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AN INTERPOLATION PROPERTY OF REFLECTIONS INVOLVING ORTHOGONAL PROJECTIONS

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Abstract. Let $\mathscr{H}$ be a complex Hilbert space. We consider the interpolation problem: describe the pair (W,L) of subspaces of $\mathscr{H}$ such that there is a reflection J on $\mathscr{H}$ satisfying $J(W) \subseteq L$. We show that two subspaces W,L have this interpolation property if and only if $dim(W \cap L^{\perp}) \leqslant dim(L \cap W^{\perp})$, which is equivalent to that there exists a conjugation C on $\mathscr{H}$ such that $C(W) \subseteq L$. Moreover, we study the least upper bound of these interpolating reflections.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be separable complex Hilbert spaces, and $\mathscr{B}(\mathscr{H},\mathscr{K})$ be the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$. An operator $A \in \mathscr{B}(\mathscr{H})$ is called positive, if $A \geqslant 0$, meaning $\langle Ax,x\rangle \geqslant 0$ for all $x \in \mathscr{H}$, where $\langle,\rangle$ is the inner product of $\mathscr{H}$. Moreover, if $P = P^* = P^2$, then P is called an (orthogonal) projection. We denote by $\mathscr{P}(\mathscr{H})$ the set of all orthogonal projections on $\mathscr{H}$. As usual, the operator order (Loewner partial order) relation $A \geqslant B$ between two self-adjoint operators is defined as $A - B \geqslant 0$. An operator $U \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is said to be unitary if U is invertible with $U^{-1} = U^*$. The set of all unitary operators from $\mathscr{H}$ onto $\mathscr{K}$ is denoted by $\mathscr{U}(\mathscr{H},\mathscr{K})$. For an operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K}), N(T), R(T)$ and $\overline{R(T)}$ denote the null space, the range of T, and the closure of $R(T)$, respectively.

An operator $J \in \mathscr{B}(\mathscr{H})$ is said to be a reflection (or self-adjoint unitary operator) if $J = J^* = J^{-1}$. In this case, $J^+ = \frac{I+J}{2}$ and $J^- = \frac{I-J}{2}$ are mutually annihilating orthogonal projections. If J is a non-scalar reflection, then an indefinite inner product is defined by

$$[x,y] := \langle Jx,y\rangle \qquad (x,y \in \mathscr{H})$$

and $(\mathscr{H},J)$ is called a Krein space ([1]). We denote by $Ref(\mathscr{H})$ the set of all reflections on $\mathscr{H}$. A map $C : \mathscr{H} \to \mathscr{H}$ is called a conjugation if (a) C is anti-linear, i.e. $C(\alpha x + y) = \overline{\alpha}Cx + Cy$ for all $x,y \in \mathscr{H}$ and $\alpha \in \mathbb{C}$, (b) C is invertible with $C^{-1} = C$ and (c) $\langle Cx,Cy\rangle = \langle y,x\rangle$ for all $x,y \in \mathscr{H}$. For $U \in \mathscr{U}(\mathscr{H})$, if both of $PU = UQ$ and $UP = QU$ hold, then U is called an intertwining operator of orthogonal projections P and Q.

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It is well-known that orthogonal projections on a Hilbert space are essential in operator theory (see [2–7, 10–11, 13–17] and therein references). Avron, Seiler and Simon ([4]) obtained that if $P,Q \in \mathscr{P}(\mathscr{H})$ with $\| P-Q \| < 1$, then there exists an intertwining operator for P and Q. A sufficient and necessary condition under which there exists an intertwining operator of P and Q has been given in [17, Theorem 6]. More recently, Simon ([16]) by "supersymmetric" approach presented a more elegant proof of [17, Theorem 6]. In particular, Dou et al. ([9]) and Böttcher et al. ([5]) have characterized the set of all intertwining operator of orthogonal projections P and Q. According to [9, Theorem 3.1] or [5, Theorem 5], an easy observation is that there exists an intertwining operator of P and Q if and only if there exists a reflection $J \in Ref(\mathscr{H})$ with $JPJ = Q$ (which is equivalent to $J(R(P)) = R(Q)$). That is, there exists a reflection $J \in Ref(\mathscr{H})$ with $J(R(P)) = R(Q))$ if and only if $dim(R(P) \cap N(Q)) = dim(N(P) \cap R(Q))$. Also, Liu et al. ([12]) have given some sufficient and necessary conditions for the existence of a conjugation C with $C(R(P)) = R(Q)$. Moreover, Jorgensen and Tian in [15] presented the reflection-positivity and structures of admissible reflection between orthogonal projections.

The aim of the present paper is to consider the interpolation problem for reflections between two projections P and Q. We mainly characterize the pairs (W,L) of subspaces of $\mathscr{H}$ such that there is a reflection $J \in Ref(\mathscr{H})$ with $J(W) \subseteq L$. The motivation to study this interpolation problem stems from the specific structures and decompositions of reflections which was been studied in [13,14]. Also, we want to know whether there is some connection between the interpolation problem of reflections and conjugations. In Section 2, we show that for the pairs (W,L) of subspaces, there is a reflection $J \in Ref(\mathscr{H})$ with $J(W) \subseteq L$ if and only if $dim(W \cap L^{\perp}) \leqslant dim(L \cap W^{\perp})$, which is also equivalent to that there exists a conjugation C on $\mathscr{H}$ such that $C(W) \subseteq L$. Moreover, the supremum (with respect to the operator order) of the reflection J with $JPJ = Q$ and $PJP \geqslant 0$ (J is called reflection positivity) are presented for two orthogonal projections P and Q. In Section 3, we mainly consider two example and a characterization of reflections involving three orthogonal projections, which has been studied in [15].

