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A NOTE ON SOME CLASSES OF G–MATRICES

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Abstract. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 A D_2$. For fixed nonsingular diagonal matrices D_1 and D_2, let $\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1 A D_2\}$, which is called a G-class. In this note, a characterization of $\mathbb{G}(D_1, D_2)$ is obtained and some properties of these G-classes are exhibited, such as conditions for equality of two G-classes. It is shown that $\mathbb{G}(D_1, D_2)$ has two or four connected components in $\mathbf{M}_n$ and that $\mathbb{G}_n = \bigcup_{D_1, D_2} \mathbb{G}(D_1, D_2)$, the set of all $n \times n$ G-matrices, has two connected components in $\mathbf{M}_n$. Sign patterns of the G-classes are also examined.

1. Introduction

All matrices in this note have real number entries. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 A D_2$, where A^{-T} denotes the transpose of the inverse of A. These matrices form a rich class and were originally studied in [3] by Fiedler and Hall. Some properties of these matrices are as follows:

All orthogonal matrices are G-matrices.

All nonsingular diagonal matrices are G-matrices.

Any n positive real numbers are the singular values and eigenvalues of a diagonal G-matrix D.

If A is a G-matrix, then both A^T and A^{-1} are G-matrices.

If A is an $n \times n$ G-matrix and D is an $n \times n$ nonsingular diagonal matrix, then both AD and DA are G-matrices.

If A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then both AP and PA are G-matrices.

Cauchy matrices have the form $C = [c_{ij}]$, where $c_{ij} = \frac{1}{x_i + y_j}$ for some numbers x_i and y_j. We shall restrict to square, say $n \times n$, Cauchy matrices – such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices i, j, and it is well known that C is nonsingular if and only if all the numbers x_i are mutually distinct and all the numbers y_j are mutually distinct. It turns out that by an observation of Fiedler [2] every nonsingular Cauchy matrix is a G-matrix. So, in particular, G-matrices arise

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naturally as the very well-defined structured nonsingular Cauchy matrices. Furthermore, G-matrices arise also in the context of "combined matrices" $C(A) = A \circ A^{-T}$, where $\circ$ denotes the Hadamard product, see [2]. For example, if A is a G-matrix, then $C(A) = A \circ (D_1 A D_2) = D_1 (A \circ A) D_2$; so if say D_1 and D_2 are nonnegative, then $C(A)$ is nonnegative. The combined matrices appear in the chemical literature where they represent the relative gain array, [10]. From a basic point of view, we can rewrite the original G-matrix equation as

$$A^{-1} = D_2 A^T D_1,$$

which says that A^{-1} and A^T are diagonally equivalent, giving a generalization of orthogonal matrices. Theorem 2.2 and Corollary 2.3 in Section 2 describe specifically how the structure of G-matrices arises from the structure of J-orthogonal matrices.

The G-matrices were later studied in two papers [9] and [4].

Denote by $J = \mathrm{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or -1. As in [8], a nonsingular real matrix Q is called J-orthogonal if

$$Q^T J Q = J,$$

or equivalently, if

$$Q^{-T} = JQJ.$$

Of course, with $J = I_n$ the identity matrix of order n, every orthogonal matrix is a J-orthogonal matrix. And clearly, from the equation $Q^{-T} = JQJ$, every J-orthogonal matrix is a G-matrix. Not every G-matrix is a J-orthogonal matrix; for example, $2I_n$ is a G-matrix but not a J-orthogonal matrix. But, a G-matrix can always be "transformed" to a J-orthogonal matrix [7].

For fixed nonsingular diagonal matrices D_1 and D_2, let the class of $n \times n$ G-matrices

$$\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1 A D_2\}.$$

We call such a class of matrices a G-class of matrices.

In this note, a characterization of $\mathbb{G}(D_1, D_2)$ is obtained and some properties of these G-classes are exhibited, such as conditions for equality of two G-classes. It is shown that $\mathbb{G}(D_1, D_2)$ has two or four connected components in $\mathbf{M}_n$ and that $\mathbb{G}_n = \bigcup_{D_1, D_2} \mathbb{G}(D_1, D_2)$, the set of all $n \times n$ G-matrices, has two connected components in $\mathbf{M}_n$. Sign patterns of the G-classes are also examined.

The following characterization of J-orthogonal matrices is contained in the article [8] by N. Higham. As stated in [8], this decomposition was first derived in [5]. As in our previous papers [11] and [12], for a fixed signature matrix J,

$$\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top J A = J\}.$$

Also, $\mathscr{O}_k$ denotes the set of all $k \times k$ orthogonal matrices.

