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EXPANSIVE OPERATORS WHICH ARE POWER BOUNDED OR ALGEBRAIC

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Abstract. Given Hilbert space operators $P,T \in B(\mathscr{H}), P \geqslant 0$ invertible, T is (m,P)-expansive (resp., (m,P)-isometric) for some positive integer m if $\triangle^m_{T^*,T}(P) = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*j}PT^j \leqslant 0$ (resp., $\triangle^m_{T^*,T}(P) = 0$). Power bounded (m,P)-expansive operators, and algebraic (m,I)-expansive operators have a simple structure. A power bounded operator T is an (m,P)-expansive operator if and only if it is a $C_{1.}$-operator such that $\|QTx\| = \|Qx\|$ (i.e., T is Q-isometric) for some invertible positive operator Q. If, instead, T is an algebraic (m,I)-expansive operator, then either the spectral radius $r(T)$ of T is greater than one or T is the perturbation of a unitary by a nilpotent such that T is $(2n-1,I)$-isometric for some positive integers $m_0 \leqslant m$, m_0 odd, and $n \geqslant \frac{m_0+1}{2}$.

1. Introduction

Let $B(\mathscr{H})$ denote the algebra of operators, i.e., bounded linear transformations, on an infinite dimensional complex Hilbert space into itself. An operator T is (m,I)-expansive, or simply m-expansive, for some positive integer m, if

$$\triangle^m_{T^*,T}(I) = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*j}T^j \leqslant 0.$$

Agler, [1, Theorem 3,1], characterized subnormality with positivity of $\triangle^m_{T^*,T}(I)$: $\triangle^m_{T^*,T}(I) \geqslant 0$ if and only if $\|T\| \leqslant 1$ and T is subnormal. Operators T such that $\triangle^m_{T^*,T}(I) \geqslant 0$ have been called m-contractive, and operators T such that $\triangle^m_{T^*,T}(I) = 0$ are said to be m-isometric [2]. Classes of m-isometric, m-expansive and m-contractive operators have attracted the attention of a large number of authors over the past three or so decades (see [4], [5], [6], [7], [8], [11], [12], [13], [14], [19] for further references): there is an extensive body of information on the structure of these classes of operators, including that on the spectral picture, preservation (or failure) of these properties under commuting products and perturbation by commuting nilpotents, available in extant literature.

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For $A,B \in B(\mathscr{H})$, let $L_A, R_B \in B(B(\mathscr{H}))$ denote respectively the left and the right multiplication operators

$$L_A(X) = AX \text{ and } R_B(X) = XB.$$

Let $\triangle_{A,B} \in B(B(\mathscr{H}))$ denote the elementary operator

$$\triangle_{A,B}(X)) = (I - L_A R_B)(X) = X - AXB.$$

Then, for positive integers m,

$$\triangle^m_{A,B}(X) = (I - L_A R_B)^m(X) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} A^j X B^j.$$

We say in the following that the pair of operators

$(A,B) \in (m,P)$-expansive if $\triangle^m_{A,B}(P) \leqslant 0$;
$(A,B) \in (m,P)$-hyperexpansive if $\triangle^t_{A,B}(P) \leqslant 0$ for all integers $1 \leqslant t \leqslant m$;
$(A,B) \in (m,P)$-contractive if $\triangle^m_{A,B}(P) \geqslant 0$;
$(A,B) \in (m,P)$-hypercontractive if $\triangle^t_{A,B}(P) \geqslant 0$ for all integers $1 \leqslant t \leqslant m$;
$(A,B) \in (m,P)$-isometric if $\triangle^m_{A,B}(P) = 0$.

Recall that an operator $T \in B(\mathscr{H})$ is power bounded if $\sup_n \|T^n\| \leqslant M$ for some scalar $M > 0$. A well known result says that power bounded m-isometric operators T (i.e., T power bounded and $(T^*,T) \in (m,I)$-isometric) are isometric; for power bounded pairs $(A,B) \in (m,I)$-isometric, A^* and B are similar to isometries [9]. Does this result extends to power bounded pairs of (m,P)-expansive operators? We prove that the answer is in the afirmative for pairs satisfying “an order preserving property”. Let us say that a pair of operators (A,B) preserves order if $L_A R_B(Q) \geqslant 0$ whenever $Q \geqslant 0$. We prove that if A,B are power boiunded operators, the pair (A,B) preserves order and $(A,B) \in (m,P)$-expansive, then there exist positive invertible operators P_1, P_2 and an isometry V such that $A = P_1^{-1} V^* P_1$ and $B = P_2^{-1} V P_2$. For power bounded $T \in (m,P)$-expansive (i.e., $(T^*,T) \in (m,P)$-expansive) operators, this translates to “T is a $C_{1\cdot}$-operator which is similar to an isometry and satisfies $T^*QT = Q$ for some positive invertible operator Q”. (Thus T is isometric in an equivalent norm: $\|x\|_Q = \langle x,x\rangle_Q^{\frac{1}{2}} = \|Q^{\frac{1}{2}}x\|$.) For operators $T \in (m,P)$-contractive, it is seen that T is similar to the direct sum of the conjugate of a $C_{0\cdot}$-contraction with a unitary. Algebraic (m,P)-expansive operators T are not Drazin invertible. We prove that for such operators T either the spectral radius $r(T) > 1$, or, T is the perturbation of a unitary operator by a commuting nilpotent such that $T \in (2n-1)$-isometric for some integer n (dependent upon m). A similar result for algebraic m-contractive operators is not possible.