2. Interpolation property of reflections

To show our main results, we need the following lemma which is another form of Halmos' two projections theorem ([10]). Also, we use P_L to denote the orthogonal projection onto the closed subspace L.

LEMMA 1. ([8, Lemma 1] or [6, Theorem 1.2]) *Let W and L be two closed subspaces of $\mathscr{H}$. Then P_W and P_L have the operator matrices*

$$P_W = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus I_5 \oplus 0I_6 \tag{2.1}$$

and

$$P_L = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}} & D^*(I_5-Q_0)D \end{pmatrix} \tag{2.2}$$

with respect to the space decomposition $\mathscr{H} = \oplus_{i=1}^{6} \mathscr{H}_i$, *respectively, where* $\mathscr{H}_1 = W \cap L$, $\mathscr{H}_2 = W \cap L^{\perp}$, $\mathscr{H}_3 = W^{\perp} \cap L$, $\mathscr{H}_4 = W^{\perp} \cap L^{\perp}$, $\mathscr{H}_5 = W \ominus (\mathscr{H}_1 \oplus \mathscr{H}_2)$ *and* $\mathscr{H}_6 = \mathscr{H} \ominus (\oplus_{j=1}^{5} \mathscr{H}_j)$, Q_0 *is a positive contraction on* $\mathscr{H}_5$, 0 *and* 1 *are not eigenvalues of* Q_0, D *is a unitary from* $\mathscr{H}_6$ *onto* $\mathscr{H}_5$ *and* I_i *is the identity on the corresponding subspace* $\mathscr{H}_i$ *for* $i = 1, \ldots, 6$.

The converse statement of the above lemma also holds.

LEMMA 2. *Let* $\mathscr{H} = \oplus_{i=1}^{6} \mathscr{H}_i$ *and* I_i *is the identity on the corresponding subspace* $\mathscr{H}_i$ *for* $i = 1, 2 \cdots, 6$. *Suppose that* P *and* Q *have the operator matrices*

$$P = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus I_5 \oplus 0I_6 \tag{2.3}$$

and

$$Q = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} D \\ D^* Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0) D \end{pmatrix}. \tag{2.4}$$

If Q_0 *is a positive contraction on* $\mathscr{H}_5$, 0 *and* 1 *are not eigenvalues of* Q_0 *and* D *is a unitary from* $\mathscr{H}_6$ *onto* $\mathscr{H}_5$, *then* $P, Q \in \mathscr{P}(\mathscr{H})$ *with* $\mathscr{H}_1 = R(P) \cap R(Q)$ *and* $\mathscr{H}_2 = R(P) \cap N(Q)$.

Proof. $P, Q \in \mathscr{P}(\mathscr{H})$ are verified directly. It is obvious that

$$R(P) = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_5$$

and

$$R(Q) = \mathscr{H}_1 \oplus \mathscr{H}_3 \oplus R \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} D \\ D^* Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0) D \end{pmatrix}.$$

If $x \in R(P) \cap R(Q)$, we get that

$$x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ x_5 \\ 0 \end{pmatrix} = Qx = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ Q_0 x_5 \\ D^* Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} x_5 \end{pmatrix},$$

where $x_i \in \mathscr{H}_i$ for $i = 1, 2, 5$. Thus $x_2 = 0$ and $x_5 = 0$, so $R(P) \cap R(Q) \subseteq \mathscr{H}_1$. Clearly, $\mathscr{H}_1 \subseteq R(P) \cap R(Q)$, which implies $R(P) \cap R(Q) = \mathscr{H}_1$. In a similar way, we have $\mathscr{H}_2 = R(P) \cap N(Q)$. $\square$

REMARK 1. We can also get from Lemma 2 that $\mathscr{H}_3 = N(P) \cap R(Q)$, $\mathscr{H}_4 = N(P) \cap N(Q)$, $\mathscr{H}_5 = R(P) \ominus (\mathscr{H}_1 \oplus \mathscr{H}_2)$ and $\mathscr{H}_6 = \mathscr{H} \ominus (\oplus_{i=1}^{5} \mathscr{H}_i)$.

LEMMA 3. *Let W and L be two closed subspaces of $\mathscr{H}$. If $dim(W\cap L^{\perp})\leqslant dim(L\cap W^{\perp})$, then there exists a closed subspace $L_0\subseteq L$ such that $dim(W\cap L_0^{\perp})=dim(L_0\cap W^{\perp})$ and $W\cap L=W\cap L_0$.*

Proof. Suppose that $\mathscr{H}_i$ is same to Lemma 1 for $i=1,2,\cdots 6$. Then Lemma 1 implies that P_W and P_L have the operator matrices

$$P_W=I_1\oplus I_2\oplus 0I_3\oplus 0I_4\oplus I_5\oplus 0I_6 \tag{2.5}$$

and

$$P_L=I_1\oplus 0I_2\oplus I_3\oplus 0I_4\oplus\begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}}D\\ D^*Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}} & D^*(I_5-Q_0)D\end{pmatrix} \tag{2.6}$$

with respect to the space decomposition $\mathscr{H}=\oplus_{i=1}^{6}\mathscr{H}_i$.

