Operators
and
Matrices
Volume 16, Number 2 (2022), 279–297 doi:10.7153/oam-2022-16-22

DISTINGUISHED SUBSPACES OF TOPELITZ OPERATORS ON N_φ–TYPE QUOTIENT MODULES

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(*Communicated by G. Misra*)

Abstract. In this paper, we show that there always exists reducing subspace M for $S_{\psi(z)}$ such that the restriction of $S_{\psi(z)}$ on M is unitarily equivalent to the Bergman shift when $\psi(z)$ is a finite Blaschke product. Moreover, we will show that only if $\psi(z)$ is a finite Blaschke product can $S_{\psi(z)}$ has distinguished reducing subspaces. We also give the form of these distinguished reducing subspaces when $\psi(z)$ is a finite Blaschke product. Finally, we show that every non-trivial minimal reducing subspace S of $S_{\psi(z)}$ is orthogonal to the direct sum of all distinguished subspaces when S is not a distinguished subspace of $S_{\psi(z)}$.

1. Introduction

Let $\mathbb{D}^2$ be the open unit bidisk in the 2-dimensional complex Euclidean space, and let Γ^2 be the distinguished boundary of $\mathbb{D}^2$. Let $L^2(\Gamma^2)$ be the Lebesgue space and $H^2(\Gamma^2)$ be the Hardy space over Γ^2. We denote by $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ the Hardy spaces on the unit circle Γ in variables z and w, respectively. A function $\varphi(w) \in H^2(\mathbb{D})$ is called inner if $|\varphi(w)| = 1$ *a.e.* on Γ. Let P be the orthogonal projection from $L^2(\Gamma^2)$ onto $H^2(\Gamma^2)$. For each function $\psi \in L^\infty$, we define the Toeplitz operator T_ψ on $H^2(\Gamma^2)$ by $T_\psi f = P(\psi f)$ for $f \in H^2(\Gamma^2)$. A closed subspace M of $H^2(\Gamma^2)$ is called a submodule if $T_z M \subseteq M$ and $T_w M \subseteq M$. There are many conclusions about submodules of the Hardy space over Γ^2 (see [9] and [11]–[13]). In $H^2(\Gamma)$, A. Beurling [2] showed an invariant subspace M of $H^2(\Gamma)$ has the form $M = \theta H^2(\Gamma)$ for some inner function θ. In $H^2(\Gamma^2)$, the structure of submodules is complicated. If M is a submodule of $H^2(\Gamma^2)$ and $N = H^2(\Gamma^2) \ominus M$, then $T_z^* N \subseteq N$ and $T_w^* N \subseteq N$. We called N is a quotient module of $H^2(\Gamma^2)$ related to M.

A reducing subspace M for an operator T on Hilbert space H is a closed subspace M of H such that $TM \subset M$ and $T^*M \subset M$. In [6], K. Guo et al show that only a multiplication operator by a finite Blaschke product on the Bergman space has a unique distinguished reduced subspace, that is, the restriction of the operator on this reduced subspace is equivalent to the Bergman shift. In [10], S. Sun et al show that

Mathematics subject classification (2020): 47B35, 47B10.
Keywords and phrases: Bergman space, Blaschke product, distinguished reducing subspace, Hardy space.
The work is supported by National Nature Science Foundation of China (grant no. 11971087).
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the multiplication operator on the Bergman space is unitarily equivalent to a weighted unilateral shift operator of finite multiplicity if and only if its symbol is a constant multiple of the N-th power of a Möbius transform.

For a subset E of $H^2(\Gamma^2)$, we denote by $[E]$ the smallest submodule of $H^2(\Gamma^2)$ containing E. Throughout this paper, let $\varphi \in H^\infty(\mathbb{D})$ be a non-constant inner function, and $N_\varphi = H^2(\Gamma^2) \ominus [z-\varphi(w)]$, a N_ϕ-type quotient module. A quotient module has a very rich structure [7, 8]. In fact, N_φ can be identified with the tensor product of two well-known classical spaces, namely, the quotient module $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L_a^2(\mathbb{D})$. In [8], K. Izuchi and R. Yang have obtained that, if φ is an one variable inner function, N_φ is essentially reduced if and only if φ is a finite Blaschke product. For a quotient module N of $H^2(\Gamma^2)$ and a function $\psi(z) \in H^\infty(\mathbb{D})$, we define a operator S_ψ on N by

$$S_\psi = P_N T_\psi|_N,$$

where P_N is the orthogonal projection from $H^2(\Gamma^2)$ onto N. In [6], the authors show that $S_{\psi(z)}$ acting on $H^2(\Gamma^2) \ominus [z-w]$ has the distinguished reducing subspace if and only if $\psi(z)$ is a finite Blaschke product. Inspired by [6], in this paper, we extend their conclusions from $H^2(\Gamma^2) \ominus [z-w]$ to the setting of the N_ϕ-type quotient module.

In this paper, we will show that only if $\psi(z)$ is a finite Blaschke product can $S_{\psi(z)}$ on N_φ has the distinguished subspace and completely described the form of those distinguished reducing subspaces when $\psi(z)$ is a finite Blaschke product. The following are our main results.

THEOREM 1.1. *Let ψ be a Blaschke product of order N, There are reducing subspace M for $S_{\psi(z)}$ such that $S_{\psi(z)}|_M \cong M_z$. In fact, M has only the following form*

$$M = \overline{span}\{P_n'(\psi)e_h : n \geqslant 0\} \tag{1}$$

where $P_n'(\psi) = \sqrt{n+1}e_n(\psi(z), \psi(\varphi(w)))$ and $e_h = h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$, $h(w) \in K_\varphi^2(\Gamma_w)$ with $\|h\| = 1$. And $\{\frac{P_n'(\psi)e_h}{\sqrt{n+1}\sqrt{N}}\}_0^\infty$ form an orthonomal basis of M.

THEOREM 1.2. *Let $\psi \in H^\infty(\mathbb{D})$. Then $S_{\psi(z)}$ acting on N_φ has the distinguished reducing subspace if and only if ψ is a finite Blaschke product.*

Let $M_k = \overline{span}\{P_n'(\psi)e_k : n \geqslant 0\}$, where $e_k = \lambda_k(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$, $k = 1,\dots,m$. And we denote $M_0 = M_1 \oplus M_2 \oplus \cdots \oplus M_m$. Then we have the following theorem.

THEOREM 1.3. *Suppose that Ω is a nontrivial minimal reducing subspace for $S_{\psi(z)}$. If Ω is not a distinguished reducing subspace, then Ω is a subspace of $M_0^\perp$.*

The paper is organized as follows. In Section 2, we give some basic facts about the space N_φ and the operator S_ϕ. In Section 3, we give the proof of Theorem 1.1 and 1.2. In Section 4, we will show that every nontrivial minimal reducing subspace Ω of $S_{\psi(z)}$ which is not distinguished reducing subspace is orthogonal to M_0.

2. Preliminaries

In this section, we lay out some basic facts about the space N_φ and the operator S_z. And in this paper, we denote Bergman shift and $H^2(\Gamma_w)\ominus\varphi(w)H^2(\Gamma_w)$ by M_z and $K^2_\varphi(\Gamma_w)$ respectively.

LEMMA 2.1. ([8]) *Let $\varphi(w)$ be a one variable non-constant inner function and $\{\lambda_k(w) : k = 1,2,\ldots,m\}$ be an orthonormal basis of $K^2_\varphi(\Gamma_w)$ and*

$$e_j(z,w) = \frac{w^j + w^{j-1}z + \ldots + z^j}{\sqrt{j+1}} \quad (j = 0,1,\ldots). \tag{2}$$

Let

$$E_{k,j} = \lambda_k(w)e_j(z,\varphi(w)). \tag{3}$$

Then $\{E_{k,j} : k = 1,2,\ldots,m; j = 0,1,\ldots\}$ (m can be infinity) is an orthonormal basis for N_φ.

