^{\frac{k}{2}-t} B^t))^{2l-1} (\lambda_1(A^{\frac{k}{2}-t} B^t))^1 = (\lambda_1(A^{\frac{k}{2}-t} B^t))^{2l} \leqslant (\lambda_1(M_4))^l (\lambda_1(M_1))^l. \tag{2.4}$$

In order to prove our result, it is enough to prove that

$$\{A^{\frac{k}{2}-t} B^t\}^{2l-1} \succ_{log} \{A^{\frac{\tilde{q}}{2}} B^t [B^{\frac{-1}{2}} (B^{\frac{vt}{2}} A B^{\frac{vt}{2}})^s B^{\frac{-1}{2}}]^{\tilde{p}} B^t A^{\frac{\tilde{q}}{2}}\}^l, \tag{2.5}$$

which is equivalent to showing that

$$B^{\frac{t}{2}}A^{\frac{k}{2}-t}B^{\frac{t}{2}} \leqslant I \Rightarrow A^{\frac{\tilde{q}}{2}}B^{t}[B^{\frac{-1}{2}}(B^{\frac{vt}{2}}AB^{\frac{vt}{2}})^{s}B^{\frac{-1}{2}}]^{\tilde{p}}B^{t}A^{\frac{\tilde{q}}{2}} \leqslant I. \tag{2.6}$$

It is clear that $B^{\frac{t}{2}}A^{\frac{k}{2}-t}B^{\frac{t}{2}} \leqslant I$ is equivalent to

$$A^{\frac{k}{2}-t} \leqslant B^{-t}. \tag{2.7}$$

Let $A_1 = B^{-t}$ and $B_1 = A^{\frac{k}{2}-t}$, (2.7) gives $A_1 \geqslant B_1$.
According to Lemma 1.2, we have

$$A_1^{(1-v+\frac{1}{t})\alpha} \geqslant [A_1^{\frac{1}{2t}}(A_1^{\frac{-v}{2}}B_1^{\frac{2}{k-2t}}A_1^{\frac{-v}{2}})^{s}A_1^{\frac{1}{2t}}]^{\tilde{p}}. \tag{2.8}$$

By using the Löwner-Heinz inequality for $-1 \leqslant (1-v+\frac{1}{t})\alpha - 2 \leqslant 0$, we have

$$A_1B_1^{(1-v+\frac{1}{t})\alpha-2}A_1 \geqslant A_1A_1^{(1-v+\frac{1}{t})\alpha-2}A_1 = A_1^{(1-v+\frac{1}{t})\alpha}. \tag{2.9}$$

Now together with (2.8) and (2.9), we can conclude

$$A_1B_1^{(1-v+\frac{1}{t})\alpha-2}A_1 \geqslant [A_1^{\frac{1}{2t}}(A_1^{\frac{-v}{2}}B_1^{\frac{2}{k-2t}}A_1^{\frac{-v}{2}})^{s}A_1^{\frac{1}{2t}}]^{\tilde{p}}. \tag{2.10}$$

Then, replacing A_1 with B^{-t} and B_1 with $A^{\frac{k}{2}-t}$, respectively, in (2.10), it is equivalent to

$$B^{-t}A^{-\tilde{q}}B^{-t} \geqslant [B^{\frac{-1}{2}}(B^{\frac{vt}{2}}AB^{\frac{vt}{2}})^{s}B^{\frac{-1}{2}}]^{\tilde{p}}, \tag{2.11}$$

and (2.11) is equivalent to

$$A^{\frac{\tilde{q}}{2}}B^{t}[B^{\frac{-1}{2}}(B^{\frac{vt}{2}}AB^{\frac{vt}{2}})^{s}B^{\frac{-1}{2}}]^{\tilde{p}}B^{t}A^{\frac{\tilde{q}}{2}} \leqslant I. \tag{2.12}$$

Thus (2.6) have been proved. This complete the proof. □

If we put $v = 0,\ s = 1$ or $\alpha = \frac{kt}{(k-2t)(1+t)}$ respectively in Theorem 2.1, we will have the following three corollaries.