Since $dim(W\cap L^{\perp})\leqslant dim(L\cap W^{\perp})$, it follows $dim\mathscr{H}_2\leqslant dim\mathscr{H}_3$. Then the subspace $\mathscr{H}_3$ can be divide into $\mathscr{H}_3=\mathscr{H}_3'\oplus\mathscr{H}_3''$, where $dim\mathscr{H}_3'=dim\mathscr{H}_2$. Let $\widetilde{\mathscr{H}_i}=\mathscr{H}_i$ for $i=1,2,5,6$, $\widetilde{\mathscr{H}_3}=\mathscr{H}_3'$, and $\widetilde{\mathscr{H}_4}=\mathscr{H}_3''\oplus\mathscr{H}_4$. So we have a new space decomposition $\mathscr{H}=\oplus_{i=1}^{6}\widetilde{\mathscr{H}_i}$. Define the operator S with respect to the space decomposition $\mathscr{H}=\oplus_{i=1}^{6}\widetilde{\mathscr{H}_i}$ as the form

$$S:=I_1\oplus 0I_2\oplus I_3'\oplus 0I_4'\oplus\begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}}D\\ D^*Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}} & D^*(I_5-Q_0)D\end{pmatrix}.$$

It is easy to see that $S\in\mathscr{P}(\mathscr{H})$ with $S\leqslant P_L$. Setting $L_0:=R(S)$, we get that $L_0\subseteq L$. Then Lemma 2 yields that

$$dim(W\cap L_0^{\perp})=dim\mathscr{H}_2=dim\mathscr{H}_3'=dim(L_0\cap W^{\perp})$$

and $W\cap L=\mathscr{H}_1=\widetilde{\mathscr{H}_1}=W\cap L_0$. $\square$

LEMMA 4. ([12, Theorem 1.7]) *Let W and L be two closed subspaces of $\mathscr{H}$. Then there exists a conjugation C on $\mathscr{H}$ such that $C(W)=L$ if and only if $dim(W\cap L^{\perp})=dim(L\cap W^{\perp})$.*

The following theorem is one of the main results of this section.

THEOREM 1. *Let W and L be two closed subspaces of $\mathscr{H}$. Then the following statements are equivalent:*
(a) There exists a $J\in Ref(\mathscr{H})$ such that $J(W)\subseteq L$,
(b) $dim(W\cap L^{\perp})\leqslant dim(L\cap W^{\perp})$,
(c) There exists a conjugation C on $\mathscr{H}$ such that $C(W)\subseteq L$.

Proof. $(a)\Rightarrow(b)$. For the self-adjoint unitary operator J, we get that $J(W^{\perp})=J(W)^{\perp}$. Indeed, for all $x\in W$ and $y\in W^{\perp}$, we have $\langle Jx,Jy\rangle=\langle x,y\rangle=0$. Thus $J(W)\subseteq J(W^{\perp})^{\perp}$ and $J(W^{\perp})\subseteq J(W)^{\perp}$, so $J(W^{\perp})=J(W)^{\perp}$. Since $J(W)\subseteq L$, it follows $L^{\perp}\subseteq J(W)^{\perp}=J(W^{\perp})$. Then

$$dim(W\cap L^{\perp})=dim(J(W\cap L^{\perp}))=dim(J(W)\cap J(L^{\perp}))\leqslant dim(L\cap W^{\perp}).$$

$(b) \Rightarrow (c)$. By Lemma 3, there exists $L_0 \subseteq L$ such that $dim(W \cap L_0^{\perp}) = dim(L_0 \cap W^{\perp})$. Then we conclude from Lemma 4 that there exists a conjugation C on $\mathscr{H}$ with $C(W) = L_0$, so $C(W) \subseteq L$.

$(c) \Rightarrow (b)$. If $L_1 := C(W) \subseteq L$, then by Lemma 4 again, we have $dim(W \cap L_1^{\perp}) = dim(L_1 \cap W^{\perp})$, which yields

$$dim(W \cap L^{\perp}) \leqslant dim(W \cap L_1^{\perp}) = dim(L_1 \cap W^{\perp}) \leqslant dim(L \cap W^{\perp}).$$

$(b) \Rightarrow (a)$. Since $dim(W \cap L^{\perp}) \leqslant dim(L \cap W^{\perp})$, it follows $dim\mathscr{H}_2 \leqslant dim\mathscr{H}_3$, where $\mathscr{H}_i$ is the same as in Lemma 1. With respect to the space decomposition $\mathscr{H} = \oplus_{i=1}^{6}\mathscr{H}_i$, we define an operator J as the form

$$J = I_1 \oplus \begin{pmatrix} 0 & V^* \\ V & I_3 - VV^* \end{pmatrix} \oplus I_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}} D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^* Q_0^{\frac{1}{2}} D \end{pmatrix},$$

where V is a isometry operator from $\mathscr{H}_2$ into $\mathscr{H}_3$, D and Q_0 are the same as in Lemma 1. By a direct calculation, we know that $J = J^* = J^{-1}$. It is easy to calculate that

$$JP_W = I_1 \oplus \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \oplus 0 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & 0 \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & 0 \end{pmatrix}$$

and

$$P_L J P_W = I_1 \oplus \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \oplus 0 \oplus \begin{pmatrix} Q_0 Q_0^{\frac{1}{2}} + Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} DD^*(I_5 - Q_0)^{\frac{1}{2}} & 0 \\ D^* Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} Q_0^{\frac{1}{2}} + D^*(I_5 - Q_0) DD^*(I_5 - Q_0)^{\frac{1}{2}} & 0 \end{pmatrix}.$$