The plan of this paper is as follows. Alongwith certain additional notation and a couple of well known complementary results, Section 2 introduces the concept of “order preserving pairs of operators”. Using simple algebraic arguments involving little more than the operators of left and right multiplication, we prove that if $(A,B) \in (m,P)$-expansive (resp., $(A,B) \in (m,P)$-contractive), then $(A^n,B^n) \in (m,P)$-expansive (resp., $(A^n,B^n) \in (m,P)$-contractive) for all positive integers n [11]. It is seen that if

(A,B) is an order preserving pair such that $(A,B) \in (m,P)$-expansive (resp., $(A,B) \in (m+1,P)$-contractive) for some positive even integer m, then $(A,B) \in (m-1,P)$-expansive (resp., $(A,B) \in (m,P)$-contractive). Section 3 is devoted to considering the structure of power bounded (m,P)-expansive and (m,P)-contractive operators. It is seen that a power bounded (m,P)-expansive operator is simiar to an isometry, and a power bounded (m,P)-contractive operator is similar to the direct sum of the adjoint of a $C_{0\cdot}$-contraction with a unitary. Algebraic (Hilbert space) operators have a well understood structure; they have a countably finite spectrum and are the perturbation of a normal operator by a commuting nilpotent. Section 4 considers algebraic (m,P)-expansive operators T to prove that if $T^*T \geqslant 1$, then $T \in (m,P)$-alternatingly expansive; if T has spectral radius less than or equal to one, then T is the perturbation of a unitary with a commuting nilpotent such that $T \in (2n-1)$-isometric for some integer $2n \geqslant m_0+1$, m_0 some odd integer satisfying $m_0 \leqslant m$. Similar analysis does not hold for algebraic (m,P)-contractive T.

2. Complementary results

Throughout the following A,B and T will denote operators in $B(\mathscr{H})$, and $P \in B(\mathscr{H})$ will denote a positive invertible operator. We shall henceforth shorten $(T^*,T) \in (m,P)-\cdots$ to $T \in (m,P)-\cdots$, and $T \in (m,I)-\cdots$ to $T \in m-\cdots$. The spectrum, the approximate point spectrum and the isolated points of the spectrum of A will be denoted by $\sigma(A)$, $\sigma_a(A)$ and $\text{iso}\sigma(A)$, respectively. T is a $C_{0\cdot}$-operator (resp., $C_{1\cdot}$-operator) if

$$\lim_{n\to\infty} \|T^n x\| = 0 \text{ for all } x \in \mathscr{H}$$

$$(\text{resp., } \inf_{n\in\mathbb{N}} \|T^n x\| > 0 \text{ for all } 0 \neq x \in \mathscr{H});$$

$T \in C_{\cdot 0}$ if $T^* \in C_{0\cdot}$, $T \in C_{\cdot 1}$ if $T^* \in C_{1\cdot}$, and $T \in C_{\alpha\beta}$ if $T \in C_{\alpha\cdot} \cap C_{\cdot\beta}$ $(\alpha,\beta = 0,1)$. The operator T is weakly $C_{0\cdot}$ (or, weakly stable [17]) if $\lim_{n\to\infty}\langle T^n x, x\rangle = 0$ for all $x \in \mathscr{H}$ (equivalently; if $\lim_{n\to\infty}\langle T^n x, y\rangle = 0$ for all $x,y \in \mathscr{H}$). It is well known, [15], that power bounded operators T have an upper triangular representation

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathscr{H}_1 \oplus \mathscr{H}_2)$$

for some decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ of $\mathscr{H}$ such that $T_1 \in C_{0\cdot}$ and $T_2 \in C_{1\cdot}$. Every isometry $V \in B(\mathscr{H})$ has a direct sum decomposition

$$V = V_{10} \oplus V_u \in B(\mathscr{H}_c \oplus \mathscr{H}_u), V_{10} \in C_{10} \text{ and } V_u \in C_{11}$$

into its completely non-unitary (i.e., unilateral shift) and unitary parts [17].

The following well known result from Douglas [10] will often be used in the sequel (without further mention).

THEOREM 2.1. *The following statements are pairwise equivalent:*

(i) $A(\mathscr{H}) \subseteq B(\mathscr{H})$.

(ii) There is a $\mu \geqslant 0$ *such that* $AA^* \leqslant \mu^2 BB^*$.

(iii) There is an operator $C \in B(\mathscr{H})$ *such that* $A = BC$.

If these conditions are satisfied, then the operator C may be chosen so that $\|C\|^2 = \inf\{\lambda : AA^* \leqslant \lambda BB^*\}$, $A^{-1}(0) \subseteq C^{-1}(0)$ *and* $C(\mathscr{H}) \subseteq B^{-1}(0)^\perp$.

Suppose that *the pair of operators* (A,B) *preserves order in the sense that* $(L_A R_B)(X) \geqslant 0$ *for all* $X \in B(\mathscr{H})$ *such that* $X \geqslant 0$. For all positive integers n,

$$\begin{aligned}\triangle^m_{A^n,B^n}(P) &= (I - L_{A^n}R_{B^n})^m(P) = (I - L^n_A R^n_B)^m(P) \\ &= \{L^{n-1}_A \triangle_{A,B}(P) R^{n-1}_B + L^{n-2}_A \triangle_{A,B}(P) R^{n-2}_B + \cdots \\ &\quad + L_A \triangle_{A,B}(P) R_B + \triangle_{A,B}(P)\}^m \\ &= \left\{L^{n-1}_A R^{n-1}_B + L^{n-2}_A R^{n-2}_B + \cdots + L_A R_B + I\right\}^m \left(\triangle^m_{A,B}(P)\right).\end{aligned}$$

Hence

$$(A,B) \in (m,P)\text{-expansive} \Longrightarrow (A^n,B^n) \in (m,P)\text{-expansive, and}$$

$$(A,B) \in (m,P)\text{-contractive} \Longrightarrow (A^n,B^n) \in (m,P)\text{-contractive}$$

for all positive integers n.

