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THE FRIEDRICHS EXTENSION OF REGULAR SYMMETRIC DIFFERENTIAL OPERATORS

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Abstract. We increase the class of regular symmetric differential operators and find, explicitly, the boundary conditions which determine the Friedrichs extension of each one of these symmetric differential operators.

1. Introduction

Consider regular differential equations

$$My = \lambda wy \text{ on } I = (a,b),\ -\infty \leqslant a < b \leqslant \infty,\ \lambda \in \mathbb{C},\ 0 < w \in L^1(I), \tag{1.1}$$

where M is a symmetric differential expression, and boundary conditions

$$UY_{a,b} = 0,\ U \in M_{l,2n}(\mathbb{C}), \tag{1.2}$$

where $M_{l,2n}(\mathbb{C})$ denotes the $l \times 2n$ matrices of complex numbers, l is an integer with $0 \leqslant l \leqslant 2n$, and $U \in M_{l,2n}(\mathbb{C})$ is a boundary condition matrix. Let

$$Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix},\ Y_{a,b} = Y_R = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix}. \tag{1.3}$$

DEFINITION 1.1. Let l be an integer $0 \leqslant l \leqslant 2n$. Any $l \times 2n$ matrix $U \in M_{l,2n}(\mathbb{C})$ with rank l is called a boundary condition matrix and the equation

$$U\, Y_R = 0,\ (y \in D_{\max}) \tag{1.4}$$

is called a boundary condition. For any such U we define an operator $T(U)$ from $L^2(J,w)$ into $L^2(J,w)$ by

$$\begin{aligned} D(T(U)) &= \{y \in D_{\max} :\ U\, Y_R = 0\} \\ T(U)\, y &= T_{\max}\, y,\ y \in D(T(U)). \end{aligned} \tag{1.5}$$

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When $l = 0$ we have $U = 0$ and $T(U) = T_{\max}$.

We study the operator realizations T

$$T_{\min} \subset T \subseteq T^* \subset T_{\max} \tag{1.6}$$

of (1.1) and (1.2) in the Hilbert space $H = L^2(I,w)$. The quasi-derivatives $y^{[r]}$, $T_{\min}$, $D_{\min}$, T, $T_{\max}$, $D_{\max}$ will be defined below in Section 2.

In 1995 [13, 14] Möller-Zettl constructed *symmetric expressions* M using the matrix $E \in M_n(\mathbb{C})$

$$E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n$$

where δ is the Kronicker delta, and proved that the minimal operator realization $T_{\min}$ of the regular equation (1.1) with positive leading coefficient is bounded below.

In 2019 Bao-Sun-Hao-Zettl [1] introduced a class of skew-diagonal matrices $C \in M_n(\mathbb{C})$ satisfying

$$C^{-1} = -C = C^* \tag{1.7}$$

and used these to construct *symmetric* differential expressions $M = M_Q$, where Q is a matrix function, to be defined in Section 2, satisfying

$$Q = -C^{-1}Q^*C \tag{1.8}$$

and used – the same matrices C – to characterize the boundary conditions (1.2) which determine *self-adjoint* operator realizations $T = T^*$ of (1.6). This class of matrices C and their use in the construction of Q and $M = M_Q$ significantly increased the class of symmetric differential equations (1.1). (The matrix E is a special case.)

In 2020 Wang-Zettl [26], used these same matrices C to characterize the boundary conditions of the *symmetric* operators realizations T of (1.6). (See also the web page `www.ams.org/bookpages/surv-245` of the book ‘Ordinary Differential Operaotrs’ by these authors.)

In this paper we

1. Prove that the minimal operator $T_{\min}$ of *this much larger class of symmetric differential expressions M is bounded below.*

2. *Given any symmetric operator realization T, $T_{\min} \subset T \subseteq T^* \subset T_{\max}$, of the larger class of equations* (1.1) *we find the boundary conditions of its Friedrichs extension T_F explicitly. The self-adjoint operators are special case.*

REMARK 1.2. The study of symmetric operators in Hilbert space has a long and interesting history dating back about a hundred years. Next we briefly review this and put it in context. Starting with the well known Von Neumann Theorem.

THEOREM 1.3. (Von Neumann) *Let T be a closed densely defined symmetric operator on a complex separable Hilbert space H, and let N_+ and N_- be the deficiency spaces of T. Then we have*

$$D(T^*) = D(T) \dotplus N_+ \dotplus N_-$$

An operator S is a closed symmetric extension of T if and only if there exist closed subspaces F_+ of N_+ and F_- of N_- and an isometric mapping V of F_+ onto F_- such that

$$D(S) = D(T) + \{g + Vg : \ g \in F_+\}.$$

Furthermore, S is self-adjoint if and only if $F_+ = N_+$ and $F_- = N_-$.

Proof. See [5], Naimark [15], or Weidmann [25]. □

In his seminal 1933 paper Friedrichs [6] proved that every symmetric operator S in a separable Hilbert space H, which is bounded below, has a self-adjoint extension which has the same lower bound as S. This came to be known as the Friedrichs extension which we denote by S_F. His proof can be described as follows:

Let $D(S)$ and $D(S^*)$ denote the domains of S and S^*, respectively. The domain $D_F(S)$ of the Friedrichs extension S_F of S consists of all y in $D(S^*)$ for which there exists a sequence y_m in $D(S)$ such that

1. $y_m \to y$ in H as $m \to \infty$,
2. $S((y_m - y_l), y_m - y_l) \to 0$ as $m, \ l \to \infty$.

Note that there is no boundary condition mentioned in this description of the domain of S_F.

In 1935 Friedrichs [7] proved that the Dirichlet boundary condition

$$y(a) = 0 = y(b)$$

determines the Friedrichs extension of

$$My = -y'' + qy = \lambda y \ on \ I = (a,b), \ -\infty < a < b < \infty.$$

In the books by Coddington-Levinson [4] and Dunford-Schwartz [5] a linear differential expression

$$My = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y \ on \ I \tag{1.9}$$

with $p_n \neq 0$ on I is defined to be symmetric if it is identical with

$$M^+ y = (-1)^n (\overline{p}_n y^{(n)}) + \cdots + \overline{p}_0 y \ on \ I \tag{1.10}$$

i.e. if $M^+ = M$. Clearly if we wish to "test" a given expression M for symmetry by this definition we need to write M^+ in the same form as M and then compare the coefficients. To do this we must assume that the coefficients of M are sufficiently smooth.

In [28] any formally self-adjoint differential operator My of order n can be expressed in the form

$$My = \sum_{j=0}^{[\frac{n}{2}]} (-1)^j (l_j y^{(j)})^{(j)} + \sum_{j=0}^{[\frac{n-1}{2}]} [(q_j y^{(j)})^{(j+1)} - (\overline{q}_j y^{(j+1)})^{(j)}], \tag{1.11}$$

where l_j are real-valued functions, and q_j are complex-valued functions, $[d]$ denotes the greatest integer $\leqslant d$. Moreover, if the coefficients p_j in (1.9) are all real, then the complex second term in (1.11) vanishes. Hence, a real symmetric expression M given by (1.9) with sufficiently smooth coefficients p_j must be of even order ($n = 2k$, $k \geqslant 1$) and have the form

$$My = \sum_{j=0}^{k} (-1)^j (l_j y^{(j)})^{(j)} \tag{1.12}$$

with l_j real, $j = 0, 1, 2, \ldots, k$.

Observe that if the coefficients l_j, q_j in (1.11) are not sufficiently differentiable, then (1.11) cannot be reduced to the form (1.9). Nevertheless, as we will see below, an analogue of (1.11) is symmetric without any differentiability conditions on the coefficients. So if one wishes to consider general symmetric differential expressions one is forced to use so-called quasi-differential forms. It turns out that there exist much more general quasi-differential forms than those analogous to (1.11) or to (1.12) in the real case.

Very general quasi-differential forms, particularly symmetric ones based on the matrix $C = E$, which contained in the recently discovered C in (1.7), were discovered by Shin in 1938 [18, 19] and 1940 [20]. They were rediscovered, in a slightly different but equivalent form, by Zettl in 1975 [29]. Special cases of these symmetric quasi-differential forms have been used extensively by many authors, including Barrett [3], Glazman [8], Hinton [9], Kogan and Rofe-Beketov [11], Naimark [15], Reid [17], Stone [21], Weyl [24], Walker [23]. For other work on differential operators, see Halperin [10], Naimark [15], Stone [21], Titchmarsh [22].