COROLLARY 2.1. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q}_1}{2}}(B^{\frac{1}{2}}A^{-s}B^{\frac{1}{2}})^{\tilde{p}_1}A^{\frac{k}{2}-t-\frac{\tilde{q}_1}{2}}\}^{l_1}$$

holds for $1 \leqslant (1+\frac{1}{t})\alpha \leqslant 2,\ 4t \leqslant k \leqslant 2t+2,\ 0 \leqslant t \leqslant 1,\ s \geqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$, *where* $\tilde{p}_1 = \frac{(1+t)(k-2t)\alpha}{2st+k-2t}$, $\tilde{q}_1 = (\frac{k}{2}-t)[2-(1+\frac{1}{t})\alpha]$, *and* $l_1 = \frac{t}{\tilde{p}_1} > 0$.

COROLLARY 2.2. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q}_2}{2}}(B^{\frac{1-vt}{2}}A^{-1}B^{\frac{1-vt}{2}})^{\tilde{p}_2}A^{\frac{k}{2}-t-\frac{\tilde{q}_2}{2}}\}^{l_2}$$

holds for $1 \leqslant (1-v+\frac{1}{t})\alpha \leqslant 2,\ 4t \leqslant k \leqslant 2t+2,\ 0 \leqslant t \leqslant 1,\ 0 \leqslant v \leqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$, *where* $\tilde{p}_2 = \frac{\alpha(1-v+\frac{1}{t})}{\frac{2}{k-2t}-v+\frac{1}{t}}$, $\tilde{q}_2 = (\frac{k}{2}-t)[2-(1-v+\frac{1}{t})\alpha]$, *and* $l_2 = \frac{(\frac{2}{k-2t}-v)t+1}{\alpha(1-v+\frac{1}{t})(1-vt)} > 0$.

COROLLARY 2.3. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q_3}}{2}}[B^{\frac{1}{2}}(B^{\frac{-vt}{2}}A^{-1}B^{\frac{-vt}{2}})^{s}B^{\frac{1}{2}}]^{\tilde{p_3}}A^{\frac{k}{2}-t-\frac{\tilde{q_3}}{2}}\}^{l_3}$$

holds for $1 \leqslant \frac{(t-vt+1)k}{(k-2t)(1+t)} \leqslant 2,\ 4t \leqslant k \leqslant 2t+2,\ 0 \leqslant t \leqslant 1,\ 0 \leqslant v \leqslant 1$ *and* $1 \leqslant s \leqslant \frac{1}{vt}$, *where* $\tilde{p_3} = \frac{kt(t-vt+1)}{(1+t)[(2-vk+2vt)st+k-2t]}$, $\tilde{q_3} = \frac{(k-4t)(1+t)+kvt}{2(1+t)}$, *and* $l_3 = \frac{t}{(1-vts)\tilde{p_3}} > 0$.

If we put $s = 1$ in Corollary 2.1, we will have the following result.

COROLLARY 2.4. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q_4}}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\tilde{p_4}}A^{\frac{k}{2}-t-\frac{\tilde{q_4}}{2}}\}^{l_4}$$

holds for $1 \leqslant (1+\frac{1}{t})\alpha \leqslant 2,\ 4t \leqslant k \leqslant 2t+2,\ 0 \leqslant t \leqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$, *where* $\tilde{p_4} = \frac{(1+t)(k-2t)\alpha}{k}$, $\tilde{q_4} = (\frac{k}{2}-t)[2-(1+\frac{1}{t})\alpha]$, *and* $l_4 = \frac{t}{\tilde{p_4}} > 0$.

If we put $\alpha = \frac{kt}{(k-2t)(1+t)}$ in Corollary 2.2, we will have the following result.

COROLLARY 2.5. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q_5}}{2}}(B^{\frac{1-vt}{2}}A^{-1}B^{\frac{1-vt}{2}})^{\tilde{p_5}}A^{\frac{k}{2}-t-\frac{\tilde{q_5}}{2}}\}^{l_2}$$

holds for $1 \leqslant \frac{k(t-vt+1)}{(k-2t)(1+t)} \leqslant 2,\ 4t \leqslant k \leqslant 2t+2,\ 0 \leqslant t \leqslant 1$ *and* $0 \leqslant v \leqslant 1$, *where* $\tilde{p_5} = \frac{kt(t-vt+1)}{(1+t)[k-vt(k-2t)]}$, $\tilde{q_5} = \frac{2(k-2t)(1+t)-k(t-vt+1)}{2(1+t)}$, *and* $l_5 = \frac{(1+t)[k-vt(k-2t)]}{k(t-vt+1)(1-vt)} > 0$.