PROPOSITION 1.1. [8, Theorem 3.2 (hyperbolic CS decomposition)] *Let $q \geqslant p$ and $J = I_p \oplus (-I_q)$. Then every $A \in \Gamma_n(J)$ is of the form*

$$(U_1 \oplus U_2)\left(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p}\right)(V_1 \oplus V_2), \tag{1.1}$$

where $U_1, V_1 \in \mathscr{O}_p$, $U_2, V_2 \in \mathscr{O}_q$ and $C, S \in \mathbf{M}_p$ are nonnegative diagonal matrices such that $C^2 - S^2 = I$. Also, any matrix of the form (1.1) is J-orthogonal.

As in our previous works, the decomposition in Proposition 1.1 will also be employed in this paper.

2. Classes of G-matrices

We note that the nonsingular diagonal matrices D_1 and D_2 satisfying $A^{-T} = D_1AD_2$ are in general not uniquely determined as we can multiply one of them by a nonzero real number and divide the other by the same number. (However, if A is a "fully indecomposable" G-matrix then D_1 and D_2 are unique up to scalar multiplies, see Theorem 2.2 in [7]. For the definition of fully indecomposable matrices, the reader can see [1, p. 112]). On the other hand, for nonsingular $n \times n$ diagonal matrices D_1 and D_2, the following known result from [3] shows that if $A^{-T} = D_1AD_2$ then D_1 and D_2 have the same inertia matrix.

PROPOSITION 2.1. *Suppose A is a G-matrix and $A^{-T} = D_1AD_2$, where D_1 and D_2 are nonsingular diagonal matrices. Then the inertia of D_1 is equal to the inertia of D_2.*

Proof. We have $A^TD_1AD_2 = I$ and so $A^TD_1A = D_2^{-1}$. Since A is nonsingular, the result follows from Sylvester's Law of Inertia. $\square$

We just mention the following in passing. A (J_1, J_2)-orthogonal matrix is defined as a nonsingular real matrix Q such that

$$Q^TJ_1Q = J_2,$$

where $J_1 = \operatorname{diag}(\pm 1)$ and $J_2 = \operatorname{diag}(\pm 1)$ are signature matrices [8]. Similar to the proof of Proposition 2.1, J_1 and J_2 have the same inertia and are just permutations of each other.

In this section we find a characterization of $\mathbb{G}(D_1, D_2)$ in terms of the matrices in $\Gamma_n(J)$, thus establishing a further specific connection between the G-matrices and the J-orthogonal matrices.

Let D be a nonsingular diagonal matrix with the inertia matrix J (a signature matrix having all its positive ones in the upper left corner). Then there exists a permutation matrix P such that $D = |D|P^TJP$, where $|D|$ is obtained by taking the absolute value on entries of D. Recall that for a fixed signature matrix J, $\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\}$. In fact,

$$\Gamma_n(J) = \mathbb{G}(J, J).$$

THEOREM 2.2. *Let D_1 and D_2 be nonsingular diagonal matrices with the inertia matrix J. Then there exist permutation matrices P and Q such that*

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2}P^TAQ|D_2|^{-1/2} : A \in \Gamma_n(J)\}.$$

This characterization shows that $\mathbb{G}(D_1, D_2)$ is in fact nonempty.

Proof. Since J is the inertia matrix for D_1 and D_2, there exist permutation matrices P and Q such that $D_1 = |D_1|P^T JP$ and $D_2 = |D_2|Q^T JQ$.

Then $J = P|D_1|^{-1/2}D_1|D_1|^{-1/2}P^T = Q|D_2|^{-1/2}D_2|D_2|^{-1/2}Q^T$. These imply that

$$\begin{aligned}
A \in \Gamma_n(J) &\Leftrightarrow A^{-T} = JAJ \\
&\Leftrightarrow A^{-T} = P|D_1|^{-1/2}D_1|D_1|^{-1/2}P^T AQ|D_2|^{-1/2}D_2|D_2|^{-1/2}Q^T \\
&\Leftrightarrow (|D_1|^{-1/2}P^T AQ|D_2|^{-1/2})^{-T} = D_1(|D_1|^{-1/2}P^T AQ|D_2|^{-1/2})D_2 \\
&\Leftrightarrow |D_1|^{-1/2}P^T AQ|D_2|^{-1/2} \in \mathbb{G}(D_1,D_2).
\end{aligned}$$

Let $X = |D_1|^{-1/2}P^T$ and $Y = Q|D_2|^{-1/2}$. We have shown that

$$A \in \Gamma_n(J) \Leftrightarrow XAY \in \mathbb{G}(D_1,D_2).$$

Now, by use of this fact we have

$$\begin{aligned}
B \in \mathbb{G}(D_1,D_2) &\Leftrightarrow X(X^{-1}BY^{-1})Y \in \mathbb{G}(D_1,D_2) \\
&\Leftrightarrow X^{-1}BY^{-1} \in \Gamma_n(J) \\
&\Leftrightarrow B \in X\Gamma_n(J)Y.
\end{aligned}$$