The development of the theory of symmetric differential operators in the books by Naimark [15] and by Akhiezer-Glazman [8] is based on the real symmetric form analogous to (1.12). Although these authors mention Shin's more general symmetric expressions they make no use of them in their books. (Perhaps because Shin's claim that there are only two deficiency indices for symmetric higher even order problems ($n = 2k$), as is the case when $k = 1$, also holds when $k > 1$ is false.) In [29] Zettl showed that the techniques used in the books of Naimark and Akhieser-Glazman, based largely on the work of Glazman using Hilbert space methods, can be applied to the much larger class of symmetric operators based on E. Recently Bao-Hao-Sun-Wang-Zettl have shown that these Hilbert space methods can be applied to the larger class of symmetric expressions generated by matrices C satisfying (1.7) used here.

For other methods of studying boundary value problems, including boundary triplets, see the recent book "Boundary Value Problems, Weyl Functions, and Differential Operators", by Behrndt, Hassi, and De Snoo [2].

The organization of this paper is as follows: In Section 2 we review the construction of the symmetric operators $M = M_Q$ where $Q = -C^{-1}Q^*C$ and the characterization of their symmetric domains. In Section 3 we prove that the minimal operator $T_{\min}$ is bounded below; this extends the corresponding theorem for $C = E$ proved by Möller-Zettl in [14]. For any given symmetric operator T, $T_{\min} \subset T \subseteq T^* \subset T_{\max}$ in (1.6) the boundary conditions determining its Friedrichs extensions are given in Section 4 and examples of these extensions are given in Section 5.

2. Symmetric operators

In this section we study the symmetric operator realizations T in (1.6). Let $M_n(X)$ denote the $n\times n$ matrices with elements from the set X for each $n=2,3,4,\dots$, i.e. $M_n(X)=M_{n,n}(X)$. For the complex number field $\mathbb{C}$, we write $\mathbb{C}^n:=M_{n,1}(\mathbb{C})$ which is the n dimensional column vector space.

Let

$$\begin{aligned}&Z_n(I):=\{(q_{r,s})_{r,s=1}^n\in M_n(L^1(I)),\\&q_{r,r+1}\neq 0\ a.e.\text{ on } I,\ q_{r,r+1}^{-1}\in L^1(I),\ 1\leqslant r\leqslant n-1,\\&q_{r,s}=0\ a.e.\text{ on } I,\ 2\leqslant r+1<s\leqslant n;\\&q_{r,s}\in L^1(I),\ s\neq r+1,\ 1\leqslant r\leqslant n-1\}\end{aligned}\tag{2.1}$$

and $C=(c_{r,s})_{1\leqslant r,s\leqslant n}\in M_n(\mathbb{C})$ denotes any skew-diagonal complex matrix with the following property:

$$C^{-1}=-C=C^*.$$

Let $AC_{loc}(I)$ denote the set of functions which are absolutely continuous on all compact subintervals of I. For $Q\in Z_n(I)$ define the quasi-derivatives $y^{[r]}$ $(0\leqslant r\leqslant n)$:

$$\begin{aligned}&V_0:=\{y:\ I\to\mathbb{C},\ y\text{ is measurable}\},\ y^{[0]}:=y\ (y\in V_0),\\&V_r:=\{y\in V_{r-1}:\ y^{[r-1]}\in AC_{loc}(I)\},\\&y^{[r]}=q_{r,r+1}^{-1}[(y^{[r-1]})'-\sum_{s=1}^{r}q_{r,s}y^{[s-1]}]\ (y\in V_r,\ r=1,2,\dots,n),\end{aligned}\tag{2.2}$$

where $q_{n,n+1}=c_{n,1}$. Finally we set

$$My=\mathrm{i}^ny^{[n]},\ y\in V_n,\ \mathrm{i}=\sqrt{-1}.\tag{2.3}$$

These expressions $M=M_Q$ are generated with Q and for the notation V_n we also use the notations $D(Q)$ and $V(M)$. Since the quasi-derivatives depend on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$, $r=1,2,\dots,n$.

For the rest of this paper we assume that

$$Q=-C^{-1}Q^*C,\tag{2.4}$$

i.e.,

$$q_{r,s}=c_{r,n+1-r}\,\overline{q}_{n+1-s,n+1-r}\,c_{n+1-s,s},\tag{2.5}$$

and call Q a C-symmetric matrix and $M=M_Q$ a C-symmetric quasi-differential or just differential expression.

Consider the Hilbert space $L^2(I,w)$ with inner product $(y,z)_w=\int_a^b y\bar{z}w\,dx$, $\|y\|_w=(y,y)_w^{\frac{1}{2}}$ and let

$$D_{\max}=\{y\in L^2(I,w):\ y\in D(Q)\quad\text{and}\quad \frac{1}{w}My\in L^2(I,w)\},\tag{2.6}$$

where $w \in L^1(I)$ is positive on I. We associate the maximal operator $T_{\max}$ and the minimal operator $T_{\min}$ with the differential expression $M = M_Q$ and note that $T^*_{\max} = T_{\min}$ and $T_{\max} = T^*_{\min}$. Let $D_{\min}$ denote the domain of $T_{\min}$ and $D_{\max}$ the domain of $T_{\max}$. It is well known that the minimal operator $T_{\min}$ is a densely defined symmetric operator in the Hilbert space $L^2(I,w)$, see [1].

For which matrices U is $T = T(U)$ a symmetric operator in $L^2(I,w)$? This question is answered by the next Theorem. Let

$$U = (A\ B),\ A,\ B \in M_n(\mathbb{C}),\ R = ACA^* - BCB^*,\ r = \operatorname{rank}(R). \tag{2.7}$$

THEOREM 2.1. *Suppose M is a regular symmetric differential expression and U is a boundary condition matrix with* $\operatorname{rank}(U) = l$, $0 \leqslant l \leqslant 2n$. *Then we have*

1. *If $l < n$, then $T(U)$ is not symmetric.*
2. *If $l = n$, then $T(U)$ is self-adjoint (and hence also symmetric) if and only if $r = 0$.*
3. *Let $l = n + s$, $0 < s \leqslant n$. Then $T(U)$ is symmetric if and only if $r = 2s$.*

Proof. See Theorem 6 and Theorem 11 in [26]. □

3. Boundedness below of the symmetric operators

In this section we extend the Möller-Zettl [13, Theorem 7.2] result that symmetric operators, with positive leading coefficient, generated by $Q = CQ^*C$ where $C = E$ are bounded below and their Friedrichs extension has the same lower bound, to the larger class of operators generated by $Q = CQ^*C$ where C is a class of skew-diagonal matrices satisfying (1.7).

Since equation (1.1) is regular it follows from the Von Neumann Theorem that every symmetric operator $T(U)$ is bounded below if the minimal operator $T_{\min}$ is bounded below and every $T(U)$ is a finite dimensional extension of $T_{\min}$:

$$T_{\min} \subseteq T(U) \subseteq T_{\max}. \tag{3.1}$$

Conseqently we can restrict our search to $l \times 2n$ boundary matrices U with $n \leqslant l = \operatorname{rank}(U) \leqslant 2n$.

THEOREM 3.1. *Suppose the regular even order C-symmetric differential equation* (1.1) *has the leading coefficient* $(-1)^k \overline{c}_{k,k+1} q_{k,k+1} > 0$ *a.e. on I. Then $T_{\min}$ is bounded below.*

Proof. This follows from the following Propositions 3.2–3.5. □

PROPOSITION 3.2. *For $r = 1,2,\dots,k$, $n = 2k$ and $y \in D_{\max}$, we have*

$$
\begin{aligned}
&(q_{n-r+1,n-r+2}y^{[n-r+1]},\overline{c}_{r,n-r+1}y^{[r-1]}) + (q_{n-r,n-r+1}y^{[n-r]},c_{n-r,r+1}y^{[r]}) \\
&= c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b - c_{r,n-r+1}\left(\sum_{s=1}^{r}(y^{[n-r]},q_{r,s}y^{[s-1]}) + \sum_{s=1}^{n-r+1}(q_{n-r+1,s}y^{[s-1]},y^{[r]})\right)
\end{aligned}
$$

with

$$(y,z) = \int_a^b y\overline{z}dx,\ y,\ z \in D_{\max}.$$

Proof. Since $Q \in Z_n(I)$ is C-symmetric, from (2.2) and (2.5) it follows that for $y \in D_{\max}$ we have

$$
\begin{aligned}
&(q_{n-r+1,n-r+2}y^{[n-r+1]},\overline{c}_{r,n-r+1}y^{[r-1]}) + (q_{n-r,n-r+1}y^{[n-r]},c_{n-r,r+1}y^{[r]}) \\
&= ((y^{[n-r]})' - \sum_{s=1}^{n-r+1} q_{n-r+1,s}y^{[s-1]},\overline{c}_{r,n-r+1}y^{[r-1]}) + (q_{n-r,n-r+1}y^{[n-r]},c_{n-r,r+1}y^{[r]}) \\
&= c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b + (y^{[n-r]},-\overline{c}_{r,n-r+1}(y^{[r-1]})' + c_{n-r,r+1}\overline{q}_{n-r,n-r+1}y^{[r]}) \\
&\quad - (\sum_{s=1}^{n-r+1} q_{n-r+1,s}y^{[s-1]},\overline{c}_{r,n-r+1}y^{[r-1]}) \\
&= c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b + (y^{[n-r]},\overline{c}_{r,n-r+1}[-(y^{[r-1]})' + q_{r,r+1}y^{[r]}]) \\
&\quad - (\sum_{s=1}^{n-r+1} q_{n-r+1,s}y^{[s-1]},\overline{c}_{r,n-r+1}y^{[r-1]}) \\
&= c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b - c_{r,n-r+1}\left(\sum_{s=1}^{r}(y^{[n-r]},q_{r,s}y^{[s-1]}) + \sum_{s=1}^{n-r+1}(q_{n-r+1,s}y^{[s-1]},y^{[r-1]})\right).
\end{aligned}
$$