The identity $(a-1)^m = a^m - \sum_{j=0}^{m-1} \binom{m}{j} (a-1)^j$ implies

$$\tilde{\triangle}^m_{A,B} = (L_A R_B - I)^m = (L_A R_B)^m - \sum_{j=0}^{m-1} \binom{m}{j} \tilde{\triangle}^{m-1}_{A,B} = (-1)^m \triangle^m_{A,B}.$$

If $\tilde{\triangle}^m_{A,B}(P) \leqslant 0$ for some positive integer m, then, since

$$\tilde{\triangle}^j_{A,B} = L_A R_B \left(\tilde{\triangle}^{j-1}_{A,B}\right) - \tilde{\triangle}^{j-1}_{A,B}$$

for all integers $j \geqslant 1$,

$$\begin{aligned}\sum_{j=0}^{m-1} \binom{n}{j} L_A R_B \left(\tilde{\triangle}^j_{A,B}\right) &= \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\triangle}^{j+1}_{A,B} + \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\triangle}^j_{A,B} \\ &= \binom{n}{m-1} \tilde{\triangle}^m_{A,B} + \sum_{j=0}^{m-1} \binom{n+1}{j} \tilde{\triangle}^j_{A,B}.\end{aligned}$$

Evidently (see above), $\tilde{\triangle}^m_{A,B}(P) \leqslant 0$ implies

$$(0 \leqslant)\ (L_A R_B)^m(P) \leqslant \sum_{j=0}^{m-1} \binom{m}{j} \tilde{\triangle}^j_{A,B}(P).$$

We prove

$$(0 \leqslant)\ (L_A R_B)^n(P) \leqslant \sum_{j=0}^{m-1} \binom{n}{j} \tilde{\triangle}^j_{A,B}(P), \text{ for all } n \geqslant m.$$

The inequality being true for $n=m$, assume it to be true for $n=t$. Then, since (A,B) preserves order,

$$
\begin{aligned}
(0\leqslant)\ (L_AR_B)^{t+1}(P) &\leqslant \sum_{j=0}^{m-1}\binom{t}{j} L_AR_B\left(\tilde{\triangle}^{j}_{A,B}(P)\right)\\
&= \binom{t}{m-1}\tilde{\triangle}^{m}_{A,B}(P)+\sum_{j=0}^{m-1}\binom{t+1}{j}\tilde{\triangle}^{j}_{A,B}(P) \qquad (1)\\
&\leqslant \sum_{j=0}^{m-1}\binom{t+1}{j}\tilde{\triangle}^{j}_{A,B}(P)
\end{aligned}
$$

(since $\tilde{\triangle}^{m}_{A,B}(P)\leqslant 0$). Thus the inequality is true for $n=t+1$, hence by induction for all integers $n\geqslant m$.

Observe from (1) that

$$(0\leqslant)\ \frac{1}{n^{m-1}}(L_AR_B)^n(P)\leqslant\frac{1}{n^{m-1}}\left\{\binom{n}{m-1}\tilde{\triangle}^{m-1}_{A,B}(P)+\sum_{j=0}^{m-2}\binom{n}{j}\tilde{\triangle}^{j}_{A,B}(P)\right\}.$$

Since $\binom{n}{m-1}$ is of the order of n^{m-1} and $\binom{n}{m-2}$ is of the order of n^{m-2} for large n, letting $n\to\infty$ we have

$$0\leqslant\tilde{\triangle}^{m-1}_{A,B}(P)\ \left(\Longleftrightarrow(-1)^m\triangle^{m-1}_{A,B}(P)\leqslant 0\right).$$

In conclusion, we have:

PROPOSITION 2.2. *If the pair (A,B) preserves order, then*

(i) m positive even and $(A,B)\in(m,P)$-expansive implies $(A,B)\in(m-1,P)$-expansive;

(ii) m positive odd and $(A,B)\in(m,P)$-contractive implies $(A,B)\in(m-1,P)$-contractive.

For pairs (T^*,T) this translates to (*cf* [13]):

PROPOSITION 2.3. *If $T\in(m,P)$-expansive for some even positive integer m (resp., $T\in(m,P)$-contractive for some odd positive integer m), then $T\in(m-1,P)$-expansive (resp., $T\in(m-1,P)$-contractive).*

3. Power bounded operators

Proposition 2.2 does not extend to odd positive integers m for (m,P)-expansive (resp., even positive integers m for (m,P)-contractive) operators T: for if it were so, then one would have that $T\in(m,P)$-expansive implies $T\in(m,P)$-hyperexpansive (resp., $T\in(m,P)$-contractive implies $T\in(m,P)$-hypercontractive). A class of operators where Proposition 2.2 does have an extension to all m is that of power bounded operators. We have:

THEOREM 3.1. *If A,B are power bounded, the pair (A,B) preserves order and $(A,B)\in(m,P)$-expansive (resp., $(A,B)\in(m,P)$-contractive), then $(A,B)\in(m,P)$-hyperexpansive (resp., $(A,B)\in(m,P)$-hypercontractive).*

Proof. In view of Proposition 2.2, we have only to prove that m odd, $(A,B)\in(m,P)$-expansive implies $(A,B)\in(m-1,P)$-expansive and m even, $(A,B)\in(m,P)$-contractive implies $(A,B)\in(m-1,P)$-contractive. And for this it is sufficient to prove that

$$\tilde{\triangle}^{m}_{A,B}(P)\geqslant 0\Longrightarrow\tilde{\triangle}^{m-1}_{A,B}(P)\leqslant 0,$$

since by definition

$$\triangle^{m}_{A,B}(P)\leqslant 0\Longleftrightarrow\tilde{\triangle}^{m}_{A,B}(P)\geqslant 0,\ m\text{ odd}$$

and

$$\triangle^{m}_{A,B}(P)\geqslant 0\Longleftrightarrow\tilde{\triangle}^{m}_{A,B}(P)\geqslant 0,\ m\text{ even}.$$