LEMMA 2.2. ([8]) *There exists a unitary operator U*

$$U : N_\varphi \to K^2_\varphi(\Gamma_w)\otimes L^2_a(\mathbb{D})$$

$$E_{k,j} \mapsto \lambda_k(w)\sqrt{j+1}z^j$$

such that

$$US_z = (I\otimes M_z)U.$$

where I is an identity map on $H^2(\Gamma_w)\ominus\varphi(w)H^2(\Gamma_w)$.

COROLLARY 2.3. *(1). For each $\psi(z)\in H^\infty(\mathbb{D})$, we have*

$$US_{\psi(z)} = (I\otimes M_{\psi(z)})U$$

(2). $S_z|_{N_\varphi} = S_{\varphi(w)}|_{N_\varphi}$.
(3). For each $\psi(z)\in H^\infty(\mathbb{D})$, we have

$$S_{\psi(z)}|_{N_\varphi} = S_{\psi(\varphi(w))}|_{N_\varphi}$$

(4). Since N_φ is a backshift invariant subspace, then we have

$$T^*_z|_{N_\varphi} = S^*_z \text{ and } T^*_{\varphi(w)}|_{N_\varphi} = S^*_{\varphi(w)}.$$

Proof. We only need to prove (1).
For any $\psi(z) = \sum_{n=0}^\infty a_n z^n \in H^\infty(\mathbb{D})$, we have

$$\begin{aligned}\langle S_{\psi(z)}E_{k,j}, E_{l,i}\rangle &= \langle \psi(z)\lambda_k(w)e_j(z,\varphi(w)), \lambda_l(w)e_i(z,\varphi(w))\rangle \\ &= \frac{1}{\sqrt{j+1}\sqrt{i+1}}\sum_{s=0}^{j}\sum_{t=0}^{i}\langle \psi(z)\lambda_k(w)\varphi(w)^{j-s}z^s, \lambda_l(w)\varphi(w)^{i-t}z^t\rangle \\ &= \frac{1}{\sqrt{j+1}\sqrt{i+1}}\sum_{s=0}^{j}\sum_{t=0}^{i}\langle \lambda_k(w), \lambda_l(w)\varphi(w)^{i+s-j-t}\rangle\langle\psi(z), z^{t-s}\rangle.\end{aligned} \tag{4}$$

Hence

$$\langle S_{\psi(z)}E_{k,j}, E_{l,i}\rangle = \frac{j+1}{\sqrt{j+1}\sqrt{i+1}}a_{i-j} \text{ if and only if } l=k \text{ and } i-j \geqslant 0.$$

Then we have

$$\begin{aligned} US_{\psi(z)}E_{k,j} &= U\sum_{l=1}^{m}\sum_{i=0}^{\infty}\langle S_{\psi(z)}E_{k,j}, E_{l,i}\rangle E_{l,i} \\ &= U\sum_{i=0}^{\infty}\langle S_{\psi(z)}E_{k,j}, E_{k,i}\rangle E_{k,i} \\ &= U\sum_{i=j}^{\infty}\frac{j+1}{\sqrt{j+1}\sqrt{i+1}}a_{i-j}E_{k,i} \\ &= \sqrt{j+1}\lambda_k(w)[a_0z^j + a_1z^{j+1} + \ldots + a_nz^{j+n} + \ldots] \\ &= \sqrt{j+1}\lambda_k(w)\psi(z)z^j \\ &= (I\otimes M_{\psi(z)})UE_{k,j}. \quad \square \end{aligned} \tag{5}$$

PROPOSITION 2.4. *If* $f \in H^2(\Gamma^2)\bigcap C(\overline{\mathbb{D}^2})$ *and* $g \in N_\varphi$, *then*

$$\langle f(z,w), g(z,w)\rangle = \langle f(\varphi(w),w), g(0,w)\rangle.$$

Proof. Since $f \in \mathbb{C}(\overline{\mathbb{D}^2})$, then there are a sequence $\{q_n\}$ of polynomials of z and w converging uniformly to $f(z,w)$ on the closed bidisk. Thus it suffices to show

$$\langle z^iw^l, \lambda_k(w)e_j(z,\varphi(w))\rangle = \langle \varphi(w)^iw^l, \lambda_k(w)e_j(0,\varphi(w)\rangle,$$

for all $i,l,j \in \mathbb{N}$, and $k = 1,2,\ldots,m$. So then the result follows from the following equalities.

$$\begin{aligned} \langle z^iw^l, \lambda_k(w)e_j(z,\varphi(w))\rangle &= \langle w^l, T^*_{z^i}\lambda_k(w)e_j(z,\varphi(w))\rangle \\ &= \langle w^l, T^*_{\varphi(w)^i}\lambda_k(w)e_j(z,\varphi(w))\rangle \\ &= \langle \varphi(w)^iw^l, \lambda_k(w)e_j(z,\varphi(w))\rangle \\ &= \frac{1}{\sqrt{j+1}}\sum_{s=0}^{j}\langle \varphi(w)^iw^l, \lambda_k(w)z^s\varphi(w)^{j-s}\rangle \\ &= \frac{1}{\sqrt{j+1}}\langle \varphi(w)^iw^l, \lambda_k(w)\varphi(w)^j\rangle \\ &= \langle \varphi(w)^iw^l, \lambda_k(w)e_j(0,\varphi(w))\rangle. \quad \square \end{aligned} \tag{6}$$

PROPOSITION 2.5. *If* $h(z,w) \in H^2(\Gamma^2)$ *and* $h \in N_\varphi^\perp = [z-\varphi(w)]$, *then* $h(\varphi(w),w) = 0$ *for all* $w \in \mathbb{D}$.

Proof. Let $w \in \mathbb{D}$, then for each $f(z,w) \in (z-\varphi(w))H^2(\Gamma^2)$, we have $f(\varphi(w),w) = 0$. For each $h \in N_\varphi^\perp = [z-\varphi(w)] = \overline{(z-\varphi(w))H^2(\Gamma^2)}$, there exists a sequence $\{g_n\} \subseteq H^2(\Gamma^2)$ such that $\| h-(z-\varphi(w))g_n \|^2 \to 0$ as $n \to \infty$.

Therefore

$$0 = \langle (z-\varphi(w))g_n, k_\alpha(w)k_{\varphi(\alpha)}(z)\rangle \to \langle h, k_\alpha(w)k_{\varphi(\alpha)}(z)\rangle = h(\varphi(\alpha),\alpha)$$

as $n \to \infty$, for each $\alpha \in \mathbb{D}$. $\square$

PROPOSITION 2.6. *Suppose $\psi(w) \in H^\infty(\mathbb{D})$, then we have $\psi(z)-\psi(\varphi(w)) \in [z-\varphi(w)]$.*

Proof. Suppose $\psi(w) = \Sigma_{n=0}^\infty a_n z^n \in H^\infty(\mathbb{D})$. It is clear that $\psi(z)-\psi(\varphi(w)) \in H^2(\Gamma^2)$. For every $E_{k,j} \in N_\varphi$, $k = 1,\ldots,m$, $j = 0,1,\ldots$, we have

$$\begin{aligned}
\langle \psi(z)-\psi(\varphi(w)), E_{k,j}\rangle &= \langle \psi(z)-\psi(\varphi(w)), \frac{1}{\sqrt{j+1}}\lambda_k(w)\Sigma_{i=0}^j z^i\varphi(w)^{j-i}\rangle \\
&= \frac{1}{\sqrt{j+1}}\Sigma_{i=0}^j\langle \psi(z), \lambda_k(w)z^i\varphi(w)^{j-i}\rangle \\
&- \frac{1}{\sqrt{j+1}}\Sigma_{i=0}^j\langle \psi(\varphi(w)), \lambda_k(w)z^i\varphi(w)^{j-i}\rangle \\
&= \frac{1}{\sqrt{j+1}}\Sigma_{i=0}^j a_i\overline{\lambda_k(0)\varphi(0)^{j-i}} \\
&- \frac{1}{\sqrt{j+1}}\langle \Sigma_{n=0}^\infty a_n\varphi(w)^n, \lambda_k(w)\varphi(w)^j\rangle \\
&= 0.
\end{aligned} \tag{7}$$

This completes the proof. $\square$

3. The distinguished reducing subspace

In this section we will show that there always exists reducing subspace M for $S_{\psi(z)}$ such that the restriction of $S_{\psi(z)}$ on M is unitarily equivalent to the Bergman shift when $\psi(z)$ is a finite Blaschke product. Moreover, we will give the concrete forms of these reduced subspaces. At last, we will prove $S_{\psi(z)}$ acting on N_φ has the distinguished reducing subspace if and only if ψ is a finite Blaschke product.