$\square$

PROPOSITION 3.3. *For $y \in D_{\max}$ and $n = 2k$, we have*

$$
\begin{aligned}
&(y^{[n]},y) - (q_{k,k+1}y^{[k]},c_{k,k+1}y^{[k]}) \\
&= \sum_{r=1}^{k}\sum_{s=1}^{k}(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]},y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b\,.
\end{aligned}
$$

Proof. Since $c_{1,n}\overline{c}_{1,n} = 1$ and $c_{n,1} = -\overline{c}_{1,n}$, from Proposition 3.2 and (2.5) we infer

$$
\begin{aligned}
&(y^{[n]},y) - (q_{k,k+1}y^{[k]},c_{k,k+1}y^{[k]}) = (c_{n,1}y^{[n]},c_{n,1}y) - (q_{k,k+1}y^{[k]},c_{k,k+1}y^{[k]}) \\
&= -\sum_{r=1}^{k}\left((q_{n-r+1,n-r+2}y^{[n-r+1]},\overline{c}_{r,n-r+1}y^{[r-1]}) + (q_{n-r,n-r+1}y^{[n-r]},c_{n-r,r+1}y^{[r]})\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{r=1}^{k} c_{r,n-r+1}\left(\sum_{s=1}^{r}(y^{[n-r]}, q_{r,s}y^{[s-1]}) + \sum_{s=1}^{n-r+1}(q_{n-r+1,s}y^{[s-1]}, y^{[r-1]})\right) \\
&\quad - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{r=1}^{k}\sum_{s=1}^{r}(c_{r,n-r+1}\overline{q}_{r,s}y^{[n-r]}, y^{[s-1]}) \\
&\quad + \sum_{r=1}^{k}\sum_{s=1}^{n-r+1}(c_{r,n-r+1}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{r=n-k+1}^{n}\sum_{s=1}^{n-r+1}(c_{n-r+1,r}\overline{q}_{n-r+1,s}y^{[r-1]}, y^{[s-1]}) \\
&\quad + \sum_{r=1}^{k}\sum_{s=1}^{n-r+1}(c_{r,n-r+1}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{r=k+1}^{n}\sum_{s=1}^{n-r+1}(\overline{c}_{n-s+1,s}q_{n-s+1,r}y^{[r-1]}, y^{[s-1]}) \\
&\quad + \sum_{r=1}^{k}\sum_{s=1}^{n-r+1}(c_{r,n-r+1}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{s=k+1}^{n}\sum_{r=1}^{n-s+1}(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) \\
&\quad + \sum_{r=1}^{k}\sum_{s=1}^{n-r+1}(c_{r,n-r+1}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{r=1}^{k}\sum_{s=k+1}^{n-r+1}(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) \\
&\quad + \sum_{r=1}^{k}\sum_{s=1}^{n-r+1}(c_{r,n-r+1}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{r=1}^{k}\left(\sum_{s=k+1}^{n-r+1} - \sum_{s=1}^{n-r+1}\right)(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b \\
&= \sum_{r=1}^{k}\sum_{s=1}^{k}(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) - \sum_{r=1}^{k} c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b. \quad \square
\end{aligned}
$$

PROPOSITION 3.4. *For $y \in D_{\min}$ and $n = 2k$, we have*

$$
\begin{aligned}
(T_{\min}y, y)_w = &\sum_{r=1}^{k}\sum_{s=1}^{k}(-1)^k(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]}) \\
&+ (-1)^k(q_{k,k+1}y^{[k]}, c_{k,k+1}y^{[k]}).
\end{aligned}
$$

Proof. From Proposition 3.3 we infer

$$\begin{aligned}(T_{\min}y,y)_w &= (M_Q y,y) = \mathrm{i}^n(y^{[n]},y)\\ &= \sum_{r=1}^{k}\sum_{s=1}^{k}(-1)^k(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]},y^{[r-1]})\\ &\quad + (-1)^k(q_{k,k+1}y^{[k]},c_{k,k+1}y^{[k]}) - (-1)^k\sum_{r=1}^{k}c_{r,n-r+1}y^{[n-r]}\overline{y^{[r-1]}}\,|_a^b\\ &= \sum_{r=1}^{k}\sum_{s=1}^{k}(-1)^k(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]},y^{[r-1]}) + (-1)^k(q_{k,k+1}y^{[k]},c_{k,k+1}y^{[k]}).\end{aligned}$$

Hence the stated identity follows. □

PROPOSITION 3.5. *Let* $u(x) = (-1)^k\overline{c}_{k,k+1}q_{k,k+1}$. *Assume that the regular* C*-symmetric even order differential expression* M_Q *generated by* Q *has a positive leading coefficient, i.e.,* $u(x)$ *is positive a.e.:*

$$(-1)^k\overline{c}_{k,k+1}q_{k,k+1} > 0 \quad a.e.. \tag{3.2}$$

Then minimal operator $T_{\min}$ *in* $L^2(I,w)$*, associated with* Q *and the weight function* w*, is bounded below. Moreover, every symmetric extension of* $T_{\min}$ *is also bounded below.*

Proof. For $y \in D_{\min}$, from Proposition 3.4 we have

$$\begin{aligned}&|(T_{\min}y,y)_w - (y^{[k]},y^{[k]})_u|\\ &= |\sum_{r=1}^{k}\sum_{s=1}^{k}(-1)^k(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]},y^{[r-1]})|\\ &\leqslant \sum_{r=1}^{k}\sum_{s=1}^{k}|(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]},y^{[r-1]})|.\end{aligned} \tag{3.3}$$

For $Q \in Z_n(I)$, let $Q_k = (q_{r,s})_{r,s=1}^k$ whose columns are the first k columns of $Q = Q_n$ with components in first k rows. Then $D(\widetilde{T}_{\min}) \subset D(T_{\min}) \subset D(T_{\max}) \subset D(\widetilde{T}_{\max})$, where $\widetilde{T}_{\min}$ and $\widetilde{T}_{\max}$ denote the minimal and maximal operator associated with Q_k in Hilbert space $L^2(I,u)$, respectively, i.e. $\widetilde{T}_{\max}y = \widetilde{T}^*_{\min}y = \mathrm{i}^k\frac{1}{u}y^{[k]}_{Q_k} = \mathrm{i}^k\frac{1}{u}\overline{c}_{k,1}q_{k,k+1}y^{[k]}_Q$ (here $y^{[k]}_Q = y^{[k]}$). By using [12, Theorem 1] we obtain that for any $\delta_1 > 0$ there exists a $K(\delta_1) > 0$ such that for all $y \in D(\widetilde{T}_{\max})$ and $r = 1,2,\ldots,k$,

$$\|y^{[r-1]}\|_\infty \leqslant \delta_1(\widetilde{T}_{\max}y,\widetilde{T}_{\max}y)_u^{\frac{1}{2}} + K(\delta_1)(y,y)_w^{\frac{1}{2}}. \tag{3.4}$$

Moreover an application of [12, Corollary 2] yields that for any $\delta_2 > 0$ there exists a $K(\delta_2) > 0$ such that for $r,\ s = 1,2,\ldots,k,$ we have

$$\|\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}\|_1 \leqslant \delta_2\|\widetilde{T}_{\max}y\|_u + K(\delta_2)\|y\|_w,$$

where $\|\cdot\|_1 = \int_a^b |\cdot|dx$. Combined with (3.4) for any $\delta_3 > 0$ we have

$$|(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]}, y^{[r-1]})| \leqslant [\delta_3(y^{[k]}, y^{[k]})_u^{\frac{1}{2}} + K(\delta_3)(y,y)_w^{\frac{1}{2}}]^2.$$

Hence together with (3.3) we infer that for each $\delta > 0$, there exists a $K(\delta) > 0$ such that

$$|(T_{\min}y,y)_w - (y^{[k]}, y^{[k]})_u| \leqslant \delta(y^{[k]}, y^{[k]})_u + K(\delta)(y,y)_w.$$

Therefore for $\delta = 1$ we have

$$(T_{\min}y,y)_w \geqslant -K(\delta)(y,y)_w,$$

i.e., $T_{\min}$ is bounded below. Then it follows from the Von Neumann Theorem that every symmetric extension of $T_{\min}$ is also bounded below. $\square$

REMARK 3.6. The statement that regular symmetric differential operators with positive leading coefficient are bounded below was proved by Möller-Zettl [14, Lemma 3.1], [13, Theorem 7.2] for the class of operators generated by $E = ((-1)^r\delta_{r,n+1-s})_{r,s=1}^n$ and $Q = EQ^*E$. In this paper we have shown that for the larger class of skew-diagonal matrices C and the corresponding Q satisfying

$$Q = CQ^*C,\ C^{-1} = -C = C^*, \tag{3.5}$$

the symmetric operators $M = M_Q$ are bounded below.