Therefore

$$\mathbb{G}(D_1,D_2) = X\Gamma_n(J)Y = \{|D_1|^{-1/2}P^T AQ|D_2|^{-1/2} : A \in \Gamma_n(J)\}. \quad \square$$

We will now incorporate the hyperbolic CS Decomposition for $\Gamma_n(J)$ into a simplified version of $\mathbb{G}(D_1,D_2)$. Assume $q \geqslant p$, the common inertia matrix J of D_1 and D_2 has the form

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

and that D_1 and D_2 have the form

$$\begin{pmatrix} +_p & 0 \\ 0 & -_q \end{pmatrix},$$

where $+_p$ $(-_q)$ denotes an order p (q) diagonal matrix with positive (negative) diagonal entries. Then $P = Q = I$, $D_1 = |D_1|J$, and $D_2 = |D_2|J$. Using Proposition 1.1 on the CS Decomposition, we then have the following result.

COROLLARY 2.3. *With the above notation,*

$$\mathbb{G}(D_1,D_2) = \left\{ |D_1|^{-1/2}(U_1 \oplus U_2)\left(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p} \right)(V_1 \oplus V_2)|D_2|^{-1/2} \right\}.$$

where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$ and $C, S \in \mathbf{M}_p$ are nonnegative diagonal matrices such that $C^2 - S^2 = I_p$.

Let A be an $n\times n$ nonsingular real matrix with Singular Value Decomposition

$$A = U\Sigma W$$

where U and W are orthogonal matrices, and Σ is a diagonal matrix with positive diagonal entries.. So, $AW^T = U\Sigma$. Now, it is easy to see that $U\Sigma \in \mathbb{G}(I,\Sigma^{-2})$. (Also: since U is an orthogonal matrix, U is a G-matrix; multiplying U by the nonsingular diagonal matrix Σ we still have a G-matrix.) Hence, $AW^T = B$, where $B \in \mathbb{G}(I,\Sigma^{-2})$, so that $A = BW$. We thus arrive at the following result.

PROPOSITION 2.4. *Every $n\times n$ nonsingular real matrix is a product of a G-matrix and an orthogonal matrix. In particular, if $U\Sigma W$ is a Singular Value Decomposition of a nonsingular matrix A, then $U\Sigma$ is a G-matrix.*

Let $\mathscr{P}_n$ denote the set of permutation matrices of order n.

LEMMA 2.5. *Let D_1 and D_2 be $n\times n$ real nonsingular diagonal matrices. If $D_1\mathscr{P}_nD_2 \subseteq \mathscr{O}_n$, then there is a positive number d such that*

$$D_1 = d\begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1\end{pmatrix}, \quad D_2 = \frac{1}{d}\begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1\end{pmatrix}.$$

Proof. For $i,j = 1,\dots,n$, let the i^{th} diagonal entry of D_1 be d_i^1 and similarly j^{th} diagonal entry of D_2 be d_j^2. Let P_{ij} be the permutation matrix obtained from the identity matrix by interchanging the i^{th} and j^{th} rows. The norm of the j^{th} column of $D_1P_{ij}D_2$ is $|d_i^1d_j^2|$. By the assumption of $D_1\mathscr{P}_nD_2 \subseteq \mathscr{O}_n$, we have $|d_i^1d_j^2| = 1$ for all $i,j = 1,\dots,n$. Letting $d = |d_1^1|$, then the result is achieved. $\square$

COROLLARY 2.6. *Let D_1 and D_2 be $n\times n$ real nonsingular diagonal matrices. Then $D_1\mathscr{O}_nD_2 = \mathscr{O}_n$ if and only if there is a positive number d such that*

$$D_1 = d\begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1\end{pmatrix}, \quad D_2 = \frac{1}{d}\begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1\end{pmatrix}.$$

Proof. The proof of sufficiency is clear. For the proof of the necessity, suppose that $D_1\mathscr{O}_nD_2 = \mathscr{O}_n$. Then we have $D_1\mathscr{P}_nD_2 \subseteq \mathscr{O}_n$. By the use of Lemma 2.5 the result is obtained. $\square$

3. Relationships between the G-classes

Given a fixed $n\times n$ inertia matrix J, we have various G-classes associated with J. We define the following relation on the collection of the $n\times n$ G-classes:

$$\mathbb{G}(D_1,D_2) \sim \mathbb{G}(D_3,D_4)$$

if and only if each class is associated with the same J, i.e. the inertia matrix of D_1, D_2, D_3, D_4 is J. (Note that $\mathbb{G}(J,J)$ is in the same equivalence class.) It is clear that $\sim$ is an equivalence relation on the collection of the $n\times n$ G-classes.