Obviously,

$$Q_0 Q_0^{\frac{1}{2}} + Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} DD^*(I_5 - Q_0)^{\frac{1}{2}} Q_0^{\frac{1}{2}} = Q_0^{\frac{1}{2}}$$

and

$$D^* Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} Q_0^{\frac{1}{2}} + D^*(I_5 - Q_0) DD^*(I_5 - Q_0)^{\frac{1}{2}} = D^*(I_5 - Q_0)^{\frac{1}{2}},$$

so $P_L J P_W = JP_W$, which yields $J(W) \subseteq L$. $\square$

LEMMA 5. ([9, Theorem 3.1] or [5, Theorem 5]) *Let $P, Q \in \mathscr{P}(\mathscr{H})$ with operator matrices (2.1) and (2.2), respectively. Then there exists a unitary $U \in \mathscr{U}(\mathscr{H})$ such that $PU = UQ$ and $UP = QU$ if and only if* $\dim \mathscr{R}(P) \cap \mathscr{N}(Q) = \dim \mathscr{N}(P) \cap \mathscr{R}(Q)$. *In this case,*

$$\{U \in \mathscr{U}(\mathscr{H}) : PU = UQ \text{ and } UP = QU\}$$
$$= \left\{ \begin{array}{l} U_1 \oplus \begin{pmatrix} 0 & C_2 \\ C_3 & 0 \end{pmatrix} \oplus U_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}} D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^* Q_0^{\frac{1}{2}} D \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & D^* U_0 D \end{pmatrix} : \\ U_1 \in \mathscr{U}(\mathscr{H}_1),\ C_2 \in \mathscr{U}(\mathscr{H}_3, \mathscr{H}_2),\ C_3 \in \mathscr{U}(\mathscr{H}_2, \mathscr{H}_3), \\ \quad U_4 \in \mathscr{U}(\mathscr{H}_4),\ U_0 \in \mathscr{U}(\mathscr{H}_5),\ U_0 Q_0 = Q_0 U_0 \end{array} \right\}.$$

By Theorem 1, if W and L are closed subspaces of $\mathscr{H}$, then there exists a reflection J such that $J(W) = L$ if and only if $\dim(W \cap L^{\perp}) = \dim(L \cap W^{\perp})$. Moreover, we have the following.

PROPOSITION 1. *Let $P, Q \in \mathscr{P}(\mathscr{H})$ and $J \in Ref(\mathscr{H})$. Then the following statements are equivalent:*

(a) $JPJ = Q$,

(b) $JP = QJP$ and $JQ = PJQ$,

(c)

$$J = J_1 \oplus \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \oplus J_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} J_5 & (I_5 - Q_0)^{\frac{1}{2}} J_5 D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} J_5 & -D^* Q_0^{\frac{1}{2}} J_5 D \end{pmatrix}. \qquad (2.7)$$

with respect to the space decomposition $\mathscr{H} = \oplus_{i=1}^{6} \mathscr{H}_i$, where $\mathscr{H}_i$ $(i = 1, 2, \cdots 6)$, Q_0 and D are the same as Lemma 2 and Remark 1, $J_i \in Ref(\mathscr{H}_i)$ for $i = 1, 4, 5$ with $Q_0 J_5 = J_5 Q_0$ and $V \in \mathscr{U}(\mathscr{H}_3, \mathscr{H}_2)$.

Proof. $(a) \Longleftrightarrow (b)$ is clear.

$(a) \Longleftrightarrow (c)$. Since $J \in Ref(\mathscr{H})$, it follows that $JPJ = Q$ if and only if $J \in \{U \in \mathscr{U}(\mathscr{H}) : PU = UQ$ and $UP = QU\}$. Then Lemma 5 and the fact of $J = J^*$ imply that $JPJ = Q$ is equivalent to J has the matrix form (2.7). $\square$

COROLLARY 1. *Let W and L be two closed subspaces of $\mathscr{H}$ with $dim(W \cap L^{\perp}) \leqslant dim(L \cap W^{\perp})$. If $\mathscr{F} := \{L_0 \subseteq L : dim(W \cap L_0^{\perp}) = dim(L_0 \cap W^{\perp})\}$, then*

$$\begin{aligned} &\{J \in Ref(\mathscr{H}) : J(W) \subseteq L\} \\ = &\bigcup_{L_0 \in \mathscr{F}} \{J_1 \oplus \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \oplus J_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} J_5 & (I_5 - Q_0)^{\frac{1}{2}} J_5 D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} J_5 & -D^* Q_0^{\frac{1}{2}} J_5 D \end{pmatrix} : J_i \in Ref(\mathscr{H}_i) \\ &\text{for } i = 1, 4, 5 \text{ with } Q_0 J_5 = J_5 Q_0 \text{ and } V \in \mathscr{U}(\mathscr{H}_3, \mathscr{H}_2)\}, \end{aligned}$$

where Q_0 and D is the same as in Lemma 1, $\mathscr{H}_1 = W \cap L_0$, $\mathscr{H}_2 = W \cap L_0^{\perp}$, $\mathscr{H}_3 = W^{\perp} \cap L_0$, $\mathscr{H}_4 = W^{\perp} \cap L_0^{\perp}$, $\mathscr{H}_5 = W \ominus (\mathscr{H}_1 \oplus \mathscr{H}_2)$, and $\mathscr{H}_6 = \mathscr{H} \ominus (\oplus_{i=1}^{5} \mathscr{H}_i)$.

Proof. Since $dim(W \cap L^{\perp}) \leqslant dim(L \cap W^{\perp})$, it follows from Lemma 3 that $\mathscr{F} \neq \emptyset$.

Let $J \in \{J \in Ref(\mathscr{H}) : J(W) \subseteq L\}$. Setting $L_0 = J(W)$, we conclude that $L_0 \subseteq L$ and $dim(W \cap L_0^{\perp}) = dim(L_0 \cap W^{\perp})$, so $L_0 \in \mathscr{F}$. Moreover, Proposition 1 implies that the inclusion $\subseteq$ holds. Another inclusion $\supseteq$ is obvious. $\square$

The following two corollaries give the simpler conditions under which there exists a $J \in Ref(\mathscr{H})$ with $J(W) \subseteq L$ for two closed subspaces W and L which satisfy certain conditions.