4. The Friedrichs extension of symmetric operators

In this section, for each symmetric differential operator, we find the boundary condition which determine its Friedrichs extension. If $T(U)$ defined by (1.5) satisfies

$$\operatorname{rank}(UGU^*) = 2(l-n),\ n < l \leqslant 2n,$$

with

$$G = (-1)^k \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \tag{4.1}$$

then $M = M_Q$ generates symmetric operators T. Suppose M_Q has a positive leading coefficient that (3.2) is satisfied.

Now we introduce some notation and several Lemmas before starting our main results.

Write

$$U := VJ, \quad V = (V_1\ V_2), \quad V_j \in M_{l,n}(\mathbb{C}),\ \ j = 1,2, \tag{4.2}$$

in (1.5), and set

$$\hat{G} = JGJ, \tag{4.3}$$

with

$$J = \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix} \in M_{2n}(\mathbb{C}),$$

where I_k denotes the $k\times k$ identity matrix. Also $\mathscr{N}(V)$ denotes the null space of the matrix V and $\mathscr{R}(V)$ denotes the range of the matrix V.

For $y\in D_{\max}$ let

$$Y_k=\begin{pmatrix} y^{[0]} \\ \vdots \\ y^{[k-1]}\end{pmatrix},\ \hat{Y}_k=\begin{pmatrix} y^{[k]} \\ \vdots \\ y^{[n-1]}\end{pmatrix}, \tag{4.4}$$

and

$$\hat{Y}_{a,b}=\begin{pmatrix} Y_k(a) \\ Y_k(b) \\ \hat{Y}_k(a) \\ \hat{Y}_k(b)\end{pmatrix}.$$

Note that

$$(-1)^kC=(-1)^k\begin{pmatrix} 0 & \hat{C}_k \\ -\hat{C}_k^* & 0\end{pmatrix}$$

with $\hat{C}_k=(c_{r,k+s})_{r,s=1}^k$ is a skew-diagonal unitary matrix, that is,

$$\begin{aligned} c_{r,k+s}\overline{c}_{r,k+s}&=1,\ \text{for } r+s=k+1,\ 1\leqslant r\leqslant k; \\ c_{r,k+s}&=0,\ \text{otherwise.}\end{aligned} \tag{4.5}$$

Hence from (4.3) we have

$$\hat{G}=(-1)^k\begin{pmatrix} 0 & G_1 \\ -G_1^* & 0\end{pmatrix}$$

with

$$G_1=\begin{pmatrix} \hat{C}_k & 0 \\ 0 & -\hat{C}_k\end{pmatrix}.$$

REMARK 4.1. Here $\hat{G}$ satisfies

$$\hat{G}^{-1}=-\hat{G}=\hat{G}^*.$$

LEMMA 4.2. *Suppose $n\leqslant l\leqslant 2n$. Then the operator T which is defined on*

$$D(T)=\left\{y\in D_{\max}:\ V\hat{Y}_{a,b}=0,\ V\in M_{l,2n}(\mathbb{C})\right\}, \tag{4.6}$$

is a symmetric operator with l dimensional restriction of $T_{\max}$ if and only if there exists a matrix $N\in M_{(2n-l),2n}(\mathbb{C})$ satisfying

$$\operatorname{rank}(N)=2n-l,\qquad N\hat{G}N^*=0, \tag{4.7}$$

and V is a complete solution of the matrix equation

$$NV^*=0, \tag{4.8}$$

i.e., V satisfies the equation (4.8) with $\operatorname{rank}(V)=l$. *Moreover, the domain of its adjoint operator T^* is characterized by*

$$D(T^*)=\left\{y\in D_{\max}:\ N\hat{G}\hat{Y}_{a,b}=0\right\}, \tag{4.9}$$

where $\hat{G}\in M_{2n}(\mathbb{C})$ is defined as (4.3).

Proof. First let us assume that $N\in M_{(2n-l),2n}(\mathbb{C})$ satisfies (4.7) and (4.8). We show that T defined on (4.6) is symmetric. Note that $T(U)=T(VJ)$ with $U=VJ$ and $J\mathscr{N}(U)=\mathscr{N}(V)$, from [26, Lemma 14] and Theorem 2.1 we only need to prove $\mathscr{N}(V)\subset\mathscr{R}(\hat{G}V^*)$, i.e.,

$$\hat{Z}^*_{a,b}\hat{G}\hat{Y}_{a,b}=0$$

for all $y,\ z\in D(T)$ with

$$\hat{Y}_{a,b}=\begin{pmatrix}Y_k(a)\\Y_k(b)\\\hat{Y}_k(a)\\\hat{Y}_k(b)\end{pmatrix},\ \hat{Z}_{a,b}=\begin{pmatrix}Z_k(a)\\Z_k(b)\\\hat{Z}_k(a)\\\hat{Z}_k(b)\end{pmatrix}.$$

If $y,\ z\in D(T)$, in view of (4.8) there exists a column vector $\vec{c}\in\mathbb{C}^{2n-l}$ such that $\hat{Z}_{a,b}=N^*\vec{c}$ and a column vector $\vec{\hat{c}}\in\mathbb{C}^{2n-l}$ such that $\hat{Y}_{a,b}=N^*\vec{\hat{c}}$. This yields

$$\hat{Z}^*_{a,b}\hat{G}\hat{Y}_{a,b}=\vec{c}^{\,*}(N\hat{G}N^*)\vec{\hat{c}}=0. \tag{4.10}$$

Since V is a complete solution of matrix equation (4.8), it follows that $\operatorname{rank}(V)=l$. From Theorem 1.3 we also see that T is a l dimensional restriction of the maximal operator $T_{\max}$. Clearly the converse also holds. In fact, if T which is defined on (4.6) is symmetric, then from Theorem 2.1 we obtain that $\operatorname{rank}(UGU^*)=2(l-n)$, $n\leqslant l\leqslant 2n$, with $U=VJ$. Combined with Naimark Patching Lemma [26, Lemma 6] and (4.10) there exists a matrix $N\in M_{(2n-l),2n}(\mathbb{C})$ such that (4.7) is satisfied. Moreover (4.8) also holds.

Next we prove that (4.9) holds. Note that

$$D_{\min}\subseteq D(T)\subseteq D(T^*)\subseteq D_{\max},$$

since T is a l dimensional restriction of the maximal operator $T_{\max}$, this shows that the deficiency index of T is $(l-n)$ and, therefore, T^* is a $2n-l$ dimensional restriction of the maximal operator $T_{\max}$. On the other hand, we obtain

$$0=(Ty,z)_w-(y,T^*z)_w=Z^*_{a,b}GY_{a,b}=\hat{Z}^*_{a,b}\hat{G}\hat{Y}_{a,b},$$

where $y\in D(T)$ and $z\in D(T^*)$. It should be noted that, for any $\vec{c}\in\mathbb{C}^{2n-l}$, there exists a function $y\in D(T)$ such that $\hat{Y}_{a,b}=N^*\vec{c}$. It leads to $(N\hat{G}^*)\hat{Z}^*_{a,b}=0$ if $z\in D(T^*)$. By the fact $\operatorname{rank}(N\hat{G}^*)=2n-l$, we know that the dimension of the space solutions of equation $(N\hat{G}^*)\hat{Z}^*_{a,b}=0$ is l. Therefore, combined with $\hat{G}^*=-\hat{G}$, we obtain that (4.9) holds. The proof is complete. □

We decompose $N=(N_1\ N_2)$ with matrices $N_1,\ N_2\in M_{(2n-l),n}(\mathbb{C})$. Then

$$N\hat{G}=(-1)^k(-N_2G_1^*\ N_1G_1).$$

Since $D(T)\subset D(T^*)$. This implies that $N\hat{G}$ can be represented by a linear combination of row vectors of V. By the elementary matrix transformation of rows, we can rewrite V as

$$V=(-1)^k\begin{pmatrix} V_{11} & V_{12}\\ -N_2G_1^* & N_1G_1\end{pmatrix}, \tag{4.11}$$

where $V_{11},\ V_{12}\in M_{(2l-2n),n}(\mathbb{C})$.