PROPOSITION 3.1. *For each $f(z,w) \in H^2(\Gamma^2)$, f is in N_φ if and only if there is a function $\widetilde{f}(z,w)$ in $\mathfrak{D} \otimes K_\varphi^2(\Gamma_w)$ such that*

$$f(z,w) = \frac{\widetilde{f}(z,w)-\widetilde{f}(\varphi(w),w)}{z-\varphi(w)} \tag{8}$$

for two points z and w with $z \neq \varphi(w)$ in the unit disk.

Proof. Since $\{E_{k,j} : k=1,\dots,m;\ j=0,1,\dots\}$ is an orthonormal basis of N_φ, then for each $f \in N_\varphi$, we can write

$$f(z,w) = \sum_{k=1}^{m}\sum_{j=0}^{\infty} a_{kj}E_{k,j}(z,w).$$

Let $\widetilde{f}(z,w) = \sum_{k=1}^{m}\sum_{j=0}^{\infty}\frac{a_{kj}}{\sqrt{j+1}}\lambda_k(w)z^{j+1}$. Then the equation (8) holds. Also we have

$$\begin{aligned}
\|\widetilde{f}\|^2_{\mathfrak{D}\otimes k^2_\varphi} &= \sum_{j=0}^{\infty}\left\langle \sum_{k=1}^{m}\frac{a_{kj}}{\sqrt{j+1}}\lambda_k(w), \sum_{k=1}^{m}\frac{a_{kj}}{\sqrt{j+1}}\lambda_k(w)\right\rangle \| z^{j+1} \|^2_{\mathfrak{D}} \\
&= \sum_{j=0}^{\infty}\sum_{k=1}^{m}\frac{|a_{kj}|^2}{j+1}(j+2) \\
&\leqslant 2\sum_{j=0}^{\infty}\sum_{k=1}^{m}|a_{kj}|^2 \\
&= 2\|f\|^2.
\end{aligned} \tag{9}$$

Hence $\widetilde{f}(z,w)$ in $\mathfrak{D}\otimes K^2_\varphi(\Gamma_w)$.

Conversely, if $f(z,w) = \frac{\widetilde{f}(z,w)-\widetilde{f}(\varphi(w),w)}{z-\varphi(w)}$, for some $\widetilde{f}(z,w)$ in $\mathfrak{D}\otimes K^2_\varphi(\Gamma_w)$. Let $\widetilde{f}(z,w) = \sum_{k=1}^{m}\sum_{j=0}^{\infty}a_{kj}\lambda_k(w)z^j$, where $\|\widetilde{f}\|^2_{k^2_\varphi\otimes\mathfrak{D}} = \sum_{k=1}^{m}\sum_{j=0}^{\infty}(j+1)|a_{kj}|^2 < +\infty$. Then

$$\begin{aligned}
f(z,w) &= \frac{\sum_{j=0}^{\infty}\sum_{k=1}^{m}a_{kj}\lambda_k(w)z^j - \sum_{j=0}^{\infty}\sum_{k=1}^{m}a_{kj}\lambda_k(w)\varphi(w)^j}{z-\varphi(w)} \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{m}a_{kj}\lambda_k(w)\frac{z^j-\varphi(w)^j}{z-\varphi(w)} \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{m}\sqrt{j}a_{kj}E_{k,j-1}.
\end{aligned} \tag{10}$$

and $\|f\|^2_{H^2} = \sum_{k=1}^{m}\sum_{j=1}^{\infty}|j||a_{kj}|^2 \leqslant \|\widetilde{f}\|^2_{k^2_\varphi\otimes\mathfrak{D}} < +\infty$. $\square$

THEOREM 3.2. *Let f be a nonzero function in N_φ, $\psi(z)$ is a function in $H^\infty(\mathbb{D})$. If $(\psi(z)+\psi(\varphi(w)))f \in N_\varphi$, then*

$$f(z,w) = ch(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)},$$

where c is a constant and $h(w)\in K^2_\varphi(\Gamma_w)$ with $\|h\|=1$.

Proof. Since $f \in N_\varphi$ and $(\psi(z)+\psi(\varphi(w)))f \in N_\varphi$, by Theorem 2.1, we have

$$f(z,w) = \frac{\widetilde{f}(z,w)-\widetilde{f}(\varphi(w),w)}{z-\varphi(w)}$$

for some $\widetilde{f}(z,w) = \sum_{k=1}^m F_k(z)\lambda_k(w) \in \mathfrak{D} \otimes K_\varphi^2(\Gamma_w)$, and

$$(\psi(z) + \psi(\varphi(w)))f(z,w) = \frac{\widetilde{g}(z,w) - \widetilde{g}(\varphi(w),w)}{z - \varphi(w)}$$

for some $\widetilde{g}(z,w) = \sum_{k=1}^m G_k(z)\lambda_k(w) \in \mathfrak{D} \otimes K_\varphi^2(\Gamma_w)$. Therefore

$$\begin{aligned} f(z,w) &= \sum_{k=1}^m \frac{F_k(z) - F_k(\varphi(w))}{z - \varphi(w)}\lambda_k(w) \\ &= \sum_{k=1}^m f_k(z,w), \end{aligned} \tag{11}$$

where $f_k(z,w) = \frac{F_k(z)-F_k(\varphi(w))}{z-\varphi(w)}\lambda_k(w)$, and

$$\begin{aligned} (\psi(z) + \psi(\varphi(w)))f(z,w) &= \sum_{k=1}^m \frac{G_k(z) - G_k(\varphi(w))}{z - \varphi(w)}\lambda_k(w) \\ &= \sum_{k=1}^m g_k(z,w). \end{aligned} \tag{12}$$

where $g_k(z,w) = \frac{G_k(z)-G_k(\varphi(w))}{z-\varphi(w)}\lambda_k(w)$. Then we have

$$(\psi(z) + \psi(\varphi(w)))f(z,w) = \sum_{k=1}^m (\psi(z) + \psi(\varphi(w)))f_k(z,w).$$

Next we want to prove $(\psi(z) + \psi(\varphi(w)))f_k(z,w) = g_k(z,w)$. Since g_k and f_k are in N_φ, and

$$(\psi(z) + \psi(\varphi(w)))f_k(z,w) = \frac{(\psi(z) + \psi(\varphi(w)))(F_k(z) - F_k(\varphi(w)))}{z - \varphi(w)}\lambda_k(w),$$

for each $i \neq j$, we have $g_i(z,w) \perp g_j(z,w)$ and $g_i(z,w) \perp f_j(z,w)$. Since $F_k(z) = \sum_{n=0}^\infty a_n^k z^n \in \mathfrak{D}$ for every $k = 1,\dots,m$, we have

$$\begin{aligned} &\left\langle \psi(z)\frac{z^n - \varphi(w)^n}{z - \varphi(w)}\lambda_i(w), \psi(z)\frac{z^m - \varphi(w)^m}{z - \varphi(w)}\lambda_j(w) \right\rangle \\ &= \sum_{t=0}^{m-1}\sum_{s=0}^{n-1} \langle \psi(z)z^s\varphi(w)^{n-1-s}\lambda_i(w), \psi(z)z^t\varphi(w)^{m-1-t}\lambda_j(w)\rangle \\ &= \sum_{t=0}^{m-1}\sum_{s=0}^{n-1} \langle \psi(z)z^s, \psi(z)z^t)(\lambda_i(w), \varphi(w)^{m+s-t-n}\lambda_j(w)\rangle = 0. \end{aligned} \tag{13}$$