COROLLARY 2. *Let W and L be two closed subspaces of $\mathscr{H}$. If $dimW < +\infty$, then there exists a $J \in Ref(\mathscr{H})$ with $J(W) \subseteq L$ if and only if $dimW \leqslant dimL$.*

Proof. Necessity is clear. Sufficiency. Suppose that $\mathscr{H}_i$ is the same as Lemma 1 for $i = 1, 2, \cdots 6$. Then $dimW < +\infty$ implies $dim\mathscr{H}_i < +\infty$ for $i = 1, 2, 5$, so $dim\mathscr{H}_6 = dim\mathscr{H}_5 < +\infty$.

Case 1. If $dimL = +\infty$, then equation (2.2) implies $dim\mathscr{H}_3 = +\infty$, so $dim\mathscr{H}_2 \leqslant dim\mathscr{H}_3$. Thus we conclude from Theorem 1 that there exists a $J \in Ref(\mathscr{H})$ with $J(W) \subseteq L$.

Case 2. If $dimL < +\infty$, then equation (2.2) induces

$$dimL = dim\mathscr{H}_1 + dim\mathscr{H}_3 + dimR\begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}} & D^*(I_5-Q_0)D \end{pmatrix}.$$

It is easy to see that $V = \begin{pmatrix} Q_0^{\frac{1}{2}} \\ D^*(I_5-Q_0)^{\frac{1}{2}} \end{pmatrix} : \mathscr{H}_5 \to R(VV^*)$ is a unitary operator. Thus

$$dimR\begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5-Q_0)^{\frac{1}{2}} & D^*(I_5-Q_0)D \end{pmatrix} = dimR(VV^*) = dim\mathscr{H}_5,$$

so

$$dimL = dim\mathscr{H}_1 + dim\mathscr{H}_3 + dim\mathscr{H}_5. \tag{2.8}$$

Since

$$dimW = dim\mathscr{H}_1 + dim\mathscr{H}_2 + dim\mathscr{H}_5, \tag{2.9}$$

we have

$$dim(W \cap L^{\perp}) - dim(L \cap W^{\perp}) = dim\mathscr{H}_2 - dim\mathscr{H}_3 = dimW - dimL \leqslant 0.$$

Hence, $J(W) \subseteq L$ follows from Theorem 1. □

COROLLARY 3. *Let W and L be two closed subspaces of $\mathscr{H}$. If $W \subseteq L^{\perp}$, then there exists a $J \in Ref(\mathscr{H})$ with $J(W) \subseteq L$ if and only if $dimW \leqslant dimL$.*

Proof. Sufficiency. Since $W \subseteq L^{\perp}$, it follows $L \subseteq W^{\perp}$, so $dim(W \cap L^{\perp}) = dimW \leqslant dimL = dim(W^{\perp} \cap L)$. Thus $J(W) \subseteq L$ follows from Theorem 1. Necessity is obvious. □

For unit vectors $x, y \in \mathscr{H}$, it is well known that there is a unitary operator $U \in \mathscr{B}(\mathscr{H})$ with $Ux = y$. The following corollary gives an equivalent condition for the existence of a reflection J with $Jx = y$.

COROLLARY 4. *Let $x, y \in \mathscr{H}$ be unit vectors. Then there exists a reflection J with $Jx = y$ if and only if $\langle x, y \rangle = \langle y, x \rangle$.*

Proof. Sufficiency. Let $\mathscr{M}$ and $\mathscr{N}$ be subspaces spanned by vectors x and y, respectively. It is easy to check that $dim(M \cap N^{\perp}) = dim(N \cap M^{\perp})$, so Theorem 1 implies that there exists a reflection J' with $J'x = e^{i\theta}y$. If $\langle x, y \rangle = \langle y, x \rangle$, then $\langle x, y \rangle$ is a real number. Moreover, $e^{i\theta}\langle y, x \rangle = \langle J'x, x \rangle$ is also a real number, which yields $e^{i\theta} = 1$ or $e^{i\theta} = -1$. Thus $J'x = y$ or $J'x = -y$. In the second case, we set $J = -J'$. Necessity is clear. □

The following result describes the partial order of a class of special orthogonal projections, which is used in the proof of Theorem 2.

PROPOSITION 2. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces with $\dim\mathscr{H} = \dim\mathscr{K}$. If $V \in \mathscr{U}(\mathscr{K},\mathscr{H})$ and $P_V := \begin{pmatrix} \frac{I}{2} & \frac{V}{2} \\ \frac{V^*}{2} & \frac{I}{2} \end{pmatrix} : \mathscr{H} \oplus \mathscr{K}$, then*

(a) $P_V \in \mathscr{P}(\mathscr{H} \oplus \mathscr{K})$ with $P_V^{\perp} = P_{-V}$, where $P_V^{\perp} := I - P_V$.

(b) If $P \in \mathscr{P}(\mathscr{H} \oplus \mathscr{K})$, then $P \leqslant P_V$ if and only if

$$P = \begin{pmatrix} \frac{P_1}{2} & \frac{P_1 V}{2} \\ \frac{V^* P_1}{2} & \frac{V^* P_1 V}{2} \end{pmatrix} : \mathscr{H} \oplus \mathscr{K}, \text{ where } P_1 \in \mathscr{P}(\mathscr{H}). \tag{2.10}$$

(c) If $Q \in \mathscr{P}(\mathscr{H} \oplus \mathscr{K})$, then $P_V \leqslant Q$ if and only if

$$Q = \begin{pmatrix} I - \frac{Q_1}{2} & \frac{Q_1 V}{2} \\ \frac{V^* Q_1}{2} & I - \frac{V^* Q_1 V}{2} \end{pmatrix} : \mathscr{H} \oplus \mathscr{K}, \text{ where } Q_1 \in \mathscr{P}(\mathscr{H}).$$

Proof. (a) is obvious.