LEMMA 4.3. *Let T be a symmetric operator as stated in Lemma 4.2. Then V can be represented as (4.11) with*

$$\operatorname{rank}(V_{12})=l-n.$$

REMARK 4.4. Let $\operatorname{rank}(N_1)=r_1(\leqslant(2n-l))$, by the above Lemma, there exists a nonsingular matrix P of order l so that the matrix V, defined by (4.11), also can be represented as follows.

$$V=P^{-1}\begin{pmatrix} \hat{V}_{11} & \hat{V}_{12}\\ \hat{V}_{21} & 0\\ -\hat{N}_{12}G_1^* & 0\\ -\hat{N}_{22}G_1^* & \hat{N}_{21}G_1\end{pmatrix}, \tag{4.12}$$

where $\hat{V}_{rs}\in M_{(l-n),n}(\mathbb{C})$, $r,\ s=1,2$ and $\operatorname{rank}(\hat{V}_{21})=\operatorname{rank}(\hat{V}_{12})=l-n$. In the following proof we use the notation " $\longrightarrow$ " to denote this multiplication process, eg. (4.13).

Proof. For $N=(N_1\ N_2)$ with $N_1,\ N_2\in M_{(2n-l),n}(\mathbb{C})$ being given, if $\operatorname{rank}(N_1)=r_1(\leqslant(2n-l))$, and $B\in M_{(n-r_1),n}(\mathbb{C})$ is a complete solution of the matrix equation $N_1B^*=0$, it is easy to see that $\operatorname{rank}(B)=n-r_1$. Since V is a complete solution of matrix equation $NV^*=0$, it follows that the row vectors of $(B\ 0)$ can be linear expressed by the row vectors of V. This implies that there exists a matrix $(\tilde{V}_{11}\ \tilde{V}_{12})$ and a nonsingular matrix of order l such that

$$V\longrightarrow\begin{pmatrix} \tilde{V}_{11} & \tilde{V}_{12}\\ B & 0\end{pmatrix}, \tag{4.13}$$

with $0\in M_{(n-r_1),n}$ and $\operatorname{rank}(\tilde{V}_{11})=\operatorname{rank}(\tilde{V}_{12})=l-n+r_1$.

On the other hand, in view of $\operatorname{rank}(N_1)=r_1$, we obtain by the elementary matrix transformation of rows that

$$N\longrightarrow\begin{pmatrix} 0 & \hat{N}_{12}\\ \hat{N}_{21} & \hat{N}_{22}\end{pmatrix}=(N_1\ N_2),$$

where $\mathrm{rank}(\hat{N}_{12}) = 2n-l-r_1$, $\hat{N}_{21}$, $\hat{N}_{22} \in M_{r_1,n}(\mathbb{C})$, 0 is a $(2n-l-r_1)\times n$ zero matrix. Moreover, it is easy to see that

$$(-N_2G_1^*\ N_1G_1) = \begin{pmatrix} -\hat{N}_{12}G_1^* & 0 \\ -\hat{N}_{22}G_1^* & \hat{N}_{21}G_1 \end{pmatrix}.$$

Since $N\hat{G}N^* = 0$ and $\mathrm{rank}(\hat{N}_{21}) = r_1$, we get that the row vectors of $(-\hat{N}_{12}G_1^*\ 0)$ and $(-\hat{N}_{22}G_1^*\ \hat{N}_{21}G_1)$ can be linear expressed by the row vectors of $(B\ 0)$ and $(\tilde{V}_{11}\ \tilde{V}_{12})$ in (4.13), respectively. This together with (4.11) and (4.13) yields (4.12), where $\mathrm{rank}(\hat{V}_{12}) = l-n$. This also shows that $\mathrm{rank}(\hat{V}_{21}) = l-n$. The proof is complete. □

LEMMA 4.5. *For $y \in D_{\max}$ and $n = 2k$, we have*

$$\begin{aligned}(T_{\max}y,y)_w = &\sum_{r=1}^{k}\sum_{s=1}^{k}(-1)^k(\overline{c}_{n-r+1,r}q_{n-r+1,s}y^{[s-1]},y^{[r-1]}) \\ &+(-1)^k(q_{k,k+1}y^{[k]},c_{k,k+1}y^{[k]}) + (Y_k^*(a)\ Y_k^*(b))G_1\begin{pmatrix}\hat{Y}_k(a)\\ \hat{Y}_k(b)\end{pmatrix}.\end{aligned}$$

Proof. From Proposition 3.3 we obtain this identity immediately. □

LEMMA 4.6. *Let T be a symmetric operator as stated in Lemma 4.2. Then there exists a constant matrix $H \in M_n(\mathbb{C})$ such that for $y \in D(T)$ we have*

$$(Y_k^*(a)\ Y_k^*(b))\,G_1\begin{pmatrix}\hat{Y}_k(a)\\ \hat{Y}_k(b)\end{pmatrix} = (Y_k^*(a)\ Y_k^*(b))H\begin{pmatrix}Y_k(a)\\ Y_k(b)\end{pmatrix}. \tag{4.14}$$

Proof. From the proof of Lemma 4.2, we have

$$D(T) = \left\{y \in D_{\max}:\ \hat{Y}_{a,b} = N^*\vec{c},\ \vec{c} \in \mathbb{C}^{2n-l}\right\}.$$

We decompose $N = (N_1\ N_2)$ with N_1, $N_2 \in M_{(2n-l),n}(\mathbb{C})$. Then

$$\begin{pmatrix}Y_k(a)\\ Y_k(b)\end{pmatrix} = N_1^*\vec{c}, \qquad \begin{pmatrix}\hat{Y}_k(a)\\ \hat{Y}_k(b)\end{pmatrix} = N_2^*\vec{c}.$$

This implies that

$$(Y_k^*(a)\ Y_k^*(b))G_1\begin{pmatrix}\hat{Y}_k(a)\\ \hat{Y}_k(b)\end{pmatrix} = \vec{c}^*(N_1G_1N_2^*)\vec{c}. \tag{4.15}$$

We prove (4.14) holds throughout the following two cases.

Case 1. $\mathrm{rank}(N_1) = 2n-l$. In this case, since $(2n-l) \leqslant n$, it follows that $(N_1N_1^*)$ is nonsingular and left multiplying $\begin{pmatrix}Y_k(a)\\ Y_k(b)\end{pmatrix} = N_1^*\vec{c}$ by $(N_1N_1^*)^{-1}N_1$ gives

$$\vec{c} = (N_1N_1^*)^{-1}N_1\begin{pmatrix}Y_k(a)\\ Y_k(b)\end{pmatrix}.$$

This together with (4.15) shows that (4.14) remains true when

$$H = ((N_1N_1^*)^{-1}N_1)^*(N_1G_1N_2^*)(N_1N_1^*)^{-1}N_1.$$

Case 2. $\operatorname{rank}(N_1) = r_1 < 2n-l$. In this case, according to the proof of Lemma 4.3, there exists an $(2n-l)\times(2n-l)$ nonsingular matrix P so that

$$PN_1 = \begin{pmatrix} 0 \\ \hat{N}_{21} \end{pmatrix}, \qquad PN_2 = \begin{pmatrix} \hat{N}_{12} \\ \hat{N}_{22} \end{pmatrix}, \tag{4.16}$$

where $\hat{N}_{21},\ \hat{N}_{22} \in M_{r_1,n}(\mathbb{C})$ and $0,\ \hat{N}_{12} \in M_{(2n-l-r_1),n}(\mathbb{C})$. Without loss of generality, we assume N_1 and N_2 have the forms of the right hand side of (4.16). For any $y \in D(T)$, there exists a column vector $\vec{c} = \begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} \in \mathbb{C}^{2n-l}$ satisfying

$$\begin{pmatrix} Y_k(a) \\ Y_k(b) \end{pmatrix} = (0\ \hat{N}_{21}^*)\begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} = \hat{N}_{21}^*\vec{c}_2,$$
$$\begin{pmatrix} \hat{Y}_k(a) \\ \hat{Y}_k(b) \end{pmatrix} = (\hat{N}_{12}^*\ \hat{N}_{22}^*)\begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} = \hat{N}_{12}^*\vec{c}_1 + \hat{N}_{22}^*\vec{c}_2,$$

where $\vec{c}_1 \in \mathbb{C}^{(2n-l-r_1)}$ and $\vec{c}_2 \in \mathbb{C}^{r_1}$. Similar to the argument in Case 1, we get

$$\vec{c}_2 = (\hat{N}_{21}\hat{N}_{21}^*)^{-1}\hat{N}_{21}\begin{pmatrix} Y_k(a) \\ Y_k(b) \end{pmatrix}. \tag{4.17}$$

