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# ON AN INEQUALITY CONJECTURED BY BESENYEI AND PETZ

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*Abstract.* In this paper we investigate the inequality $\mathrm{Tr}(T\otimes I_2)\rho_{12}(\log_q\rho_{12}-\log_q\rho_1\otimes I_2-I_1\otimes\log_q\rho_2)\geqslant 0$, where $\rho_{12}$ is a density matrix and $0\leqslant T\in\mathbb{M}_m(\mathbb{C})$. This inequality was conjectured by Besenyei and Petz in 2013, where it was proved to hold for the density matrices in $\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$. Here we prove this inequality for the density matrices in $\mathbb{M}_m(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})$ using some elementary matrix methods. We also obtain some new inequalities related to the operators (matrices) in this inequality.

## 1. Introduction

Entropy is an important notion in both classical and quantum information theories. Strong subadditivity is a basic ingredient in quantum information theory which is used in topological entanglement theory, conformal field theory and in some other research areas [8].

Entropy was first introduced by Claude Shannon [12] as a concept in mathematics. The Shannon entropy of a discrete random variable $X$ with possible values $\{x_1,\dots,x_m\}$ and probability distribution $p(x)=P(X=x)$ is defined as

$$H(X)=-\sum_{i=1}^{m}p(x_i)\log p(x_i)$$

If $Y$ is another random variable with possible values $\{y_1,y_2,\dots,y_n\}$, then the joint entropy of the pair $(X,Y)$ is defined as

$$H(X,Y)=-\sum_{i=1}^{m}\sum_{j=1}^{n}p(x_i,y_j)\log p(x_i,y_j).$$

If $Z$ is another random variable with possible values in $\{z_1,z_2,\dots,z_r\}$, $H(X,Y,Z)$ is defined similarly. It is known that Shannon entropy satisfies the following [11]:

$$H(X,Y,Z)+H(Y)\leqslant H(X,Y)+H(Y,Z).$$

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This property is called the stong subadditivity inequality. It implies the following subadditivity:

$$H(X,Y) \leqslant H(X)+H(Y).$$

If we put $p_{ijk} = p(x_i,y_j,z_k)$, strong subadditivity is equivalent to

$$\sum_{i,j,k} p_{ijk}(\log p_{ijk} + \log p_{-j-} - \log p_{ij-} - \log p_{-jk}) \geqslant 0.$$

where

$$p_{ij-} = \sum_k p_{ijk},\ \ p_{-j-} = \sum_{i,k} p_{ijk},\ \ p_{-jk} = \sum_i p_{ijk}$$

are marjinal distributions.

The $q$-extention (or one-parameter extension) of the Shannon entropy is the Tsallis entropy. Define the $q$-logarithm function $\log_q : \mathbb{R}^+ \longrightarrow \mathbb{R}$ as

$$\log_q x = \frac{x^{q-1}-1}{q-1} \qquad (q \neq 1) \tag{1}$$

Then the Tsallis entropy [3, 6, 10] is defined as

$$H_q(X) = -\sum_{i=1}^{n} p(x_i)\log_q p(x_i) = \frac{1}{1-q}\sum_{i=1}^{n}(p(x_i)^q - p(x_i)).$$

It is known that Tsallis entropy is strongly subadditive for $q \geqslant 1$ [6, 10]. Hence it is also subadditive. In [3] a new type of inequality which can be considered as 'partial (strong) subadditivity' is introduced and proved for both Shannon and Tsallis entropies. The partial strong subadditivity of the Shannon entropy is defined as the following form:

For fixed $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant r$ we have

$$\sum_{i=1}^{m} p_{ijk}(\log p_{ijk} + \log p_{-j-} - \log p_{ij-} - \log p_{-jk}) \geqslant 0.$$

In the same paper the quantum analogue of the partial subadditivity is also discussed.

Here we are intrested in the quantum versions of the above entropies. In the quantum world, instead of probability distributions one uses matrices(or states):

Let $\mathscr{H}$ be a finite dimensional Hilbert space and $0 \leqslant \rho \in B(\mathscr{H})$ be a state (or a density matrix, namely $0 \leqslant \rho \in \mathbb{M}_n(\mathbb{C}), \mathrm{Tr}\rho = 1$). Then the von Neumann entropy [3, 4, 7] is defined by

$$S(\rho) = -\mathrm{Tr}\rho\log\rho.$$

One should note that the composite systems are described by the tensor product of the corresponding Hilbert spaces and marginal distributions by the partial traces of density matrices, which is called reduced densities [11]. Von Neumann entropy is the quantum analogue of Shannon entropy. It is known that von Neumann entropy is strongly subadditive [9]. Hence it is subadditive.

Almost at the same time with the paper [3] was published, Kim proved an operator extension of the strong subadditivity of von Neumann entropy which is a kind of partial strong subadditivity [8]. In fact it is the operator version of the partial (strong) subadditivity of Shannon entropy. The importance of partial (strong) subadditivity is that it implies (strong) subadditivity.