If $\tilde{\triangle}^{m}_{A,B}(P)\geqslant 0$, then

$$\tilde{\triangle}^{m}_{A,B}(P)=(L_AR_B)^m(P)-\sum_{j=0}^{m-1}\binom{m}{j}\tilde{\triangle}^{j}_{A,B}(P)\geqslant 0.$$

By hypothesis, (A,B) preserves order. Hence, since

$$\begin{aligned}&(L_AR_B)\left\{(L_AR_B)^t-\sum_{j=0}^{m-1}\binom{t}{j}\tilde{\triangle}^{j}_{A,B}\right\}\\&=(L_AR_B)^{t+1}-\left\{\sum_{j=0}^{m-1}\binom{t+1}{j}\tilde{\triangle}^{j}_{A,B}+\binom{t}{m-1}\tilde{\triangle}^{m}_{A,B}\right\},\end{aligned}$$

an induction argument shows that

$$\begin{aligned}0&\leqslant(L_AR_B)^n(P)-\left\{\sum_{j=0}^{m-1}\binom{n}{j}\tilde{\triangle}^{j}_{A,B}(P)+\binom{n-1}{m-1}\tilde{\triangle}^{m}_{A,B}(P)\right\}\\&\leqslant(L_AR_B)^n(P)-\sum_{j=0}^{m-1}\binom{n}{j}\tilde{\triangle}^{j}_{A,B}(P)\end{aligned}\tag{2}$$

for all integer $n\geqslant m$.

The power bounded hypothesis on A,B implies

$$|\langle(L_AR_B)^n(P)x,x\rangle|\leqslant\left\|P^{\frac{1}{2}}\right\|^2\|A^{*n}\|\,\|B^n\|\,\|x\|^2\leqslant M\|x\|^2$$

for some scalar $M>0$. Hence, since

$$\sum_{j=0}^{m-1}\binom{n}{j}\tilde{\triangle}^{j}_{A,B}=\binom{n}{m-1}\tilde{\triangle}^{m-1}_{A,B}+\sum_{j=0}^{m-2}\binom{n}{j}\tilde{\triangle}^{j}_{A,B},$$

$\binom{n}{m-1}$ is of the order of n^{m-1} and $\binom{n}{j}$ is of the order of n^{m-2} $(0 \leqslant j \leqslant m-2)$ as $n \to \infty$, it follows upon dividing the inequality in (2) by n^{m-1} and letting $n \to \infty$ that

$$-\tilde{\triangle}_{A,B}^{m-1}(P) \geqslant 0 \Longleftrightarrow \tilde{\triangle}_{A,B}^{m-1}(P) \leqslant 0. \quad \square$$

The following theorem says that for power bounded order preserving pairs of operators $(A,B) \in (m,P)$-expansive, A and B have a simple form: B is similar to an isometry and A is similar to a co-isometry.

THEOREM 3.2. *Given power bounded operators A,B such that (A,B) preserves order, if $(A,B) \in (m,P)$-expansive, then there exist positive operators P_i and isometries V_i, $i = 1,2$, such that $A = P_1^{-1}V_1^*P_1$ and $B = P_2^{-1}V_2P_2$.*

Proof. Since $\triangle_{A,B}^m(P) \leqslant 0$ implies $\triangle_{A^n,B^n}^m(P) \leqslant 0$ for all positive integers n, we have:

$$\begin{aligned}
&(A,B) \in (m,P)\text{-expansive} \Longrightarrow \triangle_{A^n,B^n}^m(P) \leqslant 0\\
\Longleftrightarrow\ & P \leqslant \sum_{j=1}^m (-1)^{j+1}\binom{m}{j} A^{nj}PB^{nj}\\
\Longleftrightarrow\ & I \leqslant \sum_{j=1}^m (-1)^{j+1}\binom{m}{j}\left(P^{-\frac{1}{2}}A^nP^{\frac{1}{2}}\right)^j\left(P^{\frac{1}{2}}B^nP^{-\frac{1}{2}}\right)^{j-1}\left(P^{\frac{1}{2}}B^nP^{-\frac{1}{2}}\right)\\
\Longrightarrow\ & \|x\| \leqslant \left\|\sum_{j=1}^m (-1)^{j+1}\binom{m}{j}\left(P^{-\frac{1}{2}}A^n(P^{\frac{1}{2}}\right)^j\left(P^{\frac{1}{2}}B^n(P^{-\frac{1}{2}}\right)^{j-1}\right\|\left\|P^{\frac{1}{2}}B^nP^{-\frac{1}{2}}x\right\|\\
\Longrightarrow\ & \|x\| \leqslant M_0\left\|P^{\frac{1}{2}}B^nP^{-\frac{1}{2}}x\right\|
\end{aligned}$$

for some scalar $M_0 > 0$ and all $x \in \mathscr{H}$. The operator $P^{\frac{1}{2}}BP^{-\frac{1}{2}}$ being power bounded, there exists a scalar $M_1 > 0$ such that

$$\frac{1}{M_0}\|x\| \leqslant \left\|\left(P^{\frac{1}{2}}BP^{-\frac{1}{2}}\right)^n x\right\| \leqslant M_1\|x\|$$

for all $x \in \mathscr{H}$. Hence there exists an invertible operator S and an isometry V such that

$$P^{\frac{1}{2}}BP^{-\frac{1}{2}} = S^{-1}VS \Longleftrightarrow B = \left(SP^{\frac{1}{2}}\right)^{-1}V\left(SP^{\frac{1}{2}}\right)$$

[16]. But then

$$B^*P^{\frac{1}{2}}S^*SP^{\frac{1}{2}}B = P^{\frac{1}{2}}S^*SP^{\frac{1}{2}} \Longleftrightarrow B^*P_1^2B = P_1^2$$

for some invertible positive operator $P_1^2 = P^{\frac{1}{2}}S^*SP^{\frac{1}{2}}$.