Note that from (4.7) and (4.16) we have

$$\begin{aligned}
0 &= N\hat{G}N^* \\
&= (-1)^k \begin{pmatrix} 0 & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix}\begin{pmatrix} 0 & G_1 \\ -G_1^* & 0 \end{pmatrix}\begin{pmatrix} 0 & \hat{N}_{21}^* \\ \hat{N}_{12}^* & \hat{N}_{22}^* \end{pmatrix} \\
&= (-1)^k \begin{pmatrix} 0 & -\hat{N}_{12}G_1^*N_{21}^* \\ \hat{N}_{21}G_1\hat{N}_{12}^* & -\hat{N}_{22}G_1^*\hat{N}_{21}^* + \hat{N}_{21}G_1\hat{N}_{22}^* \end{pmatrix},
\end{aligned}$$

which implies $N_{12}G_1^*N_{21}^* = 0$. Furthermore, together with (4.16) we infer

$$\begin{aligned}
N_2G_1^*N_1^* &= \begin{pmatrix} \hat{N}_{12} \\ \hat{N}_{22} \end{pmatrix} G_1^*(0\ \hat{N}_{21}^*) \\
&= \begin{pmatrix} 0 & \hat{N}_{12}G_1^*N_{21}^* \\ 0 & \hat{N}_{22}G_1^*\hat{N}_{21}^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{N}_{22}G_1^*\hat{N}_{21}^* \end{pmatrix}.
\end{aligned}$$

Therefore, we infer

$$\begin{aligned}
\vec{c}^*(N_1G_1N_2^*)\vec{c} &= (\vec{c}_1^*\ \vec{c}_2^*)\begin{pmatrix} 0 & 0 \\ 0 & \hat{N}_{21}G_1\hat{N}_{22}^* \end{pmatrix}\begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \end{pmatrix} \\
&= \vec{c}_2^*(\hat{N}_{21}G_1\hat{N}_{22}^*)\vec{c}_2.
\end{aligned}$$

This, together with (4.15) and (4.17), shows that (4.14) remains true when

$$H = ((\hat{N}_{21}\hat{N}_{21}^*)^{-1}\hat{N}_{21})^*(\hat{N}_{21}G_1\hat{N}_{22}^*)(\hat{N}_{21}\hat{N}_{21}^*)^{-1}\hat{N}_{21}.$$

The proof is complete. □

Now we state our main Theorems:

THEOREM 4.7. *Let the notations and assumptions be as in Proposition 3.5 and assume that T defined by*

$$D(T) = \left\{y \in D_{\max} :\ V\hat{Y}_{a,b} = 0,\ V \in M_{l,2n}(\mathbb{C})\right\},\ n < l \leqslant 2n, \tag{4.18}$$

is a symmetric extension of $T_{\min}$. Then V can be rewritten as

$$V = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & 0 \\ -\hat{N}_{12}G_1^* & 0 \\ -\hat{N}_{22}G_1^* & \hat{N}_{21}G_1 \end{pmatrix},\ \hat{V}_{rs} \in M_{(l-n),n}(\mathbb{C}),\ r,\ s = 1,2 \tag{4.19}$$

with $\operatorname{rank}(\hat{V}_{21}) = \operatorname{rank}(\hat{V}_{12}) = l-n$, $n < l \leqslant 2n$ *and* $\hat{N}_{21},\ \hat{N}_{22} \in M_{r_1,n}(\mathbb{C})$ *with* $\operatorname{rank}(\hat{N}_{21}) = r_1 = \operatorname{rank}(\hat{N}_{22})$, $r_1 \leqslant 2n-l$. *Here 0 is a $(2n-l-r_1)\times n$ zero matrix and $\hat{N}_{12} \in M_{(2n-l-r_1),n}(\mathbb{C})$ in (4.12). Then T is bounded below and the Friedrichs extension T_F of T is defined on the domain*

$$D(T_F) = \left\{y \in D(T^*) :\ \hat{V}_{21}\begin{pmatrix} Y_k(a) \\ Y_k(b) \end{pmatrix} = 0\right\} \tag{4.20}$$

$$= \left\{y \in D_{\max} :\ \hat{V}_{21}\begin{pmatrix} Y_k(a) \\ Y_k(b) \end{pmatrix} = 0,\ N\hat{G}\hat{Y}_{a,b} = 0\right\}, \tag{4.21}$$

where $N \in M_{(2n-l),2n}(\mathbb{C})$ is a complete solution of the matrix equation

$$NV^* = 0,$$

and satisfies $N\hat{G}N^* = 0$.

Proof. It is obvious that (4.19) follows from Lemma 4.3 and Remark 4.4. Moreover from Proposition 3.5 we infer that T defined on (4.18) is bounded below. Now we define an operator T_s with $D(T_s)$:

$$D(T_s) = \left\{y \in D(T^*) :\ \hat{V}_{21}\begin{pmatrix} Y_k(a) \\ Y_k(b) \end{pmatrix} = 0\right\}.$$

Here $\hat{V}_{21} \in M_{(l-n),n}(\mathbb{C})$ is a submatrix of (4.19) satisfying

$$\operatorname{rank}(\hat{V}_{21}) = l-n,\ n < l \leqslant 2n.$$

Denote by

$$A = (-1)^k \begin{pmatrix} \hat{V}_{21} & 0 \\ -N_2 G_1^* & N_1 G_1 \end{pmatrix},$$

where $N = (N_1\ N_2)$ satisfies (4.7) and (4.8), then

$$D(T_s) = \left\{ y \in D_{\max} : A \begin{pmatrix} Y_k(a) \\ Y_k(b) \end{pmatrix} = 0 \right\}.$$

It is easy to ensure that $D(T) \subset D(T_s) \subset D(T_{\max}) \subset D(\widetilde{T}_{\max})$, where $\widetilde{T}_{\max}$ is defined as in the proof of Proposition 3.5. Since $NV^* = 0$, it follows from (4.19) that $N_1 \hat{V}_{21}^* = 0$. Furthermore, by Lemma 4.2 and Lemma 4.3 we infer $\operatorname{rank}(A) = n$, also combined with $G_1 G_1^* = I_n$ we have

$$\begin{aligned} A\hat{G}A^* &= \begin{pmatrix} \hat{V}_{21} & 0 \\ -N_2 G_1^* & N_1 G_1 \end{pmatrix} \begin{pmatrix} 0 & G_1 \\ -G_1^* & 0 \end{pmatrix} \begin{pmatrix} \hat{V}_{21}^* & -G_1 N_2^* \\ 0 & G_1^* N_1^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & \hat{V}_{21} G_1 \\ -N_1 G_1 G_1^* & -N_2 G_1^* G_1 \end{pmatrix} \begin{pmatrix} \hat{V}_{21}^* & -G_1 N_2^* \\ 0 & G_1^* N_1^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & \hat{V}_{21} G_1 G_1^* N_1^* \\ -N_1 G_1 G_1^* \hat{V}_{21}^* & N_1 G_1 G_1^* G_1 N_2^* - N_2 G_1^* G_1 G_1^* N_1^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & \hat{V}_{21} N_1^* \\ -N_1 \hat{V}_{21}^* & N\hat{G}N^* \end{pmatrix} = 0. \end{aligned}$$

Thus combined with [1, Theorem 1.1] we know that the operator T_s is a self-adjoint extension of T which is defined on (4.18).

On the other hand, we prove that T_s is the Friedrichs extension of T. Let $y \in D(T)$. From the proof of Proposition 3.5 combined with Lemma 4.5 and Lemma 4.6 we obtain that for each $\varepsilon > 0$, there is a $K(\varepsilon) > 0$ such that

$$(Ty, y)_w \leqslant \varepsilon (y^{[k]}, y^{[k]})_u + K(\varepsilon)(y, y)_w$$

and thus it yields

$$\|y^{[k]}(x)\|_u^2 \leqslant C_0((T+K)y, y)_w, \tag{4.22}$$

for all $y \in D(T)$, where $K = K(\varepsilon)$ and C_0 is a positive constant.

According to Lemma 4.5 and Lemma 4.6 we also infer that there exists a constant K_0 such that

$$(Ty, y)_w \geqslant (y^{[k]}, y^{[k]})_u + K_0 \sum_{r=0}^{k-1} (y^{[r]}, y^{[r]})$$

for all $y \in D(T)$. Moreover we have

$$\sum_{r=0}^{k-1} (y^{[r]}, y^{[r]}) \leqslant C_1((T+K_1)y, y)_w, \tag{4.23}$$

where $K_1 = K_1(\varepsilon)$ and C_1 is a positive constant.