Let $\psi(z)=\sum_{n=0}^{\infty} b_n z^n$ with $\sum_{n=0}^{\infty}|b_n|^2<+\infty$. Then

$$\begin{aligned}
&\left\langle \psi(\varphi(w))\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \psi(\varphi(w))\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} b_l\overline{b_k}\left\langle \varphi(w)^l\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \varphi(w)^k\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} b_l\overline{b_k}\sum_{t=0}^{m-1}\sum_{s=0}^{n-1}\langle \varphi(w)^l z^s\varphi(w)^{n-1-s}\lambda_i(w), \varphi(w)^k z^t\varphi(w)^{m-1-t}\lambda_j(w)\rangle=0
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
&\left\langle \psi(z)\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \psi(\varphi(w))\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\sum_{l=0}^{\infty}\overline{b_l}\left\langle \varphi(z)\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \varphi(w)^l\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\sum_{l=0}^{\infty}\overline{b_l}\sum_{t=0}^{m-1}\sum_{s=0}^{n-1}\langle \varphi(z) z^s\varphi(w)^{n-1-s}\lambda_i(w), \varphi(w)^l z^t\varphi(w)^{m-1-t}\lambda_j(w)\rangle=0.
\end{aligned}\tag{15}$$

Hence,

$$\begin{aligned}
&\langle \psi(z)f_i(z,w), \psi(z)f_j(z,w)\rangle\\
&=\left\langle \psi(z)\frac{F_i(z)-F_i(\varphi(w))}{z-\varphi(w)}\lambda_i(w), \psi(z)\frac{F_j(z)-F_j(\varphi(w))}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\left\langle \psi(z)\Sigma_{n=1}^{\infty}a_n^i\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \psi(z)\Sigma_{m=1}^{\infty}a_m^j\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty}a_n^i\overline{a_m^j}\left\langle \psi(z)\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \psi(z)\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle=0.
\end{aligned}\tag{16}$$

$$\begin{aligned}
&\langle \psi(\varphi(w))f_i(z,w), \psi(\varphi(w))f_j(z,w)\rangle\\
&=\left\langle \psi(\varphi(w))\frac{F_i(z)-F_i(\varphi(w))}{z-\varphi(w)}\lambda_i(w), \psi(\varphi(w))\frac{F_j(z)-F_j(\varphi(w))}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\left\langle \psi(\varphi(w))\Sigma_{n=1}^{\infty}a_n^i\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \psi(\varphi(w))\Sigma_{m=1}^{\infty}a_m^j\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=\Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty}a_n^i\overline{a_m^j}\left\langle \psi(\varphi(w))\frac{z^n-\varphi(w)^n}{z-\varphi(w)}\lambda_i(w), \psi(\varphi(w))\frac{z^m-\varphi(w)^m}{z-\varphi(w)}\lambda_j(w)\right\rangle\\
&=0.
\end{aligned}\tag{17}$$

Similarly,

$$\langle \psi(z)f_i(z,w), \psi(\varphi(w))f_j(z,w)\rangle=0.$$

So we can get, from the above discussion,

$$\begin{aligned}
&\langle(\psi(z)+\psi(\varphi(w)))f_i(z,w), (\psi(z)+\psi(\varphi(w)))f_j(z,w)\rangle\\
&=\langle \psi(z)f_i(z,w), \psi(\varphi(w))f_j(z,w)) + (\psi(\varphi(w))f_i(z,w), \psi(z)f_j(z,w)\rangle\\
&=0,
\end{aligned}\tag{18}$$

and

$$\begin{aligned}&\langle g_i(z,w),(\psi(z)+\psi(\varphi(w)))f_j(z,w)\rangle\\&=\langle g_i(z,w),\psi(z)f_j(z,w)\rangle+\langle g_i(z,w),\psi(\varphi(w)))f_j(z,w)\rangle\\&=2\langle g_i(z,w),\psi(z)f_j(z,w)\rangle\\&=0.\end{aligned}\tag{19}$$

Hence

$$\begin{aligned}(\psi(z)+\psi(\varphi(w)))f_k(z,w)&=g_k(z,w)\\&=\frac{G_k(z)-G_k(\varphi(w))}{z-\varphi(w)}\lambda_k(w).\end{aligned}\tag{20}$$

and

$$f_k(z,w)=\frac{F_k(z)-F_k(\varphi(w))}{z-\varphi(w)}\lambda_k(w).\tag{21}$$

In following we discuss it in two cases. Firstly we assume $\psi(0)=0$. Letting z tend to $\varphi(w)$ in the equations (21) and (8), respectively, we get

$$f_k(\varphi(w),w)=F_k'(\varphi(w))\lambda_k(w),$$

and

$$2\psi(\varphi(w))f_k(\varphi(w),w)=G_k'(\varphi(w))\lambda_k(w).$$

Hence

$$2\psi(\varphi(w))F_k'(\varphi(w))=G_k'(\varphi(w)).\tag{22}$$

Letting $z=0$ in (21) and (8), we have

$$\psi(\varphi(w))F_k(\varphi(w))=G_k(\varphi(w)).\tag{23}$$

Taking derivatives at two sides of (23), we get

$$\psi'(\varphi(w))F_k(\varphi(w))+\psi(\varphi(w))F_k'(\varphi(w))=G_k'(\varphi(w)).\tag{24}$$

Then by (22) and (24) we have

$$\psi(\varphi(w))F_k'(\varphi(w))=\psi'(\varphi(w))F_k(\varphi(w)).$$

Hence

$$\left(\frac{\psi(z)}{F_k(z)}\right)'\bigg|_{z=\varphi(w)}=0,$$

and so, since φ is a non-constant inner function,

$$F_k(z)=a_k\psi(z)$$

for some constant a_k. Hence

$$f_k(z,w)=a_k\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}\lambda_k(w)$$

and

$$f(z,w) = ch(w)\frac{\psi(z) - \psi(\varphi(w))}{z - \varphi(w)}.$$

where $h(w) = \frac{\sum_{k=1}^{m} a_k \lambda_k(w)}{\|\sum_{k=1}^{m} a_k \lambda_k(w)\|}$ and $c = \|\sum_{k=1}^{m} a_k \lambda_k(w)\|$.

If $\psi(0) \neq 0$, since $(\psi(z) - \psi(0) + \psi(\varphi(w)) - \psi(0))f = (\psi(z) + \psi(\varphi(w)))f - 2\psi(0)f \in N_\varphi$ and $f \in N_\varphi$, then, through the above discussion, we can conclude

$$\begin{aligned} f(z,w) &= ch(w)\frac{\psi(z) - \psi(0) - \psi(\varphi(w)) + \psi(0)}{z - \varphi(w)} \\ &= ch(w)\frac{\psi(z) - \psi(\varphi(w))}{z - \varphi(w)}. \end{aligned} \tag{25}$$

This completes the proof. □

PROPOSITION 3.3. *Suppose ψ is a nonconstant finite Blaschke product, and $f(z,w) = ch(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$ for some constant c and $h(w) \in K_\varphi^2(\Gamma_w)$ with $\|h\| = 1$. Then, for each $l \geqslant 1$,*

$$\sqrt{l+1}e_l(\psi(z), \psi(\varphi(w)))f \in N_\varphi.$$

Proof. By Theorem 3.2, let $f(z,w) = ch(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$, where c is constant and $h(w) \in K_\varphi^2(\Gamma_w)$ with $\|h\| = 1$. Then, for each $l \geqslant 1$,

$$\sqrt{l+1}e_l(\psi(z), \psi(\varphi(w)))f = ch(w)\frac{\psi(z)^{l+1} - \psi(\varphi(w))^{l+1}}{z - \varphi(w)} \in N_\varphi. \quad \square$$