(b) Sufficiency is clear. Necessity. Assume

$$P = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} : \mathscr{H} \oplus \mathscr{K},$$

where A_{11} and A_{22} are positive contraction operators. Since $0 \leqslant P \leqslant P_V$, we have

$$PP_V^{\perp} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \begin{pmatrix} \frac{I}{2} & \frac{-V}{2} \\ \frac{-V^*}{2} & \frac{I}{2} \end{pmatrix} = 0,$$

so a direct calculation yields

$$\begin{cases} \frac{A_{11}}{2} - \frac{A_{12}V^*}{2} = 0 & \text{①} \\ -\frac{A_{12}^* V}{2} + \frac{A_{22}}{2} = 0 & \text{②} \end{cases}$$

From ① and ②, we get that $A_{11} = A_{12}V^*$ and $A_{22} = A_{12}^* V$. Furthermore, $P^2 = P$ implies that

$$(A_{12}V^*)^2 + A_{12}A_{12}^* = A_{12}V^*. \tag{2.11}$$

This coupled with the fact that $A_{11} = A_{12}V^* = VA_{12}^* \geqslant 0$ gives $2(A_{12}V^*)^2 = A_{12}V^*$. Setting $P_1 := 2A_{12}V^*$, we conclude that $P_1^2 = P_1$ and (2.10) holds as desired.

(c) If $Q \in \mathscr{P}(\mathscr{H} \oplus \mathscr{K})$ and $P_V \leqslant Q$, then $Q^{\perp} \leqslant P_V^{\perp} = \begin{pmatrix} \frac{I}{2} & \frac{-V}{2} \\ \frac{-V^*}{2} & \frac{I}{2} \end{pmatrix}$. It follows from (b) that

$$Q^{\perp} = \begin{pmatrix} \frac{Q_1}{2} & -\frac{Q_1 V}{2} \\ -\frac{V^* Q_1}{2} & \frac{V^* Q_1 V}{2} \end{pmatrix} : \mathscr{H} \oplus \mathscr{K}, \text{ where } Q_1 \in \mathscr{P}(\mathscr{H}).$$

Therefore, $P_V \leqslant Q$ if and only if

$$Q = \begin{pmatrix} I - \frac{Q_1}{2} & \frac{Q_1 V}{2} \\ \frac{V^* Q_1}{2} & I - \frac{V^* Q_1 V}{2} \end{pmatrix} : \mathscr{H} \oplus \mathscr{K}, \text{ where } Q_1 \in \mathscr{P}(\mathscr{H}). \quad \square$$

We denote

$$\mathscr{F}(P,Q) := \{J : JPJ = Q,\ J \in Ref(\mathscr{H})\}$$

and

$$\widetilde{\mathscr{F}}(P,Q) := \{J : J \in \mathscr{F}(P,Q) \text{ and } PJP \geqslant 0\}.$$

Let $\emptyset \neq \Gamma \subseteq \mathscr{P}(\mathscr{H})$. We also denote by $\underset{E\in\Gamma}{\vee} E \in \mathscr{P}(\mathscr{H})$ the supremum of all projections in Γ. That is, $R(\underset{E\in\Gamma}{\vee} E) = \overline{\vee\{\underset{E\in\Gamma}{\cup} R(E)\}}$.

Another main result of this section is the following.

THEOREM 2. *Let $P,Q \in \mathscr{P}(\mathscr{H})$ with $dim(R(P)\cap N(Q)) = dim(N(P)\cap R(Q))$. Then*

(a) $\sup\{J : J \in \mathscr{F}(P,Q)\} = I.$

(b) $\sup\{J : J \in \widetilde{\mathscr{F}}(P,Q)\} = I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}} D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^* Q_0^{\frac{1}{2}} D \end{pmatrix}.$

(c) $\max\{J : J \in \widetilde{\mathscr{F}}(P,Q)\}$ *exists if and only if $dim(R(P)\cap N(Q)) = dim(N(P)\cap R(Q)) = 0$.*

Proof. (a) It is conclude from Proposition 1 that $J \in \mathscr{F}(P,Q)$ if and only if

$$J = J_1 \oplus \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \oplus J_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} J_5 & (I_5 - Q_0)^{\frac{1}{2}} J_5 D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} J_5 & -D^* Q_0^{\frac{1}{2}} J_5 D \end{pmatrix}.$$

For all $J \in \mathscr{F}(P,Q)$, define the projection

$$\widetilde{P}_J := \frac{I+J}{2} = \frac{I_1 + J_1}{2} \oplus \begin{pmatrix} \frac{I_2}{2} & \frac{V}{2} \\ \frac{V^*}{2} & \frac{I_3}{2} \end{pmatrix} \oplus \frac{I_4 + J_4}{2} \oplus \begin{pmatrix} \frac{I_5 + Q_0^{\frac{1}{2}} J_5}{2} & \frac{(I_5 - Q_0)^{\frac{1}{2}} J_5 D}{2} \\ \frac{D^*(I_5 - Q_0)^{\frac{1}{2}} J_5}{2} & \frac{I_6 - D^* Q_0^{\frac{1}{2}} J_5 D}{2} \end{pmatrix}.$$