Now suppose $y \in D(T_F)$. Then we see that $y \in D(T^*)$ and there exists a sequence $\{y_m\} \subset D(T)$ such that

$$y_m \to y \qquad \text{in} \quad L^2(I,w), \tag{4.24}$$

and

$$(T(y_m - y_l), y_m - y_l)_w \to 0, \tag{4.25}$$

as $m,\ l \to \infty$. Applying (4.22) to $(y_m - y_l)$ instead of y, and together with (4.24) and (4.25) show that $y_m^{[k]}$ converge uniformly on interval I to some continuous function z_k in $L^2(I,u)$. For $r = 0,1,2,\ldots,k-1$, from (4.23) there are continuous functions z_r such that $y_m^{[r]} \to z_r$ in $L^\infty(I) \subset L^2(I,u)$. For $r = 0$ we also have $y_m \to y$ in $L^2(I,w)$ which implies that we may assume $y = z_0$. Moreover, the sequence

$$\begin{pmatrix} Y_{km}(a) \\ Y_{km}(b) \end{pmatrix} = \left(y_m^{[0]}(a) \ \cdots \ y_m^{[k-1]}(a) \ y_m^{[0]}(b) \ \cdots \ y_m^{[k-1]}(b) \right)^T$$

is convergent in $\mathbb{C}^n$. That is, there is a unique column vector $\vec{\beta} \in \mathbb{C}^n$ satisfying

$$\lim_{m\to\infty} \begin{pmatrix} Y_{km}(a) \\ Y_{km}(b) \end{pmatrix} = \vec{\beta} = (z_0(a) \ \cdots \ z_{k-1}(a) \ z_0(b) \ \cdots \ z_{k-1}(b))^T .$$

For a fixed $t \in \bar{I}$, let

$$\phi_k(\cdot,t) = (\phi_{k,r,s}(\cdot,t))_{r,s=1}^k$$

be the fundamental matrix of $Y_k' = Q_k Y_k$ with $\phi_k(t,t) = I_k,\ Q_k = (q_{r,s})_{r,s=1}^k$. Here Q_k is defined as in the proof of Proposition 3.5. Note that

$$\begin{aligned} y_{Q_k}^{[r]} &= y_Q^{[r]}, \quad r = 0,1,\ldots,k-1, \\ y_{Q_k}^{[k]} &= \bar{c}_{k,1} q_{k,k+1} y_Q^{[k]}. \end{aligned}$$

Combined with [26, Corollary 1] and [13, Corollary 2.7] we have for $\alpha \in \bar{I}$ and $r = 0,1,\ldots,k-1$,

$$y_m^{[r]}(x) = \sum_{s=1}^k \phi_{k,r+1,s}(x,\alpha) y_m^{[s-1]}(\alpha) + \int_\alpha^x \phi_{k,r+1,k}(x,t)\bar{c}_{k,1} q_{k,k+1}(t) y_m^{[k]}(t)dt,$$

with $Y_{km}(\alpha) = \vec{c}_m$, for $\vec{c}_m \in \mathbb{C}^k,\ m \in \mathbb{N}$, and by taking limits we obtain

$$z_r(x) = \sum_{s=1}^k \phi_{k,r+1,s}(x,\alpha) z_{s-1}(\alpha) + \int_\alpha^x \phi_{k,r+1,k}(x,t)\bar{c}_{k,1} q_{k,k+1}(t) z_k(t)dt.$$

In particular, for $r = 0,\ y = z_0$ is the unique solution of the initial value problem

$$y_{Q_k}^{[k]} = \bar{c}_{k,1} q_{k,k+1} z_k,\ y^{[r]}(a) = z_r(a), \quad r = 0,1,\ldots,k-1.$$

And for $r=1,\dots,k-1$, we have $y^{[r]}=z_r$. In particular, since a is the left endpoint of I and b is the right endpoint of I, we have

$$y^{[r]}(a)=\lim_{m\to\infty} y_m^{[r]}(a)=z_r(a),\ y^{[r]}(b)=\lim_{m\to\infty} y_m^{[r]}(b)=z_r(b).$$

Hence we get

$$\vec{\beta}=\begin{pmatrix} Y_k(a)\\ Y_k(b)\end{pmatrix}.$$

Furthermore, since y_m belong to $D(T)\subset D(\widetilde{T}_{\max})$, it follows that

$$V\begin{pmatrix} Y_{km}(a)\\ Y_{km}(b)\\ \hat{Y}_{km}(a)\\ \hat{Y}_{km}(b)\end{pmatrix}=0.$$

This implies

$$0=(\hat{V}_{21}\ 0)\begin{pmatrix} Y_{km}(a)\\ Y_{km}(b)\\ \hat{Y}_{km}(a)\\ \hat{Y}_{km}(b)\end{pmatrix}=\hat{V}_{21}\begin{pmatrix} Y_{km}(a)\\ Y_{km}(b)\end{pmatrix}\to \hat{V}_{21}\begin{pmatrix} Y_k(a)\\ Y_k(b)\end{pmatrix}.$$

The fact that $\hat{V}_{21}\begin{pmatrix} Y_k(a)\\ Y_k(b)\end{pmatrix}=0$ shows $y\in D(T_s)$ and thus $D(T_F)\subset D(T_s)$. On the other hand, we have proved that $D(T_s)$ is the domain of a self-adjoint extension T_s of T. Consequently, the self-adjointness of T_F leads to $T_s=T_F$. The proof is complete. □

THEOREM 4.8. *Let the assumptions be as in Theorem 4.7 and let the symmetric operator T be given by (4.18). Then the domain $D(T_F)$ of its Friedrichs extension T_F is characterized by*

$$D(T_F)=\left\{y\in D(T_{\max}):\ \hat{Y}_{a,b}\in\mathscr{R}(\hat{G}V^*),\ \begin{pmatrix} Y_k(a)\\ Y_k(b)\end{pmatrix}\in G_1V_2^*(\mathscr{N}(V\hat{G}V^*))\right\},\quad (4.26)$$

where V_2 is given by (4.2).

Proof. Observe that $V=UJ=(V_1\ V_2)$ and $JGU^*(\mathscr{N}(UGU^*))=\hat{G}V^*(\mathscr{N}(V\hat{G}V^*))$, (4.26) is obtained easily from property (7) in [26, Lemma 14] combined with the proof of Theorem 4.7. □

REMARK 4.9. Under the assumptions of Theorem 4.8 we also have

$$D(T_F)=\left\{y\in D(T_{\max}):\ \hat{Y}_{a,b}\in\mathscr{R}(\hat{G}V^*),\ \begin{pmatrix} Y_k(a)\\ Y_k(b)\end{pmatrix}\in V_1^{-1}\mathscr{R}(V_2)\right\}.$$

REMARK 4.10. Let $C = E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n$ in Theorem 4.8 and Remark 4.9, i.e.,

$$\hat{G} = \begin{pmatrix} 0 & G_1 \\ -G_1^* & 0 \end{pmatrix}$$

with

$$G_1 = (-1)^k \begin{pmatrix} -E_k & 0 \\ 0 & E_k \end{pmatrix}, \; 2k = n,$$

then we obtain the Möller-Zettl results in [14] as a special case. Moreover in this case if $l = 2n$ in (4.18), we obtain the Friedrichs extension of the minimal operator in [13, Theorem 8.1]. Also Niessen-Zettl [16, Theorem 2.1] found a special case for this E and a certain matrix Q.

REMARK 4.11. It is clear that the characterization of the Friedrichs extension in Theorem 4.7 and Theorem 4.8 should be equivalent to each other. However Theorem 4.7 is more explicit than the result in Theorem 4.8. For a better understanding of our main results we give some simple examples for the special case $n = 2,\ 4$ in the next section.

5. Examples

In this section we consider the Friedrichs extension T_F of the symmetric operator T for some special cases. Throughout this section the minimal operator $T_{\min}$ is bounded below and thus its symmetric extensions are also bounded below.

We start to consider the Friedrichs extension T_F for regular Sturm-Liouville differential operators. In this case, $n = 2$ in (1.1) and (1.2). Assume that

$$Ty = w^{-1}(-(p_1 y')' + p_0 y) \tag{5.1}$$

on I. Here the coefficients $w,\ p_0,\ p_1$ all are real valued functions satisfying

$$\frac{1}{p_1},\ p_0,\ w \in L^1(I), \text{ and } p_1,\ w > 0 \text{ a.e. on } I.$$

It is obvious that (5.1) generates a minimal operator $T_{\min}$ and a maximal operator $T_{\max}$ with domains $D_{\min}$ and $D_{\max}$, respectively. The symmetric operator realizations T of (1.1) in the Hilbert space $L^2(I,w)$ satisfy

$$T_{\min} \subset T = T^* \subset T_{\max},$$

these operators T differ from each other only by their domains.