If we put $v = 0$ in Corollary 2.3, we will have the following result.

COROLLARY 2.6. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t-\frac{\tilde{q_6}}{2}}(B^{\frac{1}{2}}A^{-s}B^{\frac{1}{2}})^{\tilde{p_6}}A^{\frac{k}{2}-t-\frac{\tilde{q_6}}{2}}\}^{l_3}$$

holds for $1 \leqslant \frac{k}{k-2t} \leqslant 2,\ 4t \leqslant k \leqslant 2t+2,\ 0 \leqslant t \leqslant 1$ *and* $s \geqslant 1$, *where* $\tilde{p_6} = \frac{kt}{2st+k-2t}$, $\tilde{q_6} = \frac{k-4t}{2}$, *and* $l_6 = \frac{t}{\tilde{p_6}} > 0$.

REMARK 2.1. If we put $v = 0,\ s = 1$ and $\alpha = \frac{kt}{(k-2t)(1+t)}$ in Theorem 2.1, it is just Theorem 1.1 in the case of $4t \leqslant k \leqslant 2t+2$.

3. Generalized Ghabries-Abbas-Mourad log-majorization in the case of $k \geqslant 2t+2$

In this section, we will show a different generalization of Theorem 1.1 in the case of $k \geqslant 2t+2$.

THEOREM 3.1. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t+\frac{\tilde{r}}{2}}[B^{\frac{1}{2}}(B^{\frac{rt}{2}}A^{(t-\frac{k}{2})p}B^{\frac{rt}{2}})^{\frac{1+r}{p+r}\alpha}B^{\frac{1}{2}}]^{\beta}A^{\frac{k}{2}-t+\frac{\tilde{r}}{2}}\}^{h} \tag{3.1}$$

holds for $1 \leqslant [(1+r)\alpha + \frac{1}{t}]\beta \leqslant 2,\ k \geqslant 2t+2,\ 0 \leqslant t \leqslant 1,\ 0 \leqslant r \leqslant 1,\ p \geqslant 1,\ 0 \leqslant \beta \leqslant min\{1, \frac{2t}{rt\alpha\frac{1+r}{p+r}+1}\}$ *and* $\alpha \in [0,1]$, *where* $\tilde{r} = (\frac{k}{2} - t)[((1+r)\alpha + \frac{1}{t})\beta - 2]$, *and* $h = \frac{t}{(rt\alpha\frac{1+r}{p+r}+1)\beta} > 0.$

Proof. According to Schur's complement, we have

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \geqslant 0, \tag{3.2}$$

where $M_1 = A^{-\frac{\tilde{r}}{2}} B^t [B^{\frac{-1}{2}} (B^{\frac{-rt}{2}} A^{(\frac{k}{2}-t)p} B^{\frac{-rt}{2}})^{\frac{1+r}{p+r}\alpha} B^{\frac{-1}{2}}]^{\beta} B^t A^{-\frac{\tilde{r}}{2}}$, $M_2 = A^{-\frac{\tilde{r}}{2}} B^t A^{\frac{k}{2}-t+\frac{\tilde{r}}{2}}$, $M_3 = A^{\frac{k}{2}-t+\frac{\tilde{r}}{2}} B^t A^{-\frac{\tilde{r}}{2}}$, $M_4 = A^{\frac{k}{2}-t+\frac{\tilde{r}}{2}} [B^{\frac{1}{2}} (B^{\frac{rt}{2}} A^{(t-\frac{k}{2})p} B^{\frac{rt}{2}})^{\frac{1+r}{p+r}\alpha} B^{\frac{1}{2}}]^{\beta} A^{\frac{k}{2}-t+\frac{\tilde{r}}{2}}$.