Then

$$\bigvee_{J\in\mathscr{F}(P,Q)} \widetilde{P}_J = I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus I_6.$$

Indeed, if $J \in \mathscr{F}(P,Q)$, then $-J \in \mathscr{F}(P,Q)$ and $\widetilde{P}_{-J} = \widetilde{P}_J^{\perp}$. This means that both $\widetilde{P}_J$ and $\widetilde{P}_J^{\perp}$ are in $\bigvee_{J\in\mathscr{F}(P,Q)} \widetilde{P}_J$. Hence,

$$\sup\{J \mid J \in \mathscr{F}(P,Q)\} = 2 \underset{J\in\mathscr{F}(P,Q)}{\vee} \widetilde{P}_J - I = I.$$

(b) It is easy to see that if $J \in \mathscr{F}(P,Q)$, then $PJP \geqslant 0$ iff $QJQ \geqslant 0$, which is equivalent to $J_5 = I_5$. Thus

$$\bigvee_{J\in\widetilde{\mathscr{F}}(P,Q)} \widetilde{P}_J = I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus \begin{pmatrix} \frac{I_5 + Q_0^{\frac{1}{2}}}{2} & \frac{(I_5 - Q_0)^{\frac{1}{2}} D}{2} \\ \frac{D^*(I_5 - Q_0)^{\frac{1}{2}}}{2} & \frac{I_6 - D^* Q_0^{\frac{1}{2}} D}{2} \end{pmatrix},$$

so

$$\sup\{J \mid J \in \widetilde{\mathscr{F}}(P,Q)\} = 2 \bigvee_{J \in \widetilde{\mathscr{F}}(P,Q)} \widetilde{P}_J - I$$

$$= I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}} D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^* Q_0^{\frac{1}{2}} D \end{pmatrix}.$$

(c) It follows from (b) that $\max\{J : J \in \widetilde{\mathscr{F}}(P,Q)\}$ exists if and only if

$$I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}} D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^* Q_0^{\frac{1}{2}} D \end{pmatrix} \in \widetilde{\mathscr{F}}(P,Q). \tag{2.12}$$

By Proposition 1, (2.12) is equivalent to $dim(R(P) \cap N(Q)) = dim(N(P) \cap R(Q)) = 0$. □

3. Some examples and applications

In [15], some interpolation relations of the three orthogonal projections and a reflection were considered. Here, we use the same notations as that in [15]. That is, $\varepsilon := (E_0, E_\pm)$,

$$R(\varepsilon) := \{J \in Ref(\mathscr{H}) : E_- J E_+ = J E_+\} \tag{3.1}$$

and

$$R_0(\varepsilon) := \{J \in Ref(\mathscr{H}) : J E_0 = E_0, E_- J E_+ = J E_+, E_+ J E_- = J E_-\}, \tag{3.2}$$

where E_0, E_+ and E_- are orthogonal projections. Then Theorem 1 implies that $R(\varepsilon) \neq \emptyset$ if and only if $dim(R(E_+) \cap N(E_-)) \leqslant dim(R(E_-) \cap N(E_+))$. However, the condition for $R_0(\varepsilon) \neq \emptyset$ was not given in [15]. As an application, we present a characterization of $R_0(\varepsilon) \neq \emptyset$.

THEOREM 3. *Let* $\varepsilon = (E_0, E_\pm)$ *and* $R_0(\varepsilon)$ *be as above. Then* $R_0(\varepsilon) \neq \emptyset$ *if and only if* $dim(R(E_+) \cap N(E_-)) = dim(N(E_+) \cap R(E_-))$ *and* $R(E_0) \subseteq M_1 \oplus R\begin{pmatrix} V \\ I_3 \end{pmatrix} \oplus M_4 \oplus R\begin{pmatrix} (I_5 + Q_0^{\frac{1}{2}} J_5)^{\frac{1}{2}} \\ D^* J_5 (I_5 - Q_0^{\frac{1}{2}} J_5)^{\frac{1}{2}} \end{pmatrix}$, *where* $M_1 \subseteq \mathscr{H}_1$ *and* $M_4 \subseteq \mathscr{H}_4$ *are two closed subspaces,* V *is a unitary operator from* $\mathscr{H}_3$ *onto* $\mathscr{H}_2$, $J_5 \in Ref(\mathscr{H}_5)$ *with* $J_5 Q_0 = Q_0 J_5$.

Proof. For all $J \in Ref(\mathscr{H})$, it is clear that $E_- J E_+ = J E_+$ and $E_+ J E_- = J E_-$ if and only if $J E_+ J = E_-$, which is equivalent to $J(R(E_+)) = R(E_-)$. By Proposition 1, there exists a $J \in Ref(\mathscr{H})$ with $J E_+ J = E_-$ if and only if

$$dim(R(E_+) \cap N(E_-)) = dim(R(E_-) \cap N(E_+)).$$

Furthermore, $JE_0 = E_0$ is equivalent to

$$R(E_0) \subseteq R\left(\frac{I+J}{2}\right). \tag{3.3}$$

Considering that J has form (2.7), we have

$$\frac{I+J}{2} = P_1 \oplus \begin{pmatrix} \frac{I_1}{2} & \frac{V}{2} \\ \frac{V^*}{2} & \frac{I_3}{2} \end{pmatrix} \oplus P_4 \oplus \begin{pmatrix} \frac{I_5+Q_0^{\frac{1}{2}}J_5}{2} & \frac{(I_5-Q_0)^{\frac{1}{2}}J_5D}{2} \\ \frac{D^*(I_5-Q_0)^{\frac{1}{2}}J_5}{2} & \frac{I_6-D^*Q_0^{\frac{1}{2}}J_5D}{2} \end{pmatrix},$$