We first consider $\mathbb{G}(D_1,D_2)\sim\mathbb{G}(D_3,D_4)$ in the specific case where the common inertia matrix of D_1, D_2, D_3, D_4 is $\pm I$, and we give the condition for equality of the two classes.

THEOREM 3.1. *Assume D_1, D_2, D_3 and D_4 are real nonsingular diagonal matrices with the inertia matrix I or $-I$. Then*

$$\mathbb{G}(D_1,D_2)=\mathbb{G}(D_3,D_4)$$

if and only if there exists a positive number d such that $D_3=dD_1$ and $D_4=\frac{1}{d}D_2$.

Proof. Since $\mathbb{G}(D_1,D_2)=\mathbb{G}(-D_1,-D_2)$, we can assume without loss of generality that the common inertia matrix is I. We have by using Theorem 2.2 that

$$\begin{aligned}\mathbb{G}(D_1,D_2)=\mathbb{G}(D_3,D_4) &\Leftrightarrow D_1^{-1/2}\mathscr{O}_nD_2^{-1/2}=D_3^{-1/2}\mathscr{O}_nD_4^{-1/2}\\ &\Leftrightarrow D_3^{1/2}D_1^{-1/2}\mathscr{O}_nD_2^{-1/2}D_4^{1/2}=\mathscr{O}_n,\end{aligned}$$

which by Corollary 2.6 holds

$$\Leftrightarrow D_3^{1/2}D_1^{-1/2}=kI, D_2^{-1/2}D_4^{1/2}=\frac{1}{k}I,$$

where k is a positive number. Then for $d=k^2$, the latter two equations become

$$D_3D_1^{-1}=dI, D_2^{-1}D_4=\frac{1}{d}I,$$

which then completes the proof. $\square$

We next consider a second case where $\mathbb{G}(D_1,D_2)\sim\mathbb{G}(D_3,D_4)$.

Let $J=\begin{pmatrix}I_p & 0\\ 0 & -I_q\end{pmatrix}$. For $A\in\mathbf{M}_n$, we say that A is conformal to J if $A=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}$ with $A_1\in\mathbf{M}_p$ and $A_2\in\mathbf{M}_q$.

THEOREM 3.2. *Let $q\geqslant p$, $J=I_p\oplus(-I_q)$ and, suppose D_1, D_2, D_3 and D_4 are real nonsingular diagonal matrices with the inertia matrix J such that $D_i=|D_i|J$ for $i=1,\dots,4$. Then*

$$\mathbb{G}(D_1,D_2)=\mathbb{G}(D_3,D_4)$$

if and only if there exists a positive number d such that $D_3=dD_1$ and $D_4=\frac{1}{d}D_2$.

Proof. The proof of the sufficiency is straightforward. Suppose $D_3=dD_1$ and $D_4=\frac{1}{d}D_2$. Hence if $A\in\mathbb{G}(D_3,D_4)$, then $A\in\mathbb{G}(D_1,D_2)$. Similarly $\mathbb{G}(D_1,D_2)\subseteq\mathbb{G}(D_3,D_4)$.

To prove the necessity, suppose

$$\mathbb{G}(D_1,D_2)=\mathbb{G}(D_3,D_4).$$

By using Theorem 2.2 we have

$$|D_1|^{-1/2}\,\Gamma_n(J)\,|D_2|^{-1/2}=|D_3|^{-1/2}\,\Gamma_n(J)\,|D_4|^{-1/2}.$$

That is equivalent to

$$|D_3|^{1/2}|D_1|^{-1/2}\,\Gamma_n(J)\,|D_2|^{-1/2}|D_4|^{1/2}=\Gamma_n(J). \tag{3.1}$$

The set of $P\oplus Q$ such that $P\in\mathscr{P}_p$ and $Q\in\mathscr{P}_q$ is a subset of $\Gamma_n(J)$. So

$$|D_3|^{1/2}|D_1|^{-1/2}(P\oplus Q)\,|D_2|^{-1/2}|D_4|^{1/2}\in\Gamma_n(J),$$

for all $P\in\mathscr{P}_p$ and $Q\in\mathscr{P}_q$. Matrices of the form $|D_3|^{1/2}|D_1|^{-1/2}(P\oplus Q)\,|D_2|^{-1/2}|D_4|^{1/2}$ are conformal to J. That is to say, they have the form

$$B=\begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}$$

where B_1 is $p\times p$ and B_2 is $q\times q$. Since $B\in\Gamma_n(J)$, $BJB^T=J$. Thus we have $B_1\in\mathscr{O}_p$ and $B_2\in\mathscr{O}_q$. That implies

$$|D_3|^{1/2}|D_1|^{-1/2}(P\oplus Q)\,|D_2|^{-1/2}|D_4|^{1/2}\in\mathscr{O}_p\oplus\mathscr{O}_q.$$