Let the operator T defined by

$$D(T) = \left\{ y \in D_{\max} :\ V\hat{Y}_{a,b} = 0,\ \operatorname{rank}(V) = 3,\ V \in M_{3,4}(\mathbb{C}) \right\} \tag{5.2}$$

be a symmetric operator (not minimal) with 3 dimensional restriction of $T_{\max}$, where

$$\hat{Y}_{a,b} = \begin{pmatrix} y(a) \\ y(b) \\ (py')(a) \\ (py')(b) \end{pmatrix}.$$

Then by Lemma 4.2 there exists a matrix $N = (a_1\ a_2\ a_3\ a_4) \in M_{1,4}(\mathbb{C})$ satisfying $\operatorname{rank}(N) = 1$ and $NF_4N^* = 0$, where

$$F_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{5.3}$$

and $V \in M_{3,4}(\mathbb{C})$ is a complete solution of matrix equation $NV^* = 0$. Moreover from Lemma 4.3 it follows that V in (5.2) has the form

$$V = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} \\ -a_3 & a_4 & a_1 & -a_2 \end{pmatrix},\ b_{rs} \in \mathbb{C} \tag{5.4}$$

with

$$\begin{aligned} \bar{a}_1 b_{11} + \bar{a}_2 b_{12} &= 0, \\ \bar{a}_1 b_{21} + \bar{a}_2 b_{22} + \bar{a}_3 b_{23} + \bar{a}_4 b_{24} &= 0, \\ \bar{a}_1 a_3 - \bar{a}_2 a_4 - \bar{a}_3 a_1 + \bar{a}_2 a_4 &= 0. \end{aligned}$$

Then from Theorem 4.7 we obtain the following Corollary:

COROLLARY 5.1. *Let T be a symmetric operator defined by* (5.1)–(5.4), *which is a* 3 *dimensional restriction of $T_{\max}$. Then the boundary conditions of its Friedrichs extension T_F are characterized by*

$$\left.\begin{aligned} b_{11}y(a) + b_{12}y(b) &= 0, \\ -a_3y(a) + a_4y(b) + a_1(py')(a) - a_2(py')(b) &= 0. \end{aligned}\right\} \tag{5.5}$$

Obviously these are equivalent to

$$D(T_F) = \left\{ y \in D_{\max} : \hat{V}_{21}\begin{pmatrix} y(a) \\ y(b) \end{pmatrix} = 0,\ NF_4 \begin{pmatrix} y(a) \\ y(b) \\ (py')(a) \\ (py')(b) \end{pmatrix} = 0 \right\},$$

where $\hat{V}_{21} = (b_{11}\ b_{12})$.

It is easy to see that if both b_{11} and b_{12} in (5.5) are not vanishing, then the boundary conditions involve coupled self-adjoint conditions. We give an example for this case.

EXAMPLE 5.1. Let $N = (-2\ 1\ \ -5+\mathrm{i}\ \ -7-2\mathrm{i})$. Then T defined by (5.2) with

$$V = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 5+\mathrm{i} & 1 & 0 \\ 5-\mathrm{i} & -7-2\mathrm{i} & -2 & -1 \end{pmatrix}$$

is a symmetric operator with 3 dimensional restriction of the maximal operator $T_{\max}$. By using Corollary 5.1, the boundary conditions of its Friedrichs extension are:

$$\left.\begin{aligned} y(a)+2y(b)=0, \\ (5-\mathrm{i})y(a)+(-7-2\mathrm{i})y(b)-2(py')(a)-(py')(b)=0. \end{aligned}\right\} \tag{5.6}$$

By a simple calculation, we obtain that the canonical form of (5.6) is the real coupled self-adjoint boundary conditions:

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \text{ with } K = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{3}{2} & -2 \end{pmatrix}.$$

Next we consider the case $n = 4$. Assume that M has the familiar form

$$My = [(p_2 y'')' - (p_1 y')]' + p_0 y = \lambda w y \quad \text{on} \quad I, \tag{5.7}$$

where the coefficients $w,\ p_j$ for $j = 0,1,2$ all are real valued functions defined on the interval I satisfying

$$\frac{1}{p_2},\ p_1,\ p_0,\ w \in L^1(I), \text{ and } p_2,\ w > 0, \text{ a.e. on } I.$$

For a given matrix $N \in M_{(8-l),8}(\mathbb{C})$ satisfying

$$\operatorname{rank}(N) = 8-l, \quad NF_8N^* = 0, \quad 4 \leqslant l \leqslant 8, \tag{5.8}$$

where

$$F_8 = \begin{pmatrix} 0 & \hat{J}_4 \\ -\hat{J}_4 & 0 \end{pmatrix} \qquad \textit{with} \quad \hat{J}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

If $V \in M_{l,8}(\mathbb{C})$ is a complete solution of the matrix equation $NV^* = 0$, then by Lemma 4.2 T generated by (5.7) is a symmetric operator with l dimensional restriction of $T_{\max}$. The domain of T is equivalent to

$$D(T) = \left\{ y \in D_{\max} :\ V\hat{Y}_{a,b} = 0 \right\}, \tag{5.9}$$

with the boundary matrix

$$V = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & 0 \\ -N_2\hat{J}_4 & N_1\hat{J}_4 \end{pmatrix}, \quad \hat{V}_{rs} \in M_{(l-4),8}(\mathbb{C}), \quad r,\ s = 1,2 \tag{5.10}$$

with $\operatorname{rank}(\hat{V}_{12}) = \operatorname{rank}(\hat{V}_{21}) = l-4$ for $4 \leqslant l \leqslant 8,$ and

$$\hat{Y}_{a,b} = \left(y^{[0]}(a)\, y^{[1]}(a)\, y^{[0]}(b)\, y^{[1]}(b)\, y^{[2]}(a)\, y^{[3]}(a)\, y^{[2]}(b)\, y^{[3]}(b)\right)^T.$$

Here

$$y^{[0]} = y,\ y^{[1]} = y',\ y^{[2]} = p_2(y^{[1]})',\ y^{[3]} = p_1 y^{[1]} - (y^{[2]})'.$$

By Lemma 4.2 we also have

$$D(T^*) = \left\{y \in D_{\max} :\ NF_8\hat{Y}_{a,b} = 0\right\}. \tag{5.11}$$

Combined with Theorem 4.7 we infer:

COROLLARY 5.2. *Let $4 < l \leqslant 8$. Assume that T defined by (5.9) is a symmetric operator with l dimensional restriction of the maximal operator $T_{\max}$. Then the boundary conditions of its Friedrichs extensions T_F can be characterized by*

$$\left.\begin{aligned} \hat{V}_{21}\begin{pmatrix} y^{[0]}(a) \\ y^{[1]}(a) \\ y^{[0]}(b) \\ y^{[1]}(b) \end{pmatrix} &= 0, \\ \left(-N_2\hat{J}_4\ N_1\hat{J}_4\right)\hat{Y}_{a,b} &= 0. \end{aligned}\right\} \tag{5.12}$$

Here $\hat{V}_{21}$ is a submatrix of (5.10).

In the following, we give an example for Sturm-Liouville operator of order four.

EXAMPLE 5.2. Let

$$N = \begin{pmatrix} -2 & -2 & -2 & -2 & -2 & -2 & 5 & 5 \\ -3 & -3 & -3 & -3 & -3 & -3 & 5 & 10 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 3 & 2 & 2 \\ 2 & 2 & 5 & 5 & -2 & -2 & 2 & 2 \\ 3 & 3 & 10 & 5 & -3 & -3 & 3 & 3 \end{pmatrix}. \tag{5.13}$$

Observe that N satisfies (5.8) and V is a complete solution of matrix equation $NV^* = 0$. The operator $Ty = \frac{1}{w}My$, where My is defined by (5.7), is a symmetric operator with 6 dimensional restriction of $T_{\max}$ with the domain

$$D(T) = \left\{y \in D_{\max} :\ V\hat{Y}_{a,b} = 0\right\},$$

where V is denoted by (5.13). Furthermore according to (5.11) we obtain that

$$D(T^*) = \left\{ y \in D_{\max} : \begin{pmatrix} 2 & 2 & 5 & 5 & -2 & -2 & 2 & 2 \\ 3 & 3 & 10 & 5 & -3 & -3 & 3 & 3 \end{pmatrix} \hat{Y}_{a,b} = 0 \right\}.$$

Then using Corollary 5.2 we obtain

$$\hat{V}_{21} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Thus the Friedrichs extension T_F of T is characterized by the following mixed boundary conditions

$$\left.\begin{aligned} y(a) - y(b) = 0, \\ y^{[1]}(a) - y(b) = 0, \\ 2y(a) + 2y^{[1]}(a) + 5y(b) + 5y^{[1]}(b) - 2y^{[2]}(a) - 2y^{[3]}(a) + 2y^{[2]}(b) + 2y^{[3]}(b) = 0, \\ 3y(a) + 3y^{[1]}(a) + 10y(b) + 5y^{[1]}(b) - 3y^{[2]}(a) - 3y^{[3]}(a) + 3y^{[2]}(b) + 3y^{[3]}(b) = 0. \end{aligned}\right\}$$

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