PROPOSITION 3.4. *Let $\psi(z)$ be an inner function satisfying $\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} \in H^2(\Gamma^2)$, then*

$$\frac{\psi(z) - \psi(\varphi(w))}{z - \varphi(w)} \perp \psi(z)H^2(\Gamma^2).$$

Proof. Let $h(z,w) = \frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$. For any polynomial $p(z,w)$, we have

$$\begin{aligned} &\langle h(z,rw), \psi(z)p(z,w)\rangle \\ &= \langle (\psi(z) - \psi(\varphi(rw))) \sum_{n=0}^{\infty} \overline{z}^{n+1} \varphi(rw)^n, \psi(z)p(z,w)\rangle \\ &= \sum_{n=0}^{\infty} [\langle \varphi(rw)^n \psi(z), z^{n+1}\psi(z)p(z,w)\rangle - \langle \varphi(rw)^n \psi(\varphi(rw)), z^{n+1}\psi(z)p(z,w)\rangle] \\ &= 0. \end{aligned} \tag{26}$$

This implies that $h(z,rw) \perp \varphi(z)H^2(\Gamma^2)$. Since $h(z,rw)$ converges to $h(z,w)$ in the norm of $H^2(\Gamma^2)$ as $r \to 1^-$. Hence $h(z,w) \perp \psi(z)H^2(\Gamma^2)$, that is, $h(z,w) \in kerT_{\psi(z)}^*$. This completes the proof. □

From the above proposition and $T^*_{\psi(z)}|_{N_\varphi} = T^*_{\psi(\varphi(w))}|_{N_\varphi}$, we also can get

$$\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} \perp \psi(\varphi(w))H^2(\Gamma^2)$$

when $\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} \in H^2(\Gamma^2)$.

PROPOSITION 3.5. *Suppose $\psi(z)$ is an inner function and $h(w) \in K^2_\varphi(\Gamma_w)$ with $\|h\| = 1$. Then $e_h = h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$ is in $H^2(\Gamma^2)$ if and only if ψ is a finite Blaschke product. Moreover, $\|e_h\|^2 = N$, the order of ψ.*

Proof. If $\psi(z) = \frac{z-a}{1-\overline{a}z}$ is a Blaschke product of order 1, by $h(w) \in K^2_\varphi(\Gamma_w)$ with $\|h\| = 1$, then

$$\begin{aligned}
\|e_h\|^2 &= \int_{\Gamma^2} \left| h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} \right|^2 dm \\
&= \int_\Gamma |h(w)|^2 \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} dm(w) \int_\Gamma \frac{1-|a|^2}{|1-\overline{a}z|^2} dm(z) \\
&= 1.
\end{aligned} \tag{27}$$

If $\psi = \psi_1\psi_2\ldots\psi_N = \psi_1 f$ is a finite Blaschke product, then

$$\begin{aligned}
e_h &= h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} \\
&= h(w)\psi_1(z)\frac{f(z)-f(\varphi(w))}{z-\varphi(w)} + h(w)f(\varphi(w))\frac{\psi_1(z)-\psi_1(\varphi(w))}{z-\varphi(w)}.
\end{aligned} \tag{28}$$

Since, by Proposition 3.4,

$$\begin{aligned}
&\left\langle h(w)\psi_1(z)\frac{f(z)-f(\varphi(w))}{z-\varphi(w)}, h(w)f(\varphi(w))\frac{\psi_1(z)-\psi_1(\varphi(w))}{z-\varphi(w)} \right\rangle \\
&= \left\langle h(w)\frac{f(z)-f(\varphi(w))}{z-\varphi(w)}, h(w)f(\varphi(w))T^*_{\psi_1(z)}\frac{\psi_1(z)-\psi_1(\varphi(w))}{z-\varphi(w)} \right\rangle \\
&= 0,
\end{aligned} \tag{29}$$

we have

$$\begin{aligned}
\|e_h\|^2 &= \left\| h(w)\psi_1\frac{f(z)-f(\varphi(w))}{z-\varphi(w)} \right\|^2 + \left\| h(w)f(\varphi(w))\frac{\psi_1(z)-\psi_1(\varphi(w))}{z-\varphi(w)} \right\|^2 \\
&= \left\| h(w)\frac{f(z)-f(\varphi(w))}{z-\varphi(w)} \right\|^2 + \left\| h(w)\frac{\psi_1(z)-\psi_1(\varphi(w))}{z-\varphi(w)} \right\|^2 \\
&= \left\| h(w)\frac{f(z)-f(\varphi(w))}{z-\varphi(w)} \right\|^2 + 1.
\end{aligned} \tag{30}$$

By induction, we therefore have $\|e_h\|^2 = N$.

If $\psi(z)$ is a Blaschke product of infinite order. Then, similar to the above discussion, we have $\|e_h\| = \infty$.

If $\psi(z)$ is a general inner function which is not a finite Blaschke product, by Frostman's Theorem [5, p. 75], there exists a $\lambda \in \mathbb{D}$ such that $\frac{\psi(z)-\lambda}{1-\overline{\lambda}\psi(z)}$, denoted by $B(z)$, is an infinite Blaschke product. Then $\psi(z) = \frac{\lambda+B(z)}{1+\overline{\lambda}B(z)}$ and

$$\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} = (1-|\lambda|^2)\frac{B(z)-B(\varphi(w))}{(z-\varphi(w))(1+\overline{\lambda}B(z))(1+\overline{\lambda}B(\varphi(w)))}$$

Since $\lambda \in \mathbb{D}$, it is clear that $h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)} \in H^2(\Gamma^2)$ if and only if $h(w)\frac{B(z)-B(\varphi(w))}{z-\varphi(w)} \in H^2(\Gamma^2)$. Hence e_h is not in $H^2(\Gamma^2)$ in this case. $\square$

THEOREM 3.6. *Suppose ψ be a Blaschke product of order N. Then there are reducing subspaces M for $S_{\psi(z)}$ such that $S_{\psi(z)}|_M \cong M_z$. Moreover, each M has the following form*

$$M = \overline{span}\{P_n'(\psi)e_h : n \geqslant 0\} \tag{31}$$

where $P_n'(\psi) = \sqrt{n+1}e_n(\psi(z),\psi(\varphi(w)))$ and $e_h = h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$, $h(w) \in K_\varphi^2(\Gamma_w)$ with $\|h\| = 1$. And $\{\frac{P_n'(\psi)e_h}{\sqrt{n+1}\sqrt{N}}\}_0^\infty$ form an orthonomal basis of M.

Proof. For each $h(w) \in H^2(\Gamma_w) \ominus \varphi(w)H^2(\Gamma_w)$ with $\|h\| = 1$, let

$$e_h = h(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}$$

and $M = \overline{span}\{P_n'(\psi)e_h : n \geqslant 0\}$. By Theorem 3.3, we have $P_n'(\psi)e_h \in N_\varphi$, and then M is a closed subspace of N_φ. For each $n \geqslant 0$,

$$\begin{aligned} S_{\psi(z)}P_n'(\psi)e_h &= P_{N_\varphi}(\psi(z)P_n'(\psi)e_h) \\ &= \frac{n+1}{n+2}P_{n+1}'(\psi)e_h + \frac{1}{n+2}P_{n+1}'(\psi)e_h - P_{N_\varphi}(\psi(\varphi(w))^{n+1}e_h \\ &= \frac{n+1}{n+2}P_{n+1}'(\psi)e_h + P_{N_\varphi}\left(\frac{1}{n+2}P_{n+1}'(\psi)e_h - \psi(\varphi(w))^{n+1}e_h\right) \\ &= \frac{n+1}{n+2}P_{n+1}'(\psi)e_h. \end{aligned} \tag{32}$$

The last equation is obtained by $P_{n+1}'(\psi)e_h - \psi(\varphi(w))^{n+1}e_h \in [z-\varphi(w)]$.