Then we have

$$(\lambda_1(A^{\frac{k}{2}-t} B^t))^2 \leqslant \lambda_1(M_4)\lambda_1(M_1). \tag{3.3}$$

It follows that

$$(\lambda_1(A^{\frac{k}{2}-t} B^t))^{2h-1} (\lambda_1(A^{\frac{k}{2}-t} B^t))^1 = (\lambda_1(A^{\frac{k}{2}-t} B^t))^{2h} \leqslant (\lambda_1(M_4))^h (\lambda_1(M_1))^h. \tag{3.4}$$

In order to prove our result, it is enough to prove that

$$\{A^{\frac{k}{2}-t} B^t\}^{2h-1} \succ_{log} \{A^{-\frac{\tilde{r}}{2}} B^t [B^{\frac{-1}{2}} (B^{\frac{-rt}{2}} A^{(\frac{k}{2}-t)p} B^{\frac{-rt}{2}})^{\frac{1+r}{p+r}\alpha} B^{\frac{-1}{2}}]^{\beta} B^t A^{-\frac{\tilde{r}}{2}}\}^h, \tag{3.5}$$

which is equivalent to showing that

$$B^{\frac{t}{2}} A^{\frac{k}{2}-t} B^{\frac{t}{2}} \leqslant I \Rightarrow A^{-\frac{\tilde{r}}{2}} B^t [B^{\frac{-1}{2}} (B^{\frac{-rt}{2}} A^{(\frac{k}{2}-t)p} B^{\frac{-rt}{2}})^{\frac{1+r}{p+r}\alpha} B^{\frac{-1}{2}}]^{\beta} B^t A^{-\frac{\tilde{r}}{2}} \leqslant I. \tag{3.6}$$

It is clear that $B^{\frac{t}{2}} A^{\frac{k}{2}-t} B^{\frac{t}{2}} \leqslant I$ is equivalent to

$$A^{\frac{k}{2}-t} \leqslant B^{-t}. \tag{3.7}$$

Let $A_1 = B^{-t}$ and $B_1 = A^{\frac{k}{2}-t}$, (3.7) gives $A_1 \geqslant B_1$.

According to Lemma 1.1, we have

$$A_1^{(1+r)\alpha} \geqslant (A_1^{\frac{r}{2}} B_1^p A_1^{\frac{r}{2}})^{\frac{1+r}{p+r}\alpha}, \tag{3.8}$$

and then we have

$$(A_1^{\frac{1}{2t}} A_1^{(1+r)\alpha} A_1^{\frac{1}{2t}})^{\beta} \geqslant (A_1^{\frac{1}{2t}} (A_1^{\frac{r}{2}} B_1^p A_1^{\frac{r}{2}})^{\frac{1+r}{p+r}\alpha} A_1^{\frac{1}{2t}})^{\beta}. \tag{3.9}$$

By using the Löwner-Heinz inequality for $-1 \leqslant [(1+r)\alpha + \frac{1}{t}]\beta - 2 \leqslant 0$, we have

$$A_1 B_1^{[(1+r)\alpha+\frac{1}{t}]\beta-2} A_1 \geqslant A_1^{[(1+r)\alpha+\frac{1}{t}]\beta}. \tag{3.10}$$

Now together with (3.9) and (3.10), we can conclude

$$A_1 B_1^{[(1+r)\alpha+\frac{1}{t}]\beta-2} A_1 \geqslant (A_1^{\frac{1}{2t}} (A_1^{\frac{r}{2}} B_1^p A_1^{\frac{r}{2}})^{\frac{1+r}{p+r}\alpha} A_1^{\frac{1}{2t}})^{\beta}. \tag{3.11}$$