By applying Lemma 2.5 to each part of the latter direct sum, there are positive numbers k_1 and k_2 such that

$$|D_3|^{1/2}|D_1|^{-1/2}=k_1I_p\oplus k_2I_q,\ |D_2|^{-1/2}|D_4|^{1/2}=\frac{1}{k_1}I_p\oplus\frac{1}{k_2}I_q.$$

Next we show that $k_1=k_2$. Let $C=2I_p$ and $S=\sqrt{3}I_p$. Then by Proposition 1.1,

$$\begin{pmatrix} C & -S\\ -S & C\end{pmatrix}\oplus I_{q-p}\in\Gamma_n(J).$$

From Equation 3.1, we have

$$(k_1I_p\oplus k_2Iq)\left(\begin{pmatrix} C & -S\\ -S & C\end{pmatrix}\oplus I_{q-p}\right)\left(\frac{1}{k_1}I_p\oplus\frac{1}{k_2}I_q\right)\in\Gamma_n(J).$$

Hence, letting

$$A=(k_1I_p\oplus k_2Iq)\left(\begin{pmatrix} C & -S\\ -S & C\end{pmatrix}\oplus I_{q-p}\right)\left(\frac{1}{k_1}I_p\oplus\frac{1}{k_2}I_q\right),$$

and multiplying this expression out we obtain

$$A = \begin{pmatrix} C & -\frac{k_1}{k_2}S \\ -\frac{k_2}{k_1}S & C \end{pmatrix} \oplus I_{q-p}$$

From $A^T JA = J$ it can be seen that $(\frac{k_1}{k_2})^2 = 1$. So $k_1 = k_2$. Letting $k = k_1 = k_2$ and $d = k^2$, then $|D_3||D_1|^{-1} = dI$ and $|D_2|^{-1}|D_4| = \frac{1}{d}I$.

We can observe that $D_i = |D_i|J$ for $i = 1,\ldots,4$ if and only if D_1, D_2, D_3, D_4 all have the same sign pattern (the corresponding (i,i) entries have the same sign) as J. So, $|D_3||D_1|^{-1} = dI$ and $|D_2|^{-1}|D_4| = \frac{1}{d}I$ are equivalent to $D_3 = dD_1$ and $D_4 = \frac{1}{d}D_2$, respectively. $\square$

Suppose D_1, D_2, D_3, D_4 have the same sign pattern, but not all $+$ or all $-$ on the diagonals (the case where the common inertia matrix is $\pm I$ is covered in Theorem 3.1). Assume $\mathbb{G}(D_1,D_2) = \mathbb{G}(D_3,D_4)$. Since $\mathbb{G}(D_1,D_2) = \mathbb{G}(-D_3,-D_4)$, we can assume that the number q of negative diagonal entries is greater than or equal to the number p of positive diagonal entries. With $A \in \mathbb{G}(D_1,D_2)$ and P a permutation matrix, we have

$$(P^T AP)^{-T} = (P^T D_1 P)(P^T AP)(P^T D_2 P)$$

and similarly for $\mathbb{G}(D_3,D_4)$. Let P be the permutation matrix such that all $P^T D_i P$ have the p positive diagonal entries in the upper left corner so that $J = I_p \oplus (-I)_q$ is the common inertia matrix of D_1, D_2, D_3, D_4. Similar to the proof of Theorem 3.2 we would reach

$$|P^T D_3 P|\,|P^T D_1 P|^{-1} = dI,$$

so that $|D_3|\,|D_1|^{-1} = dI$, and $|P^T D_2 P|^{-1}|P^T D_4 P| = \frac{1}{d}I$ so that $|D_2|^{-1}\,|D_4| = \frac{1}{d}I$. Since D_1, D_2, D_3, D_4 have the same sign pattern, these are equivalent to $D_3 = dD_1$ and $D_4 = \frac{1}{d}D_2$. Thus, we have the final culminating result.

COROLLARY 3.3. *Suppose D_1, D_2, D_3, and D_4 are real $n \times n$ nonsingular diagonal matrices with the same sign pattern. Then*

$$\mathbb{G}(D_1,D_2) = \mathbb{G}(D_3,D_4)$$

if and only if there exists a positive number d such that $D_3 = dD_1$ and $D_4 = \frac{1}{d}D_2$.

For the general case, when $D_i = |D_i|P_i^T JP_i$ with $P_i \in \mathscr{P}_n$, for $i = 1,\ldots,4$, we can have different scenarios depending on the P_i. Note that the selection of the P_i is not unique. Without loss of generality we can consider permutations P_i such that if P_i permutes row i to row j, then it permutes row j to row i. Thus without loss of generality we can assume that such permutation matrices are equal to their transposes.