Since $S_{\psi(z)}^*e_h = T_{\psi(z)}^*e_h = 0$ and, for each $n \geqslant 1$, $S_{\psi(z)}^*P_n'(\psi)e_h = P_{n-1}'(\psi)e_h$, we have M is a reducing subspace of $S_{\psi(z)}$. Since $\|P_n'(\psi)e_h\|^2 = (n+1)\|e_h\|^2 = (n+1)N$ and $\langle P_n'(\psi)e_h, P_m'(\psi)e_h\rangle = 0$ for all $n \neq m$, then $\{\frac{P_n'(\psi)e_h}{\sqrt{n+1}\sqrt{N}}\}_0^\infty$ form an orthonomal basis of M. Since $S_{\psi(z)}\frac{P_n'(\psi)e_h}{\sqrt{n+1}\sqrt{N}} = \sqrt{\frac{n+1}{n+2}}\frac{P_{n+1}'(\psi)e_h}{\sqrt{n+2}\sqrt{N}}$, then M is a reducing subspace for $S_{\psi(z)}$ such that $S_{\psi(z)}|_M \cong M_z$.

Suppose that M is a reducing subspace of $S_{\psi(z)}$ and $S_{\psi(z)}|_M \cong M_z$, we will show that M has the form of (20). Since $S_{\psi(z)}|_M \cong M_z$, i.e. there exist an orthonomal basis $\{F_n\}_0^\infty$ of M such that

$$S_{\psi(z)}F_n = \sqrt{\frac{n+1}{n+2}}F_{n+1}$$

Observe $P_{N_\varphi}(\psi(z)+\psi(\varphi(w)))F_0 = S_{\psi(z)}F_0 + S_{\psi(\varphi(w))}F_0 = \sqrt{2}F_1$. Then

$$\|P_{N_\varphi}(\psi(z)+\psi(\varphi(w)))F_0\|^2 = 2.$$

We also have

$$\begin{aligned} &\|(\psi(z)+\psi(\varphi(w))F_0\|^2 \\ &= \|\psi(z)F_0\|^2 + \|\psi(\varphi(w))F_0\|^2 + \langle T_{\psi(z)}T^*_{\psi(\varphi(w))}F_0, F_0\rangle + \langle T_{\psi(\varphi(w))}T^*_{\psi(z)}F_0, F_0\rangle \\ &= 2. \end{aligned} \tag{33}$$

Thus $(\psi(z)+\psi(\varphi(w)))F_0 \in N_\varphi$. Then by Theorem 2.2, we have

$$F_0 = ch(w)\frac{\psi(z)-\psi(\varphi(w))}{z-\varphi(w)}.$$

for some constant c and some function $h(w) \in H^2(\Gamma_w) \ominus \varphi(w)H^2(\Gamma_w)$ with $\|h\| = 1$, and so $e_h \in M_1$. Then by propositon 3.3, for each $l \geqslant 0$, we have $P_l'(\psi)e_h = (l+1)S^l_{\psi(z)}e_h \in M$. Therefore

$$M_0 = \overline{span}\{P_n'(\psi)e : n \geqslant 0\} \subseteq M.$$

By previous discussion, we know that M_0 is a reducing subspace of $S_{\psi(z)}|_M \cong M_z$. But M_z is irreducible. Therefore we conclude $M_0 = M$. This completes the proof. $\square$

THEOREM 3.7. *Suppose $\psi \in H^\infty(\mathbb{D})$. Then $S_{\psi(z)}$ acting on N_φ has the distinguished reducing subspace if and only if ψ is a finite Blaschke product.*

Proof. We only need to prove that if S_ψ has the distinguished reducing subspace, then ψ is a finite Blaschke product.

Assume S_ψ has the distinguished reducing subspace M such that $S_\psi|_M \cong M_z$. i.e. there exist a unitary operator $U : M \to L_a^2(\mathbb{D})$ such that $U^*M_zU = S_\psi|_M$. Let K_λ^M be the reproducing kernel of M for $\lambda = (\lambda_1,\lambda_2) \in \mathbb{D}^2$. Then $\|K_\lambda^M\|^2 \neq 0$ except for at most a countable set about variable λ_1. Since

$$\begin{aligned} |\langle S_\psi K_\lambda^M, K_\lambda^M\rangle| &= |\langle \psi K_\lambda^M, K_\lambda^M\rangle| \\ &= |\psi(\lambda_1)|\|K_\lambda^M\|^2 \end{aligned} \tag{34}$$

and $\|S_\psi\| = \|M_z\| = 1$, we have that $|\psi(\lambda_1)| \leqslant 1$ except for at most a countable set, and so $\|\psi\|_\infty \leqslant 1$.

Set $e_n = U^* e_n'$, where $e_n'(z) = \sqrt{n+1} z^n$ for $n = 0, 1, \ldots$. Then

$$S^*_{\psi(z)} e_0 = U^* M_z^* U e_0 = U^* M_z^* e_0' = 0$$

and $T^*_{\psi(\varphi(w))} e_0 = T^*_\psi e_0 = S^*_\psi e_0 = 0$. By Corollary 3.2 (3), we have

$$\begin{aligned} \|P_{N_\varphi}(\psi(z) + \psi(\varphi(w))) e_0\|^2 &= \|2 S_{\psi(z)} e_0\|^2 \\ &= 4\|U^* M_z U e_0\|^2 \\ &= 4\|M_z e_0'\|^2 \\ &= 2. \end{aligned} \tag{35}$$

and

$$\begin{aligned} &\|(\psi(z) + \psi(\varphi(w))) e_0\|^2 \\ &= \|\psi(z) e_0\|^2 + \|\psi(\varphi(w)) e_0\|^2 + \langle T_{\psi(z)} T^*_{\psi(\varphi(w))} e_0, e_0 \rangle + \langle T_{\psi(\varphi(w))} T^*_{\psi(z)} e_0, e_0 \rangle \\ &= 2. \end{aligned} \tag{36}$$

Hence

$$(\psi(z) + \psi(\varphi(w))) e_0 \in N_\varphi.$$

It follows from Theorem 3.2 that

$$e_0 = c h(w) \frac{\psi(z) - \psi(\varphi(w))}{z - \varphi(w)}$$

for some constant c and some function $h(w) \in H^2(\Gamma_w) \ominus \varphi(w) H^2(\Gamma_w)$ with $\|h\| = 1$. Since

$$\|(\psi(z) + \psi(\varphi(w))) e_0\|^2 = 2$$

and $\|\psi\|_\infty \leqslant 1$, then we have $\|\psi(z) e_0\|^2 = 1$ and

$$\|\psi(z) e_0\|^2 - \|e_0\|^2 = \int_{\Gamma^2} (|\psi(z)|^2 - 1) |e_0|^2 dm_2 = 0.$$

Thus $|\psi(z)| = 1$ almost all on the unit circle and ψ is an inner function. Proposition 3.5 therefore implies that ψ is a finite Blaschke product. This completes the proof. $\square$

4. Minimal reducing subspaces

In this section we will show that every nontrivial minimal reducing subspace Ω of $S_{\psi(z)}$ is orthogonal to the subspace M_0 if Ω is not a distinguished reducing subspace, where M_0 is the union of all distinguished reducing subspaces.

Let $L_0 = ker T^*_{\psi(z)} \cap ker T^*_{\psi(\varphi(w))} \cap N_\varphi$, where ψ is a finite Blaschke product.