Conclusion: there exists an isometry V_1 and a positive invertible operator P_1 such that

$$B^*P_1 = P_1V_1^* \Longleftrightarrow B = P_1^{-1}V_1P_1.$$

To complete the proof, we apply the above argument to

$$\triangle^m_{B^*,A^*}(P) \leqslant 0 \left(\Longleftrightarrow \triangle^m_{A,B}(P) \leqslant 0\right)$$

to conclude the existence of an invertible positive operator P_2 and an isometry V_2 such that $A = P_2^{-1}V_2^*P_2$. □

For (m,P)-contractive pairs (A,B) of power bounded operators Theorem 3.1 implies

$$\triangle_{A,B}(P) \geqslant 0 \Longleftrightarrow \left(P^{-\frac{1}{2}}A(P^{\frac{1}{2}}\right)\left(P^{\frac{1}{2}}B(P^{-\frac{1}{2}}\right) \leqslant I.$$

Letting $A = B^* = T^*$, it then follows that: if $T \in (m,P)$-expansive (resp., $T \in (m,P)$-contractive), then T is similar to an isometry (resp., T is similar to a contraction, hence similar to a part of a co-isometric operator [17, Lemma 7.1]).

More is true. Since $T^{*p}QT^p = L^p_{T^*}R^p_t(Q) \geqslant 0$ for all positive integers p and operators $Q \geqslant 0$, the pair (T^*,T) is order preserving. The following theorem says that a power bounded (m,P)-isometric operator T is indeed an isometry (hence n-isometric for all $n \geqslant 1$) in an equivalent norm.

THEOREM 3.3. *The following conditions are pairwise equivalent for (m,P)-expansive operators $T \in B(\mathscr{H})$.*

(i) T is power bounded.

(ii) T is (a $C_{1\cdot}$-operator which is) similar to an isometry.

(iii) There exists a positive invertible operator Q such that $T \in (n,Q)$-isometric for all integers $n \geqslant 1$.

(iv) There exists a positive invertible operator Q and an equivalent norm $||.||_Q$ on $\mathscr{H}$ induced by the inner product $\langle .,. \rangle_Q = \langle Q.,. \rangle$ such that T is n-isometric for all integers $n \geqslant 1$ in this new norm.

Proof. $(i) \Longrightarrow (ii)$. If $T \in (m,P)$-expansive, then (see above) there exists a positive invertible operator $P_1 \in B(\mathscr{H})$ and an isometry $V_1 \in B(\mathscr{H})$ such that $P_1T = V_1P_1$. The operator T being power bounded, there exists a direct sum decomposition $\mathscr{H} = \mathscr{H}_{11} \oplus \mathscr{H}_{12}$ of $\mathscr{H}$ such that

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathscr{H}_{11} \oplus \mathscr{H}_{12}),\ T_1 \in C_{0\cdot} \text{ and } T_2 \in C_{1\cdot}$$

[15]. Decompose V_1 into its completely non-unitary (i.e., forward unilateral shift) and unitary parts by

$$V_1 = V_{10} \oplus V_{1u} \in B(\mathscr{H}_{10} \oplus \mathscr{H}_{20}).$$

Let $P_1 \in B(\mathscr{H}_{11} \oplus \mathscr{H}_{12}, \mathscr{H}_{10} \oplus \mathscr{H}_{20})$ have the matrix representation

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} P_1T &= V_1P_1 \Longrightarrow P_{12}^*T_1 = V_{1u}P_{12}^* \Longrightarrow P_{12}^*T_1^n = V_{1u}^nP_{12}^* \text{ (all positive integers } n) \\ &\Longrightarrow \|P_{12}^*x\| = \|V_{1u}^nP_{12}^*x\| = \|P_{12}^*T_1^nx\| \leqslant \|P_{12}^*\|\,\|T_1^nx\| \end{aligned}$$

for all $x \in \mathscr{H}_{11}$. Since $T_1 \in C_{0\cdot}$,

$$\|P_{12}^*x\| \to 0 \text{ as } n \to \infty \Longleftrightarrow P_{12}^* = 0.$$

Hence

$$P = P_{11} \oplus P_{22},\ P_{11} \text{ and } P_{22} \geqslant 0 \text{ invertible},$$

and

$$P_1T = V_1P_1 \Longrightarrow P_{11}T_3 = 0,\ P_{11}A_1 = V_{10}P_{11}.$$

Consequently, $T_3 = 0$ and

$$\begin{aligned} & P_{11}A_1 = V_{10}P_{11} \Longrightarrow P_{11}A_1^n = V_{10}^nP_{11} \text{ (all positive integers } n) \\ &\Longrightarrow \|P_{11}x\| = \|V_{10}^nP_{11}x\| = \|P_{11}A_1^nx\| \leqslant \|P_{11}\|\,\|A_1^nx\| \to 0 \text{ as } n \to \infty \\ &\qquad (\text{since } A_1 \in C_{0\cdot}) \\ &\Longrightarrow \|P_{11}x\| = 0 \Longleftrightarrow P_{11} = 0 \text{ or } x = 0. \end{aligned}$$

Since P_{11} is invertible, we must have $\mathscr{H}_{11} = \{0\}$, and then T is a $C_{1\cdot}$-operator such that $T = P_1^{-1}V_1P_1$.