One of the cases is that all but one of the P_i are I. Suppose $P_1 = P_2 = P_3 = I$ and $P_4 \neq I$. For simplicity assume J is the same as in Theorem 3.2. Then $\mathbb{G}(D_1,D_2) \neq \mathbb{G}(D_3,D_4)$. Otherwise, by applying Theorem 2.2, we have

$$|D_3|^{1/2}|D_1|^{-1/2}\Gamma_n(J)\,|D_2|^{-1/2}|D_4|^{1/2}P_4^T = \Gamma_n(J).$$

Note that P_4 is not conformal to J. That's true because there are at least two diagonal entries of D_4 say d_i^4 and d_j^4 such that their signs are different from the sings of the corresponding entries in J. Let's say that permutation matrix P_4 exchanges the rows and columns of these entries. By knowing the form of J, either $i \leqslant p$ and $j > p$, or $i > p$ and $j \leqslant p$. Thus if P_4 is partitioned such that

$$P_4 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where P_{11} is $p \times p$, then we have $P_{12} = P_{21}^T \neq 0$. Since $I \in \Gamma_n(J)$, we have

$$|D_3|^{1/2}|D_1|^{-1/2} I |D_2|^{-1/2}|D_4|^{1/2} P_4 \in \Gamma_n(J).$$

If

$$A = |D_3|^{1/2}|D_1|^{-1/2} I |D_2|^{-1/2}|D_4|^{1/2} P_4,$$

then A has the same partitioned form as P_4. From $A^T JA = J$ we would have $P_{12} = P_{21}^T = 0$ which doesn't hold. Thus, indeed, $\mathbb{G}(D_1,D_2) \neq \mathbb{G}(D_3,D_4)$!

With having the same inertia matrix J, two G-classes do not necessarily have nonempty intersection. Consider $\mathbb{G}(I,I) = \mathscr{O}_n$ and $\mathbb{G}(2I,I)$. We can observe by using Theorem 2.2 that

$$\mathbb{G}(I,I) \cap \mathbb{G}(2I,I) = \emptyset.$$

The following is an example of two 2×2 G-classes with finite intersection.

EXAMPLE 3.4. Let $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $D_3 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$, and $D_4 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Then I is the inertia matrix of D_1, D_2, D_3 and D_4. By Theorem 3.1, $\mathbb{G}(D_1,D_2) \neq \mathbb{G}(D_3,D_4)$. Suppose $A \in \mathbb{G}(D_1,D_2) \cap \mathbb{G}(D_3,D_4)$. By Theorem 2.2, there are $V, W \in \mathscr{O}_2$ such that

$$A = D_1^{-1/2} V D_2^{-1/2} = D_3^{-1/2} W D_4^{-1/2}.$$

That implies

$$D_3^{1/2} D_1^{-1/2} V D_2^{-1/2} D_4^{1/2} \in \mathscr{O}_2.$$

Thus $v_{1,2} = v_{2,1} = 0$. So A only can be of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

The following question arises.

OPEN QUESTION. Do there exist two $n \times n$ G-classes having finite intersection when $n \geqslant 3$.

REMARK 3.5. Of course, there are other similar questions for G-matrices with a common inertia matrix. One could also consider various relationships between $n \times n$ G-classes $\mathbb{G}(D_1,D_2)$ and $\mathbb{G}(D_3,D_4)$ with inertia matrices J_1 and J_2 respectively, where $J_1 \neq J_2$.The latter becomes more difficult to analyze.

4. The connected components and sign patterns

In this section we show that $\mathbb{G}(D_1, D_2)$ has two or four connected components in $\mathbf{M}_n$. Also we show that

$$\mathbb{G}_n = \bigcup_{D_1, D_2} \mathbb{G}(D_1, D_2),$$

the set of all $n \times n$ G-matrices, has two connected components in $\mathbf{M}_n$. Let $\mathscr{O}_n$ be the set of all $n \times n$ orthogonal matrices, $\mathscr{O}_n^+$ be the set of all $n \times n$ orthogonal matrices with determinant 1, and $\mathscr{O}_n^-$ be the set of all $n \times n$ orthogonal matrices with determinant -1.