LEMMA 4.1. *If M is a nontrivial reducing subspace for $S_{\psi(z)}$, then the wandering subspace of M is contained in L_0.*

Proof. Let M be a nontrivial reducing subspace for $S_{\psi(z)}$. Since

$$T^*_{\psi(z)}|_{N_\varphi} = T^*_{\psi(\varphi(w))}|_{N_\varphi} = S^*_{\psi(z)}.$$

For each $g \in M \ominus S_{\psi(z)}M$, it is easy to see that $T^*_{\psi(z)}g = T^*_{\psi(\varphi(w))}g = S^*_{\psi(z)}g = 0$, and then g is in L_0. This completes the proof. $\square$

LEMMA 4.2. *If ψ is a nonconstant finite Blaschke product and M is a reducing subspace for $S_{\psi(z)}$, then $S^*_{\psi(z)}M = M$.*

Proof. Note that $\psi(z)$ is a Blaschke product with finite order, the multiplicity operator M_ψ on $L^2_a(\mathbb{D})$ is a Fredholm operator and $M^*_\psi L^2_a(\mathbb{D}) = L^2_a(\mathbb{D})$. Since $S_{\psi(z)}$ on N_φ is unitarily equivalent to $I \otimes M_{\psi(z)}$ on $K^2_\varphi(\Gamma_w) \otimes L^2_a(\mathbb{D})$, then

$$S^*_{\psi(z)}N_\varphi = N_\varphi.$$

Since M is a reducing subspace for S_ψ, we have

$$S^*_{\psi(z)}M = M.$$

This completes the proof. $\square$

Let $k_\psi = \overline{span}\{\psi^l(z)\psi^k(\varphi(w))N_\varphi : l,k \geqslant 0\}$, and $\mathfrak{L}_\psi = kerT^*_{\psi(z)} \cap kerT^*_{\psi(\varphi(w))} \cap k_\psi$.

PROPOSITION 4.3. *Suppose M is a reducing subspace for $S_{\psi(z)}$, For a given g in the wandering subspace of M, there are a unique family of functions $\{d^{l-k}_g\} \subseteq \mathfrak{L}_\psi \ominus L_0$ such that*

(i) $P'_l(\psi(z),\psi(\varphi(w)))g + \sum_{k=0}^{l-1} P'_k(\psi(z),\psi(\varphi(w)))d^{l-k}_g$ is in M, for each $l \geqslant 0$,

(ii) $P'_{N_\psi}[P'_l(\psi(z),\psi(\varphi(w)))d^k_g]$ is in M for each $k \geqslant 1$ and $l \geqslant 0$.

Proof. For a given $g \in M \ominus S_{\psi(z)}M$, first we will use mathematical induction to construct a family of functions $\{d^k_g\}$.

By Lemma 4.1 and $g \in L_0$, then $T^*_{\psi(z)}[(\psi(z)+\psi(\varphi(w))g] = T^*_{\psi(\varphi(w))}[(\psi(z)+\psi(\varphi(w))g] = g$. By Lemma 4.2, there is a unique function $\widetilde{g} \in M \ominus L_0$ such that

$$T^*_{\psi(z)}\widetilde{g} = T^*_{\psi(\varphi(w))}\widetilde{g} = S^*_{\psi(z)}\widetilde{g} = g.$$

This gives

$$T^*_{\psi(z)}[\widetilde{g} - (\psi(z)+\psi(\varphi(w)))g] = g - g = 0$$

and

$$T^*_{\psi(\varphi(w))}[\widetilde{g} - (\psi(z)+\psi(\varphi(w)))g] = g - g = 0.$$

Letting $d^1_g = \widetilde{g} - (\psi(z)+\psi(\varphi(w)))$, then $d^1_g \in kerT^*_{\psi(z)} \cap kerT^*_{\psi(\varphi(w))}$ and

$$P'_1(\psi(z),\psi(\varphi(w)))g + d^1_g = (\psi(z)+\psi(\varphi(w)))g + d^1_g = \widetilde{g} \in M.$$

Because both $\widetilde{g}$ and g are in M, we have that $d_g^1 \in k_\psi$ and hence $d_g^1 \in \mathfrak{L}_\psi$.

Next we show that d_g^1 is orthogonal to L_0. Let $f \in L_0$, then we have

$$\begin{aligned}\langle d_g^1, f\rangle &= \langle \widetilde{g} - (\psi(z) + \psi(\varphi(w)))g, f\rangle \\ &= \langle \widetilde{g}, f\rangle - \langle (\psi(z) + \psi(\varphi(w)))g, f\rangle \\ &= 0 - \langle g, (T^*_{\psi(z)} + T^*_{\psi(\varphi(w))})f\rangle \\ &= 0.\end{aligned} \tag{37}$$

This gives that $d_g^1 \in \mathfrak{L}_\psi \ominus L_0$.

Assume that for $n < l$, there are a family of functions $\{d_g^k\}_{k=1}^n \in \mathfrak{L}_\psi \ominus L_0$ such that

$$P_n'(\psi(z), \psi(\varphi(w)))g + \sum_{k=0}^{n-1} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n-k} \in M.$$

Let $G = P_n'(\psi(z), \psi(\varphi(w)))g + \Sigma_{k=0}^{n-1} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n-k}$. By Lemma 4.2 again, there is a unique function $\widetilde{G} \in M \ominus L_0$ such that

$$S^*_{\psi(z)}\widetilde{G} = T^*_{\psi(z)}\widetilde{G} = T^*_{\psi(\varphi(w))}\widetilde{G} = S^*_{\psi(\varphi(w))}\widetilde{G} = G.$$

Let $F = P_{n+1}'(\psi(z), \psi(\varphi(w)))g + \Sigma_{k=1}^n P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}$, since

$$T^*_{\psi(z)}[P_k'(\psi(z), \psi(\varphi(w)))f] = T^*_{\psi(\varphi(w))}[P_k'(\psi(z), \psi(\varphi(w)))f] = P_{k-1}'(\psi(z), \psi(\varphi(w)))f,$$

for each $f \in \mathfrak{L}_\psi$ and $k \geqslant 1$, then

$$T^*_{\psi(z)}F = T^*_{\psi(\varphi(w))}F = G.$$

Thus $T^*_{\psi(z)}(\widetilde{G} - F) = T^*_{\psi(\varphi(w))}(\widetilde{G} - F) = G - G = 0$. So letting $d_g^{n+1} = \widetilde{G} - F$, then $d_g^{n+1} \in ker T^*_{\psi(z)} \cap ker T^*_{\psi(\varphi(w))}$.

Noting $\widetilde{G}$ is orthogonal to L_0, we have that for each $f \in L_0$,

$$\begin{aligned}\langle d_g^{n+1}, f\rangle &= \langle \widetilde{G}, f\rangle - \langle F, f\rangle \\ &= -\langle P_{n+1}'(\psi(z), \psi(\varphi(w)))g, f\rangle - \sum_{k=1}^n \langle P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}, f\rangle \\ &= 0.\end{aligned} \tag{38}$$

to get that $d_g^{n+1} \in \mathfrak{L}_\psi \ominus L_0$. Hence

$$P_{n+1}'(\psi(z), \psi(\varphi(w)))g + \sum_{k=1}^n P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k} + d_g^{n+1} = \widetilde{G} \in M.$$

This gives a family of function $\{d_g^k\} \in \mathfrak{L}_\psi \ominus L_0$, satisfying property (i).

Lastly to finish the proof we need only to show that property (ii) holds. Since

$$\begin{aligned}2S_{\psi(z)}g &= P_{N_\varphi}(P_1'(\psi(z), \psi(\varphi(w)))g) \\ &= P_{N_\varphi}(P_1'(\psi(z), \psi(\varphi(w))g + d_g^1) - P_{N_\psi}d_g^1 \\ &= P_1'(\psi(z), \psi(\varphi(w)))g + d_g^1 - P_{N_\psi}d_g^1.\end{aligned} \tag{39}$$

we have $P_{N_\psi} d_g^1 = P_1'(\psi(z), \psi(\varphi(w)))g + d_g^1 - 2S_{\psi(z)} g \in M$.