$(ii) \Longrightarrow (iii)$. Evident, since (ii) holds implies

$$T = P_1^{-1}VP_1 \Longrightarrow T^*QT = Q, Q = P_1^2 \Longrightarrow T \in (n,Q)\text{-isometric}$$

for all positive integers $n \geqslant 1$.

$(iii) \Longrightarrow (iv)$. The operator $Q \geqslant 0$ being invetible, $||.||_Q$ is an equivalent norm on $\mathscr{H}$ [18] such that $\sum_{j=0}^n (-1)^j \binom{n}{j} ||T^jx||_Q^2 = 0$ for integers $n \geqslant 1$ and all $x \in \mathscr{H}$.

$(iv) \Longrightarrow (i)$. Evident, since $T \in (n,Q)$-isometric implies $T^p \in (n,Q)$-isometric for all integers $p \geqslant 1$, in particular

$$\begin{aligned} & 0 = ||x||_Q^2 - ||T^px||_Q^2 = \langle (Q - T^{*p}QT^p)x, x\rangle \text{ for all } x \in \mathscr{H} \Longleftrightarrow Q = T^{*p}QT^p \\ \Longleftrightarrow & \text{ there exists an isometry } V \text{ such that } T^{*p}Q^{\frac12} = Q^{\frac12}V^* \Longleftrightarrow T^p = Q^{-\frac12}VQ^{\frac12} \\ \Longrightarrow & \sup_p ||T^p|| \leqslant ||Q^{-\frac12}||\,||Q^{\frac12}|| < \infty. \end{aligned}$$

This completes the proof. □

For (m,P)-contractive power bounded operators, we have:

THEOREM 3.4. *If T is a power bounded (m,P)-contractive operator in $B(\mathscr{H})$, then T is similar to the direct sum of the adjoint of a $C_{0\cdot}$-contraction with a unitary.*

Proof. If $T \in (m,P)$-contractive is power bounded, then $T \in (m,P)$-hypercontractive (by Theorem 2.1) and hence

$$\triangle_{T^*,T}(P) \geqslant 0 \Longleftrightarrow P \geqslant T^*PT.$$

Consequently, there exists a contraction $C \in B(\mathscr{H})$ such that

$$P^{\frac{1}{2}}C = T^*P^{\frac{1}{2}}.$$

The contraction C has a decomposition, the Foguel decomposition [17],

$$C = Z \oplus U \in B(\mathscr{H}_c \oplus \mathscr{H}_c^{\perp}),$$

$$\mathscr{H}_e = \{x \in \mathscr{H} : \langle C^n x, y\rangle \to 0 \text{ as } n \to \infty, \text{ all } y \in \mathscr{H}\},$$

where U is unitary and

$$\lim_{n\to\infty} \langle Z^n x, x\rangle = 0 \text{ for all } x \in \mathscr{H}_c$$

(i.e., $Z \in B(\mathscr{H}_c)$ is weakly $C_{0\cdot}$). Letting, as before

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix} \in B(\mathscr{H}_{11} \oplus \mathscr{H}_{12}),\ T_1 \in C_{0\cdot} \text{ and } T_2 \in C_{1\cdot},$$

and letting

$$P^{\frac{1}{2}} = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix} \in B(\mathscr{H}_{11} \oplus \mathscr{H}_{12}, \mathscr{H}_c \oplus \mathscr{H}_c^{\perp}).$$

the equality

$$\begin{aligned} & P_{12}U = T_1^*P_{12} \Longleftrightarrow U^*P_{12}^* = P_{12}^*T_1 \Longrightarrow U^{*n}P_{12}^* = P_{12}^*T_1^n \\ \Longrightarrow\ & \|P_{12}^*x\| = \|U^{*n}P_{12}^*x\| = \|P_{12}^*T_1^n x\| \leqslant \|P_{12}^*\|\,\|T_1^n x\| \ (\text{all } x \in \mathscr{H}_{11}) \\ \Longrightarrow\ & \|P_{12}^*x\| \leqslant \|P_{12}^*\| \lim_{n\to\infty} \|T_1^n x\| = 0 \\ \Longrightarrow\ & P_{12} = 0,\ P^{\frac{1}{2}} = P_{11} \oplus P_{22},\ P_{11} \text{ and } P_{22} \geqslant 0 \text{ invertible.} \end{aligned}$$

Considering now $T_3^*P_{11} = 0$ it follows that

$$T_3 = 0,\ T = T_1 \oplus T_2,\ T_1^* = P_{11}ZP_{11}^{-1},\ T_2^* = P_{22}UP_{22}^{-1}$$

and T is similar to the direct sum of the adjoint of a $C_{0\cdot}$-contraction (hence, a weakly $C_{0\cdot}$-contraction) with a unitary. □

It is clear from the above that in the case in which $T \in (m,P)$-isometric, then ($T \in (m,P)$-expansive $\wedge$ (m,P)-contractive) $P_1 - T^*P_1T = 0$, where the similarity P_1 may be chosen to be the operator P. In particular, if $P = I$, then T is isometric.