PROPOSITION 4.1. [13, Theorem 3.67] *For every $n \geqslant 1$, $\mathscr{O}_n$ has two connected components, $\mathscr{O}_n^+$ and $\mathscr{O}_n^-$.*

PROPOSITION 4.2. [11, Theorem 3.5] *Let J be an $n \times n$ signature matrix. If $J \neq \pm I$ then $\Gamma_n(J)$ has four connected components.*

COROLLARY 4.3. *For every $n \times n$ signature matrix J, $\mathscr{O}_n \cup \Gamma_n(J)$ has two connected components.*

Proof. Since every component of $\Gamma_n(J)$ has some orthogonal matrices (this is because each component has signature matrices, which in fact are orthogonal matrices), by the use of Proposition 4.1, the result is obtained. $\square$

THEOREM 4.4. *Let D_1 and D_2 be nonsingular diagonal matrices with the inertia matrix J.*

(i) *If $J \neq \pm I$, then $\mathbb{G}(D_1, D_2)$ has four connected components.*

(ii) *If $J = \pm I$, $\mathbb{G}(D_1, D_2)$ has two connected components.*

Proof. Let P and Q be as in the proof of Theorem 2.2. Consider the linear operator $T : \mathbf{M}_n \longrightarrow \mathbf{M}_n$ defined by

$$T(A) = |D_1|^{-1/2} P^T A Q |D_2|^{-1/2}.$$

Both T and T^{-1} are continuous and $T(\Gamma_n(J)) = \mathbb{G}(D_1, D_2)$ by Theorem 2.2. So the number of connected components of $\Gamma_n(J)$ and $\mathbb{G}(D_1, D_2)$ are the same. Now, by the use of Propositions 4.1, 4.2, the proof is complete. $\square$

THEOREM 4.5. *The set $\mathbb{G}_n$ of all $n \times n$ G-matrices has two connected components.*

Proof. We present the proof in two steps.

Step 1: First we show that every component of $\mathbb{G}(D_1, D_2)$ intersects one of the components of $\mathbb{G}(|D_1|, |D_2|)$ and hence $\mathbb{G}(D_1, D_2) \bigcup \mathbb{G}(|D_1|, |D_2|)$ has two connected components, when $J \neq \pm I$ is the inertia matrix of D_1 and D_2 (if $J = \pm I$ we have $\mathbb{G}(D_1, D_2) = \mathbb{G}(|D_1|, |D_2|)$. By Theorem 2.2, there exist permutation matrices P and Q such that

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2} P^T AQ |D_2|^{-1/2} : A \in \Gamma_n(J)\},$$
$$\mathbb{G}(|D_1|, |D_2|) = \{|D_1|^{-1/2} P^T AQ |D_2|^{-1/2} : A \in \mathscr{O}_n\}.$$

We know that (see [11, Proposition 3.8]) every component of $\Gamma_n(J)$ has 2^{n-2} signature matrices say $J_1, \dots, J_{2^{n-2}}$. So the corresponding component of $\mathbb{G}(D_1, D_2)$ has the following matrices:

$$|D_1|^{-1/2} P^T J_1 Q |D_2|^{-1/2}, \dots, |D_1|^{-1/2} P^T J_{2^{n-2}} Q |D_2|^{-1/2}.$$

On the other hand it is clear that

$$|D_1|^{-1/2} P^T J_1 Q |D_2|^{-1/2}, \dots, |D_1|^{-1/2} P^T J_{2^{n-2}} Q |D_2|^{-1/2} \in \mathbb{G}(|D_1|, |D_2|).$$

So, the first part of the claim is proved. Since $\mathbb{G}(|D_1|, |D_2|) = T(\mathscr{O}_n)$, where T is the linear operator in the proof of Theorem 4.4, $\mathbb{G}(|D_1|, |D_2|)$ has two connected components. By these two facts we obtain that $\mathbb{G}(D_1, D_2) \cup \mathbb{G}(|D_1|, |D_2|)$ has two connected components.

Step 2: Let $\mathbb{D}_n$ and $\mathbb{D}_n^+$ be the sets of all $n \times n$ diagonal matrices with nonzero and positive diagonal entries respectively. It is clear that $\mathbb{D}_n$ and $\mathbb{D}_n^+$ are connected sets. For every $D_1, D_2 \in \mathbb{D}_n^+$, we have $\mathbb{G}(D_1, D_2) = D_1^{-1/2} \mathscr{O}_n^+ D_2^{-1/2} \dot{\cup} D_1^{-1/2} \mathscr{O}_n^- D_2^{-1/2}$. Then we have

$$\mathbb{G}_n^+ := \bigcup_{D_1, D_2 \in \mathbb{D}_n^+} \mathbb{G}(D_1, D_2) = [\bigcup_{D_1, D_2 \in \mathbb{D}_n^+} D_1^{-1/2} \mathscr{O}_n^+ D_2^{-1/2}] \dot{\cup} [\bigcup_{D_1, D_2 \in \mathbb{D}_n^+} D_1^{-1/2} \mathscr{O}_n^- D_2^{-1/2}].$$

Since $\mathbb{D}_n^+$, $\mathscr{O}_n^+$ and $\mathscr{O}_n^-$ are connected sets,

$$\bigcup_{D_1, D_2 \in \mathbb{D}_n^+} D_1^{-1/2} \mathscr{O}_n^+ D_2^{-1/2}, \quad \bigcup_{D_1, D_2 \in \mathbb{D}_n^+} D_1^{-1/2} \mathscr{O}_n^- D_2^{-1/2}$$

are disjoint connected sets. Then $\mathbb{G}_n^+$ has two connected components.