Noting that $(d_g^1 - P_{N_\psi} d_g^1) \in N_\varphi^\perp$ and $[z - \varphi(w)]$ is an invariant subspace for analytic Toeplitz operators, we have that

$$[P_{l-1}'(\psi(z), \psi(\varphi(w)))(d_g^1 - P_{N_\psi} d_g^1)] \in N_\varphi^\perp,$$

and so $P_{N_\varphi}[P_{l-1}'(\psi(z), \psi(\varphi(w)))(d_g^1 - P_{N_\psi} d_g^1)] = 0$. Then

$$\begin{aligned} P_{N_\varphi}[P_{l-1}'(\psi(z), \psi(\varphi(w))d_g^1)] &= P_{N_\varphi}(P_{l-1}'(\psi(z), \psi(\varphi(w)))P_{N_\varphi} d_g^1)] \\ &= lS_{\psi(z)}^{l-1} P_{N_\psi} d_g^1 \in M. \end{aligned} \tag{40}$$

Assume that $P_{N_\varphi}[P_l'(\psi(z), \psi(\varphi(w)))d_g^k] \in M$ for $k \leqslant n$ and any $l \geqslant 0$. To finish the proof by induction we need only to show that

$$P_{N_\varphi}[P'(\psi(z), \psi(\varphi))d_g^{n+1}] \in M,$$

for any $l \geqslant 0$. Since

$$\begin{aligned} (n+2)S_{\psi(z)}^{n+1} g &= P_{N_\varphi}[P_{n+1}'(\psi(z), \psi(\varphi(w)))g] \\ &= P_{N_\varphi}[P_{n+1}'(\psi(z), \psi(\varphi(w)))g + \sum_{k=0}^{n} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}] \\ &\quad - P_{N_\varphi} d^{n+1-k} - P_{N_\varphi}[\sum_{k=1}^{n} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}] \end{aligned} \tag{41}$$

Thus $P_{N_\varphi} d_g^{n+1} = P_{N_\varphi}[P_{n+1}'(\psi(z), \psi(\varphi(w)))g + \sum_{k=0}^{n} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}] - (n+2)S_{\psi(z)}^{n+1} g - P_{N_\varphi}[\sum_{k=1}^{n} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}]$.

By property (i) we have

$$P_{N_\varphi}[P_{n+1}'(\psi(z), \psi(\varphi(w)))g + \sum_{k=0}^{n} P_k'(\psi(z), \psi(\varphi(w)))d_g^{n+1-k}] \in M.$$

The induction hypothesis gives that the last term is in M and the second term belongs to M, since $g \in M$ and M is a reducing subspace for $S_{\psi(z)}$. So $P_{N_\varphi} d_g^{n+1} \in M$. Therefore we conclude

$$\begin{aligned} P_{N_\varphi}[P_l'(\psi(z), \psi(\varphi(w)))d_g^{n+1}] &= P_{N_\varphi}[P_l'(\psi(z), \psi(\varphi(w)))P_{N_\varphi} d_g^{n+1}] \\ &= (l+1)S_{\psi(z)}^{l}(P_{N_\varphi} d_g^{n+1}) \in M. \end{aligned} \tag{42}$$

This completes the proof. □

In particular, N_φ is a reducing subspace of $S_{\psi(z)}$. By Theorem 4.3 we immediately get the following theorem.

PROPOSITION 4.4. *For a given $g \in L_0$, there are a unique family of functions $\{d_g^k\} \subset \mathfrak{L}_\psi \ominus L_0$ such that*

$$P_l'(\psi(z), \psi(\varphi(w)))g + \sum_{k=0}^{l-1} P_k'(\psi(z), \psi(\varphi(w)))d_g^{l-k} \in N_\varphi$$

for each $l \geqslant 1$.

The next theorem we will show that every nontrivial minimal reducing subspace Ω of $S_{\psi(z)}$ is orthogonal to M_0 if Ω is not in the form of Theorem 3.6.

THEOREM 4.5. *Suppose that Ω is a nontrivial minimal reducing subspace for $S_{\psi(z)}$. If Ω is not distinguished reducing subspace then Ω is a subspace of $M_0^{\perp}$.*

Proof. By Lemma 4.1, there is a function $g \in \Omega \cap L_0$ such that $g = f + h$ for some function $f = \sum_{k=1}^{m} \lambda_k e_k \in M_0 \cap L_0$ and $h \in M_0^{\perp} \cap L_0$, where λ_k, $k = 1, \ldots, m$, are constant. By proposition 3.3, $P_1'(\psi(z), \psi(\varphi(w)))g + d_g^1 \in \Omega$. Here d_g^1 is the function constructed in proposition 4.3. Let

$$G = S^*_{\psi(z)}[S_{\psi(z)g}] - \frac{1}{2}g \in \Omega.$$

Since $P_l'(\psi(z), \psi(\varphi(w)))f \in N_{\varphi}$, we obtain

$$S_{\psi(z)}f = \frac{P_1'(\psi(z), \psi(\varphi(w)))}{2}f.$$

Here

$$\begin{aligned}
G &= S^*_{\psi(z)}[S_{\psi(z)}(f+h)] - \frac{1}{2}(f+h) \\
&= \left(S^*_{\psi(z)}S_{\psi(z)}f - \frac{1}{2}f\right) + S^*_{\psi(z)}S_{\psi(z)}h - \frac{h}{2} \\
&= S^*_{\psi(z)}S_{\psi(z)}h - \frac{1}{2}h \\
&= \frac{1}{2}\{S^*_{\psi(z)}[P_{N_{\varphi}}(P_1'(\psi(z), \psi(\varphi(w)))h + d_h^1 - d_h^1)] - h\} \\
&= \frac{1}{2}\{S^*_{\psi(z)}[P_1'(\psi(z), \psi(\varphi(w)))h + d_h^1] - S^*_{\psi(z)}P_{N_{\varphi}}d_h^1)] - h\} \\
&= \frac{1}{2}\{h - S^*_{\psi(z)}P_{N_{\varphi}}d_h^1 - h\} \\
&= -\frac{1}{2}S^*_{\psi(z)}P_{N_{\varphi}}d_h^1.
\end{aligned} \tag{43}$$

The sixth equality holds because that $P_1'(\psi(z), \psi(\varphi(w)))h + d_h^1 \in N_{\varphi}$, the seventh equality follows from that $d_h^1 \in \mathfrak{L}_{\psi} \ominus L_0$. We claim that $G \neq 0$, if this is not true, we would have $\frac{1}{2}S^*_{\psi(z)}P_{N_{\varphi}}d_h^1 = 0$. This gives that $P_{N_{\varphi}}d_h^1 \in L_0$, and

$$\begin{aligned}
0 &= \langle P_{N_{\varphi}}d_h^1, d_h^1\rangle \\
&= \langle P_{N_{\varphi}}d_h^1, P_1'(\psi(z), \psi(\varphi(w)))h + d_h^1\rangle \\
&= \langle d_h^1, P_1'(\psi(z), \psi(\varphi(w)))h + d_h^1\rangle \\
&= \langle d_h^1, d_h^1\rangle \\
&= \| d_h^1 \|^2 .
\end{aligned} \tag{44}$$

This gives that $d_h^1 = 0$. Thus we have that $P_1'(\psi(z), \psi(\varphi(w)))h \in N_\varphi$. By theorem 3.2, $h \in M_0$. This contradicts that $h \in M_0^\perp$. By proposition 4.3, $P_{N_\varphi} d_h^1 \in M_0^\perp$ and so $G = -\frac{1}{2} S^*_{\psi(z)} P_{N_\varphi} d_h^1$.

This implies that $G \in \Omega \cap M_0^\perp$. We conclude that $\Omega \cap M_0^\perp = \Omega$, since Ω is minimal. Hence Ω is a subspace of $M_0^\perp$. $\square$

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(Received August 12, 2021)

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