Then, replacing A_1 with B^{-t} and B_1 with $A^{\frac{k}{2}-t}$, respectively, in (3.11), it is equivalent to

$$B^{-t}A^{\tilde{r}}B^{-t} \geqslant [B^{\frac{-1}{2}}(B^{\frac{-rt}{2}}A^{(\frac{k}{2}-t)p}B^{\frac{-rt}{2}})^{\frac{1+r}{p+r}\alpha}B^{\frac{-1}{2}}]^{\beta}, \tag{3.12}$$

and (3.12) is equivalent to

$$A^{-\frac{\tilde{r}}{2}}B^{t}[B^{\frac{-1}{2}}(B^{\frac{-rt}{2}}A^{(\frac{k}{2}-t)p}B^{\frac{-rt}{2}})^{\frac{1+r}{p+r}\alpha}B^{\frac{-1}{2}}]^{\beta}B^{t}A^{-\frac{\tilde{r}}{2}} \leqslant I. \tag{3.13}$$

Thus (3.6) have been proved. This complete the proof. □

If we put $r=0$, $\beta=t$ or $\alpha=\frac{2}{k-2t}$ respectively in Theorem 3.1, we will have the following three corollaries.

COROLLARY 3.1. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t+\frac{\tilde{r_1}}{2}}(B^{\frac{1}{2}}A^{(t-\frac{k}{2})\alpha}B^{\frac{1}{2}})^{\beta}A^{\frac{k}{2}-t+\frac{\tilde{r_1}}{2}}\}^{h_1}$$

holds for $1 \leqslant (\alpha+\frac{1}{t})\beta \leqslant 2$, $k \geqslant 2t+2$, $0 \leqslant t \leqslant 1$, $0 \leqslant \beta \leqslant min\{1,2t\}$ *and* $\alpha \in [0,1]$, *where* $\tilde{r}_1 = (\frac{k}{2}-t)[(\alpha+\frac{1}{t})\beta-2]$, *and* $h_1 = \frac{t}{\beta} > 0$.

COROLLARY 3.2. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t+\frac{\tilde{r_2}}{2}}[B^{\frac{1}{2}}(B^{\frac{rt}{2}}A^{(t-\frac{k}{2})p}B^{\frac{rt}{2}})^{\frac{1+r}{p+r}\alpha}B^{\frac{1}{2}}]^{t}A^{\frac{k}{2}-t+\frac{\tilde{r_2}}{2}}\}^{h_2}$$

holds for $0 \leqslant (1+r)\alpha t \leqslant 1$, $k \geqslant 2t+2$, $0 \leqslant t \leqslant 1$, $0 \leqslant r \leqslant 1$, $p \geqslant 1$ *and* $\alpha \in [0,1]$, *where* $\tilde{r}_2 = (\frac{k}{2}-t)[(1+r)\alpha t-1]$, *and* $h_2 = \frac{p+r}{rt\alpha(1+r)+p+r} > 0$.

COROLLARY 3.3. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} \{A^{\frac{k}{2}-t+\frac{\tilde{r_3}}{2}}[B^{\frac{1}{2}}(B^{\frac{rt}{2}}A^{(t-\frac{k}{2})p}B^{\frac{rt}{2}})^{\frac{2(1+r)}{(k-2t)(p+r)}}B^{\frac{1}{2}}]^{\beta}A^{\frac{k}{2}-t+\frac{\tilde{r_3}}{2}}\}^{h_3}$$

holds for $1 \leqslant [\frac{2(1+r)}{k-2t}+\frac{1}{t}]\beta \leqslant 2$, $k \geqslant 2t+2$, $0 \leqslant t \leqslant 1$, $0 \leqslant r \leqslant 1$, $p \geqslant 1$ *and* $0 \leqslant \beta \leqslant min\{1, \frac{2t(k-2t)(p+r)}{2rt(1+r)+(k-2t)(p+r)}\}$, *where* $\tilde{r}_3 = \frac{[2t(1+r)+(k-2t)]\beta-2t(k-2t)}{2t}$, *and* $h_3 = \frac{t(k-2t)(p+r)}{[2rt(1+r)+(k-2t)(p+r)]\beta} > 0$.

If we put $\beta=t$ in Corollary 3.1, we will have the following result.

COROLLARY 3.4. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^{t} \prec_{log} A^{\frac{k}{2}-t+\frac{\tilde{r_4}}{2}}(B^{\frac{1}{2}}A^{(t-\frac{k}{2})\alpha}B^{\frac{1}{2}})^{t}A^{\frac{k}{2}-t+\frac{\tilde{r_4}}{2}}$$

holds for $k \geqslant 2t+2$, $0 \leqslant t \leqslant 1$ *and* $\alpha \in [0,1]$, *where* $\tilde{r}_4 = (\frac{k}{2}-t)(t\alpha-1)$.