For every $D_1, D_2 \in \mathbb{D}_n$, by the use of Step 1, every component of $\mathbb{G}(D_1, D_2)$ intersects one of the components of $\mathbb{G}_n^+$. By Step 2, $\mathbb{G}_n^+$ has two connected components and hence $\mathbb{G}_n$ has two connected components. $\square$

Finally in this section, we make another connection with the connected components of $\Gamma_n(J)$.

LEMMA 4.6. [11, Proposition 3.6] *Let J be an $n \times n$ signature matrix and $J \neq \pm I$. Then for every i $(2 \leqslant i \leqslant 4)$, the component $\mathscr{C}_i$ of $\Gamma_n(J)$ is homeomorphic and group isomorphic to $\mathscr{C}_1$.*

As in the proof of Theorem 4.4, let P and Q be as in the proof of Theorem 2.2. Consider the linear operator $T : \mathbf{M}_n \longrightarrow \mathbf{M}_n$ defined by $T(A) = |D_1|^{-1/2} P^T AQ |D_2|^{-1/2}$. Both T and T^{-1} are continuous and $T(\Gamma_n(J)) = \mathbb{G}(D_1, D_2)$ by Theorem 2.2. We thus obtain the following result on the connected components of $\mathbb{G}(D_1, D_2)$.

PROPOSITION 4.7. *If J is the inertia matrix of D_1, D_2 and $J \neq I$, then the four connected components of $\mathbb{G}(D_1,D_2)$ are homeomorphic.*

The second part of this section will be some observations on sign pattern matrices. In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of the entries in the matrix. An $m \times n$ matrix whose entries are from the set $\{+,-,0\}$ is called a sign pattern matrix (or a sign pattern). A sign pattern is said to be full if it does not have a 0 entry. For a real matrix B, $\mathrm{sgn(B)}$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the sign pattern class of A is defined by

$$Q(A) = \{B : \mathrm{sgn(B)} = \mathrm{A}\}.$$

A sign pattern matrix P is called a permutation sign pattern if exactly one entry in each row and column is equal to + and all the other entries are 0. A permutation equivalence of the $n \times n$ sign pattern A has the form PAQ, where P and Q are permutation patterns.

Suppose $\mathscr{P}$ is a property referring to a real matrix. A sign pattern A is said to allow $\mathscr{P}$ if some real matrix in $Q(A)$ has property $\mathscr{P}$. We can then speak of the set or class of sign patterns allowing $\mathscr{P}$ as the class of sign pattern A with property $\mathscr{P}$.

Theorem 1.8 in [6] is a general result which says the following: the set of all $n \times n$ sign patterns that allow a G-matrix is the same as the set of all permutation equivalences of the $n \times n$ sign patterns allowing J-orthogonality. However, from our Theorem 2.2 we obtain the following specific result.

THEOREM 4.8. *The set of sign patterns of the matrices in an $n \times n$ G-class $\mathbb{G}(D_1,D_2)$ is the same as a permutation equivalence of the set of sign patterns of the J-orthogonal matrices in $\Gamma_n(J)$, where J is the common inertia matrix of D_1 and D_2.*

Then, from our equivalence relation $\sim$ we obtain:

COROLLARY 4.9. *If $\mathbb{G}(D_1,D_2) \sim \mathbb{G}(D_3,D_4)$, then the set of sign patterns of the matrices in $\mathbb{G}(D_1,D_2)$ is the same as a permutation equivalence of the set of sign patterns of the matrices in $\mathbb{G}(D_3,D_4)$.*

We next recall Theorem 3.2 from [3]: the class of sign patterns of $n \times n$ nonsingular Cauchy matrices is the same as the class of $n \times n$ sign patterns each of which is permutation equivalent to a “staircase pattern”. Using this and the previous corollary we have the next result.

COROLLARY 4.10. *If $G(D_1,D_2) \sim G(D_3,D_4)$, and the class of sign patterns of $G(D_1,D_2)$ contains a sign pattern of a nonsigular Cauchy matrix, then so does $G(D_3,D_4)$.*

We take this opportunity to recall the following.

Open questions. Does every full $n \times n$ sign pattern allow a J-orthogonal matrix? As shown in [6] and [7] the answer is yes for $n \leqslant 4$. For $n > 4$, we can more generally ask the following: does every $n \times n$ full sign pattern even allow a G-matrix?

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