which implies

$$R\left(\frac{I+J}{2}\right) = M_1 \oplus\ R\begin{pmatrix} V \\ I_3 \end{pmatrix} \oplus M_4 \oplus\ R\begin{pmatrix} (I_5+Q_0^{\frac{1}{2}}J_5)^{\frac{1}{2}} \\ D^*J_5(I_5-Q_0^{\frac{1}{2}}J_5)^{\frac{1}{2}} \end{pmatrix},$$

so $R(E_0) \subseteq M_1 \oplus R\begin{pmatrix} V \\ I_3 \end{pmatrix} \oplus M_4 \oplus R\begin{pmatrix} (I_5+Q_0^{\frac{1}{2}}J_5)^{\frac{1}{2}} \\ D^*J_5(I_5-Q_0^{\frac{1}{2}}J_5)^{\frac{1}{2}} \end{pmatrix}$ as desired. □

The following example follows from [15, Theorem 7.2], which gives the equivalent condition for $E_+E_0E_- = E_+E_-$. Here, we use the example to describe the specific forms of $\mathscr{R}_0(\varepsilon)$.

EXAMPLE 1. Let $C \in \mathscr{B}(\mathscr{H},\mathscr{K})$ be a contraction. Suppose that E_+ and E_- are orthogonal projections with $R(E_+) = Graph(C)$ and $R(E_-) = Graph(-C)$, respectively. If

$$E_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{H} \oplus \mathscr{K}, \tag{3.4}$$

then

$$R_0(\varepsilon) = \left\{ \begin{pmatrix} I & 0 \\ 0 & J_{22} \end{pmatrix} : J_{22} \in Ref(\mathscr{K}) \text{ with } C^*J_{22} = -C^* \right\}.$$

Indeed, it is clear that

$$R(E_+) = Graph(C) = \{x+Cx : x \in \mathscr{H}\}$$

and

$$R(E_-) = Graph(-C) = \{x-Cx : x \in \mathscr{H}\}.$$

Then a direct calculation implies that E_+ and E_- with respect to space decomposition $\mathscr{H} \oplus \mathscr{K}$ have the operator matrix form

$$E_+ = \begin{pmatrix} (I+C^*C)^{-1} & (I+C^*C)^{-1}C^* \\ C(I+C^*C)^{-1} & C(I+C^*C)^{-1}C* \end{pmatrix}$$

and

$$E_- = \begin{pmatrix} (I+C^*C)^{-1} & -(I+C^*C)^{-1}C^* \\ -C(I+C^*C)^{-1} & C(I+C^*C)^{-1}C* \end{pmatrix},$$

respectively. Let $J \in R_0(\varepsilon)$. Then $JE_0 = E_0$ if and only if $J = \begin{pmatrix} I & 0 \\ 0 & J_{22} \end{pmatrix}$, where $J_{22} \in Ref(\mathscr{K})$. In this case, the equation $JE_+J = E_-$ is equivalent to $C^*J_{22} + C^* = 0$.

Let $\varepsilon := (E_0, E_\pm)$ as above. In [15], the sets

$$\mathscr{S}_{OS}(J) := \{(E_0, E_\pm) \mid E_+JE_+ \geqslant 0\},$$

$$\varepsilon(Markov) := \{(E_0, E_\pm) \mid E_+E_0E_- = E_+E_-\}$$

and

$$\mathscr{S}(\varepsilon) := \{J \in Ref(\mathscr{H}) : E_-JE_+ = JE_+, JE_0 = E_0J\}$$

are also defined. In the following, we give an example to illustrate $\bigcap\limits_{J \in \mathscr{S}(\varepsilon)} \mathscr{S}_{OS}(J) \nsubseteq \varepsilon(Markov)$. Thus, there is a gap in [15, Theorem 6.4].

EXAMPLE 2. Let $E_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{H} \oplus \mathscr{H}$, $E_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} : \mathscr{H} \oplus \mathscr{H}$, and $E_0 = \begin{pmatrix} \frac{I}{2} & \frac{U}{2} \\ \frac{U^*}{2} & \frac{I}{2} \end{pmatrix} : \mathscr{H} \oplus \mathscr{H}$, where $U \in B(\mathscr{H})$ is a unitary operator. Let $J \in Ref(\mathscr{H} \oplus \mathscr{H})$. Then a direct calculation implies that equations $E_-JE_+ = JE_+$ and $JE_0 = E_0J$ are equivalent to $J = \begin{pmatrix} 0 & J_1U \\ U^*J_1 & 0 \end{pmatrix}$, where $J_1 \in Ref(\mathscr{H})$. That is

$$\mathscr{S}(\varepsilon) = \left\{ \begin{pmatrix} 0 & J_1U \\ U^*J_1 & 0 \end{pmatrix}, J_1 \in Ref(\mathscr{H}) \right\}.$$

It is easy to check that $E_+JE_+ = 0$ for any $J \in \mathscr{S}(\varepsilon)$ and $E_+E_0E_- \neq E_+E_-$. Thus $\varepsilon = (E_0, E_\pm) \in \bigcap\limits_{J \in \mathscr{S}(\varepsilon)} \mathscr{S}_{OS}(J)$, whereas $\varepsilon = (E_0, E_\pm) \notin \varepsilon(Markov)$. So $\bigcap\limits_{J \in \mathscr{S}(\varepsilon)} \mathscr{S}_{OS}(J) \nsubseteq \varepsilon(Markov)$.

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