If we put $\alpha=\frac{2}{k-2t}$ in Corollary 3.2, we will have the following result.

COROLLARY 3.5. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^t \prec_{log} \{A^{\frac{k}{2}-t+\frac{\tilde{r_5}}{2}}[B^{\frac{1}{2}}(B^{\frac{rt}{2}}A^{(t-\frac{k}{2})p}B^{\frac{rt}{2}})^{\frac{2(1+r)}{(k-2t)(p+r)}}B^{\frac{1}{2}}]^t A^{\frac{k}{2}-t+\frac{\tilde{r_5}}{2}}\}^{h_5}$$

holds for $0 \leqslant \frac{2(1+r)t}{k-2t} \leqslant 1,\ k \geqslant 2t+2,\ 0 \leqslant t \leqslant 1,\ 0 \leqslant r \leqslant 1$ *and* $p \geqslant 1$, *where* $\tilde{r_5} = \frac{2(1+r)t-k+2t}{2}$, *and* $h_5 = \frac{(p+r)(k-2t)}{2rt(1+r)+(p+r)(k-2t)} > 0$.

If we put $r=0$ in Corollary 3.3, we will have the following result.

COROLLARY 3.6. *Let A and B be two positive definite matrices. Then we have*

$$A^{\frac{k}{2}-t}B^t \prec_{log} \{A^{\frac{k}{2}-t+\frac{\tilde{r_6}}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\beta}A^{\frac{k}{2}-t+\frac{\tilde{r_6}}{2}}\}^{h_6}$$

holds for $1 \leqslant \frac{k\beta}{t(k-2t)} \leqslant 2,\ k \geqslant 2t+2,\ 0 \leqslant t \leqslant 1$ *and* $0 \leqslant \beta \leqslant min\{1,2t\}$, *where* $\tilde{r_6} = \frac{k\beta-2t(k-2t)}{2t}$, *and* $h_6 = \frac{t}{\beta} > 0$.

REMARK 3.1. If we put $r=0,\ \beta=t$ and $\alpha = \frac{2}{k-2t}$ in Theorem 3.1, it is just Theorem 1.1 in the case of $k \geqslant 2t+2$.

Acknowledgements. The authors thank anonymous reviewers for their helpful comments on an earlier draft of this paper. The corresponding author Jian Shi is supported by Science and Technology Project of Hebei Education Department (QN2020145, ZD2021307) and President Foundation of Hebei University (No. XZJJ201902). Zesheng Feng is supported by Post-graduate's Innovation Fund Project of Hebei Province (No. CXZZSS2021005).

REFERENCES

[1] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{\frac{1}{q}} \geqslant B^{\frac{p+2r}{q}}$ for $r \geqslant 0,\ p \geqslant 0,\ q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc. 101 (1987), 85–88.

[2] T. FURUTA, *Invitation to Linear Operators: From Matrices to Bounded Linear Operators on a Hilbert Space*, CRC Press, Taylor & Francis Group, 2001.

[3] M. M. GHABRIES, H. ABBAS, B. MOURAD, *On some open questions concerning determinantal inequalities*, Linear Algebra Appl., 596 (2020), 169–183.

[4] M. M. GHABRIES, H. ABBAS, B. MOURAD, A. ASSI, *A proof of a conjectured determinantal inequality*, Linear Algebra Appl., 605 (2020), 21–28.

[5] F. ZHANG, *Matrix Theory: Basic Results and Techniques*, Springer, New York, 2nd ed., 2011.

(Received November 23, 2020)

Zesheng Feng
Key Laboratory of Machine Learning and Computational Intelligence
College of Mathematics and Information Science
Hebei University
Baoding, 071002, P.R. China
e-mail: fengzesheng0227@qq.com

Jian Shi
Key Laboratory of Machine Learning and Computational Intelligence
College of Mathematics and Information Science
Hebei University
Baoding, 071002, P.R. China
e-mail: mathematic@126.com