4. Algebraic T

If $T \in B(\mathscr{H})$ is an algebraic operator (i.e., there exists a polynomial q such that $q(T) = 0$), then T has a representation

$$T = \bigoplus_{i=1}^{t} T\mid_{\mathscr{H}_0(T-\lambda_i I)},\ \mathscr{H} = \bigoplus_{i=1}^{t} \mathscr{H}_0(T-\lambda_i I)$$

for some positive integer t and scalars λ_i, where

$$\begin{aligned}\mathscr{H}_0(T-\lambda_i I) &= \left\{x \in \mathscr{H} : \lim_{n\to\infty} \|(T-\lambda_i I)^n x\|^{\frac{1}{n}} = 0\right\}\\ &= (T-\lambda_i I)^{-p_i}(0)\end{aligned}$$

for some positive integer p_i. The points λ_i are poles of the resolvent of T of order p_i and (therefore) each $T_i = T\mid_{\mathscr{H}_0(T-\lambda_i I)}$ has a representation

$$T_i = \lambda_i I_i + N_i,\ 1 \leqslant i \leqslant t,$$

where I_i is the identity of $B(\mathscr{H}_0(T-\lambda_i I))$ and N_i is p_i-nilpotent. Evidently,

$$T = \bigoplus_{i=1}^{t} T_i = \bigoplus_{i=1}^{t} (\lambda_i I_i + N_i) = T_0 + N,$$

where T_0 is a normal operator with

$$\sigma(T_0) = \sigma_a(T_0) = \sigma(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_t\}$$

and N is a nilpotent of order $p = \max\{p_i : 1 \leqslant i \leqslant t\}$.

Assume now that $T \in (m,P)$-expansive, $P \geqslant 0$ invertible (as before).

If $\lambda \in \sigma_a(T)$, then there exists a sequence of unit vectors $\{x_n\} \subset \mathscr{H}$ such that $\lim_{n\to\infty} \|(T-\lambda I)x_n\| = 0$ and

$$\begin{aligned}\lim_{n\to\infty} \langle \triangle^m_{T^*,T}(P)x_n, x_n\rangle &= \lim_{n\to\infty} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\|P^{\frac{1}{2}} T^j x\right\|^2\\ &= \lim_{n\to\infty} \sum_{j=0}^{m} (-1)^j \binom{m}{j} |\lambda|^{2j} \left\|P^{\frac{1}{2}} x_n\right\|^2\\ &= \lim_{n\to\infty} \left(1-|\lambda|^2\right)^m \left\|P^{\frac{1}{2}} x_n\right\|^2 \leqslant 0.\end{aligned}$$

Since $P \geqslant 0$ is invertible, we must have

$$|\lambda| = 1 \text{ if } m \text{ is even } (\Longrightarrow \sigma_a(T) \subseteq \partial\mathbb{D} \text{ if } m \text{ is even})$$

and

$$|\lambda| \geqslant 1 \text{ if } m \text{ is odd } (\Longrightarrow \sigma_a(T) \subseteq \mathbb{C}\setminus\mathbb{D} \text{ if } m \text{ is odd}).$$

Algebraic (m,P)-expansive operators cannot be Drazin invertible (hence are invertible). To see this, let T be an algebraic (m,P)-expansive Drazin invertible operator. Then there exists a decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ of $\mathscr{H}$, a decomposition $T = T|_{\mathscr{H}_1} \oplus T|_{\mathscr{H}_2} = T_1 \oplus T_2$ of T such that T_1 is invertible and T_2 is p-nilpotent for some positive integer p. Since

$$T \in (m,P)\text{-expansive} \Longrightarrow T^p \in (m,P)\text{-expansive},$$

letting $P \in B(\mathscr{H}_1 \oplus \mathscr{H}_2)$ have the representation $P = [P_{ik}]_{i,k=1}^2$, we have

$$\begin{aligned} 0 &\geqslant \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*pj} P T^{pj} \\ &= \left[\sum_{j=0}^{m} (-1)^j \binom{m}{j} T_i^{*pj} P_{ik} T_k^{pj} \right]_{i,k=1}^2 \\ &= \begin{pmatrix} \sum_{j=0}^{m} (-1)^j \binom{m}{j} T_1^{*pj} P_{11} T_1^{pj} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \\ &\Longrightarrow P_{22} = 0 \text{ (since } P_{22} \geqslant 0). \end{aligned}$$

But then, by the positivity of P, $P_{12} = P_{21}^* = 0$ ([3], Theorem I.1). Since P is invertible, this is a contradiction.

The following theorem says that for an algebraic (m,P)-expansive operator T, either $r(T) > 1$ or T is the direct sum of a unitary with a nilpotent $(2n-1)$-isometric operator for some positive integer n.

THEOREM 4.1. *If $T \in B(\mathscr{H})$ is an algebraic m-expansive operator such that $r(T) \leqslant 1$, then:*

(i) T is a perturbation of a unitary by a commuting nilpotent;

(ii) there exist positive integers m_0 and n,

$$m_0 \leqslant m, \; m_0 \text{ odd}, \; n \geqslant \frac{m_0+1}{2},$$

such that

$$T \in (2n-1)\text{-isometric}.$$

Proof. We consider m even and m odd cases separately. If m is even, then (as seen above)

$$\sigma(T) = \sigma_a(T) \subseteq \partial\mathbb{D} \Longrightarrow \sigma(T_0) \subseteq \partial\mathbb{D}$$

hence the normal operator T_0 is a unitary (and $T = T_0 + N$, $[T_0, N] = 0$, is the perturbation of T_0 by a nilpotent). Then

$$\begin{aligned}\triangle_{T^*,T} &= (I - L_{T^*}R_T) = \left(I - L_{T_0^*+N^*}R_{T_0+N}\right) \\ &= \left(I - L_{T_0^*}R_{T_0}\right) - \left\{L_{N^*}R_{T_0} + L_{T_0^*+N^*}R_N\right\} \\ &= \triangle_{T_0^*,T_0} - \left\{L_{N^*}R_{T_0} + L_{T_0^*+N^*}R_N\right\} \\ &= \triangle_{T_0^*,T_0} - S \text{ (say)}\end{aligned}$$

and

$$\begin{aligned}\triangle^t_{T^*,T}(I) &= \left(\sum_{j=0}^{t}(-1)^j \binom{t}{j} \triangle^{t-j}_{T_0^*,T_0} S^t\right)(I) \\ &= \sum_{j=0}^{t}(-1)^j \binom{t}{j} S^j \triangle^{t-j}_{T_0^*,T_0}(I)\end{aligned}$$

(since $[T_0,N] = 0$). Evidently,

$$T_0 \in 1\text{-isometric } (\Longleftrightarrow \triangle_{T_0^*,T_0}(I) = 0);$$

hence

$$\triangle^t_{T^*,T}(I) = (-1)^t S^t = (-1)^t \left(\sum_{k=0}^{t} \binom{t}{k} R^{t-k}_{T_0^*} L^k_{T_0^*+N^*} L^{t-k}_{N^*} R^k_N\right)(I)$$

This implies that if N is n-nilpotent and $t = 2n-1$, then $S = 0$ and, consequently, $T \in (2n-1)$-isometric. We prove that $n \geqslant \frac{m_0+1}{2}$. By hypothesis (above) the odd integer m_0 is the smallest positive integer such that

$$\begin{aligned}\langle \triangle^{m_0-1}_{T^*,T}(I)x_0, x_0\rangle &= \sum_{j=0}^{m_0-1}(-1)^j \binom{m_0-1}{j} \left\|T^j x_0\right\|^2 > 0 \\ \Longleftrightarrow \langle S^{m_0-1}x_0, x_0\rangle &= \sum_{k=0}^{m_0-1}(-1)^j \binom{m_0-1}{k} N^{*m_0-1-k}\langle T_0^* + N^{*m_0-1} T_0^{m_0-1-1} x_0, x_0\rangle N^k \\ &> 0.\end{aligned}$$

Since $n = \frac{m_0-1}{2}$ forces

$$\langle S^{m_0-1}x_0, x_0\rangle = 0,$$

we must have $N^n \neq 0$ for all $n \leqslant \frac{m_0-1}{2}$.

If m is odd, then

$$\sigma(T) = \sigma_a(T) \subseteq \mathbb{C} \setminus \mathbb{D}$$

and the spectral radius

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$$

satisfies $r(T) = 1$ or $r(T) > 1$. If $r(T) = 1$, then $\sigma(T) = \sigma_a(T) \subseteq \partial\mathbb{D}$ and $T = T_0 + N$ is the perturbation of a unitary by a commuting nilpotent. The argument above applies, and the proof follows. $\square$

The theorem fails in the case in which m is odd and $r(T) > 1$. Consider, for example, the operator $T = \alpha I$, where $|\alpha| > 1$. Then

$$\triangle_{T^*,T}^{2m+1}(I) = \sum_{j=0}^{2m+1} (-1)^j \binom{2m+1}{j} |\alpha|^{2j} = \left(1-|\alpha|^2\right)^{2m+1} < 0$$

for all integers $m \geqslant 0$. Observe here that

$$\tilde{\triangle}_{T^*,T}^{m}(I) > 0$$

for all positive integers m (i.e., the operator T is m-alternatingly expansive [12, Definition 1.1(7)]). Is this typical of operators $T \in m$-expansive for some odd positive integer m with $r(T) > 1$? The operator T of the example evidently satisfies $T^*T > I$: The following proposition proves that invertible operators T such that $T \in (m,P)$-expansive, $P \geqslant 0$ invertible and $T^*T \geqslant 1$ are indeed (m,P)-alternatingly expansive.

PROPOSITION 4.2. *If an invertible operator $T \in (m,P)$-expansive, $P \geqslant 0$ invertible, satisfies $T^*T \geqslant 1$, then $T \in (m,P)$-alternatingly expansive.*

Proof. The hypotheses imply that T^{-1} is a contraction, hence power bounded, such that

$$\tilde{\triangle}_{T^{*-1},T^{-1}}^{m}(P) \leqslant 0.$$

Consequently,

$$\tilde{\triangle}_{T^{*-1},T^{-1}}^{m}(P) = \begin{cases} \triangle_{T^{*-1},T^{-1}}^{m}(P) \leqslant 0 \text{ if } m \text{ is even} \\ \triangle_{T^{*-1},T^{-1}}^{m}(P) \geqslant 0 \text{ if } m \text{ is odd,} \end{cases}$$

and this (by Proposition 2.3) implies

$$T^{-1} \in \begin{cases} (m,P)\text{-hyperexpansive if } m \text{ is even} \\ (m,P)\text{-hypercontractive if } m \text{ is odd.} \end{cases}$$

Since

$$\triangle_{T^{*-1},T^{-1}}^{t}(P) \leqslant 0 \Longrightarrow \begin{cases} \triangle_{T^*,T}^{t}(P) \leqslant 0 \text{ if } t \text{ is even} \\ \triangle_{T^*,T}^{t}(P) \geqslant 0 \text{ if } t \text{ is odd} \end{cases}$$

and

$$\triangle_{T^{*-1},T^{-1}}^{t}(P) \geqslant 0 \Longrightarrow \begin{cases} \triangle_{T^*,T}^{t}(P) \geqslant 0 \text{ if } t \text{ is even} \\ \triangle_{T^*,T}^{t}(P) \leqslant 0 \text{ if } t \text{ is odd,} \end{cases}$$

the proof follows. □

REMARK 4.3. We remark in closing that a similar analysis does not hold for (m,P)-contractive algebraic operators. Thus $T = \alpha I \oplus 0 \in B(\mathscr{H} \oplus \mathscr{H})$ is Drazin invertible $(m, P_1 \oplus P_2)$-contractive operator, P_1 and $P_2 \in B(\mathscr{H})$ are positive invertible, for all scalars α if m is even and for scalars α such that $|\alpha| \leqslant 1$ if m is odd.

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