A one-parameter extension of the von Neumann entropy is the (quantum) Tsallis entropy [3, 7, 10]. From now on when we say Tsallis entropy we mean the quantum Tsallis entropy. It is defined by the formula

$$S_q(\rho) = -\operatorname{Tr}\rho \log_q(\rho) \qquad (q > 1)$$

The von Neumann entropy is the limit of the Tsallis entropy as $q \to 1$ [7]. It is known that the Tsallis entropy is subadditive [1], but not strongly subadditive [10]. The general picture is as follows:

$$\begin{array}{ccc} PSSA & \Longrightarrow & SSA \\ \Downarrow & & \Downarrow \\ PSA & \Longrightarrow & SA \end{array}$$

with the abbreviations:

SSA: strong subaddivity
SA: subadditivty
PSSA: partial strong subadditivity,
PSA: partial subaddivity

The above narrative can be summarized in the following table:

| | SA | PSA | SSA | PSSA |
|---|---|---|---|---|
| Shannon | Yes | Yes | Yes | Yes |
| Tsallis | Yes | Yes | Yes | Yes |
| von Neumann | Yes | Yes | Yes | Partly |
| quantum Tsallis | Yes | ? | No | No |

In [3] the following inequality related to the partial subadditivity of the quantum Tsallis entropy is conjectured: If $\rho_{12}$ is a density operator in $\mathbb{B}(\mathscr{H}) \otimes \mathbb{B}(\mathscr{K})$ with reduced densities $\rho_1$ and $\rho_2$, one has

$$\operatorname{Tr}(T \otimes I_2)\rho_{12}(\log_q \rho_{12} - \log_q(\rho_1 \otimes I_2) - I_1 \otimes \log_q \rho_2) \geqslant 0$$

whenever $T \geqslant 0$ and $q > 1$.

This conjecture was proved in the same article for the following cases:

1. $\rho_{12} = \rho_1 \otimes \rho_2$ (The case of product states), and
2. The case $m = n = 2$ when $q = 2$. ($\dim(\mathscr{H}) = m$ and $\dim(\mathscr{K}) = n$)

In this article first we present a proof of this inequality for arbitrary $m$ and $n$ when $q=2$. This completes the table above when $q=2$. Then, motivated by this proof and some numerical examples we conjecture an operator inequality which can be considered to be a kind of partial subadditivity of Tsallis entropy. We also obtain some new results related to the operators in this inequality.

## 2. Main results

Let $\rho_{12}$ be a density matrix in the tensor product space $\mathbb{M}_m(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})$. Then the reduced densities $\rho_1\in\mathbb{M}_m(\mathbb{C})$ and $\rho_2\in\mathbb{M}_n(\mathbb{C})$ are defined by the equalities

$$\operatorname{Tr}(Y\otimes I_2)\rho_{12}=\operatorname{Tr}(Y\rho_1)\qquad \operatorname{Tr}(I_1\otimes X)\rho_{12}=\operatorname{Tr}(X\rho_2)$$

for $X\in\mathbb{M}_m(\mathbb{C})$, $Y\in\mathbb{M}_n(\mathbb{C})$. For $m=2,\ n=2$

$$\rho_{12}=\begin{bmatrix}A & B\\ B^* & C\end{bmatrix}\qquad (A,B,C\in\mathbb{M}_2(\mathbb{C}))$$

and

$$\rho_1=\operatorname{Tr}_2\rho_{12}=\begin{bmatrix}\operatorname{Tr}A & \operatorname{Tr}B\\ \operatorname{Tr}B^* & \operatorname{Tr}C\end{bmatrix},\qquad \rho_2=\operatorname{Tr}_1\rho_{12}=A+C$$

where $\rho_1,\rho_2\in\mathbb{M}_2(\mathbb{C})$ (see in [3]).
The subadditivity of the Tsallis entropy is

$$S_q(\rho_{12})\leqslant S_q(\rho_1)+S_q(\rho_2)$$

or, equivalently

$$\operatorname{Tr}\rho_{12}(\log_q\rho_{12}-\log_q\rho_1\otimes I_2-I_1\otimes\log_q\rho_2)\geqslant 0.$$

The following inequality was conjectured in [3]:

$$\operatorname{Tr}(T\otimes I_2)\,\rho_{12}(\log_q\rho_{12}-\log_q\rho_1\otimes I_2-I_1\otimes\log_q\rho_2)\geqslant 0 \tag{2}$$

for all $0\leqslant T\in\mathbb{M}_m(\mathbb{C})$. In the same paper there is a proof of this inequality for any product state and in the case $q=2,\ \rho_{12}\in\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$. Here we prove the inequality (2) for the density matrices in $\mathbb{M}_m(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})$ and $q=2$:

THEOREM 1. *Let $\rho_{12}\in\mathbb{M}_m(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})$ be a density matrix, $\rho_1\in\mathbb{M}_m(\mathbb{C}),\rho_2\in\mathbb{M}_n(\mathbb{C})$ be its reduced densities and $0\leqslant T\in\mathbb{M}_m(\mathbb{C})$. Then*

$$\operatorname{Tr}(T\otimes I_2)\,\rho_{12}(\log_2\rho_{12}-\log_2\rho_1\otimes I-I\otimes\log_2\rho_2)\geqslant 0.$$

*or equivalently*

$$\operatorname{Tr}T\rho_1+\operatorname{Tr}(T\otimes I_2)\rho_{12}^2-\operatorname{Tr}T\rho_1^2-\operatorname{Tr}(T\otimes\rho_2)\rho_{12}\geqslant 0. \tag{3}$$

*Proof.* We will prove the inequality (3). One can write the density matrix $\rho_{12}$ as a block matrix:

$$\rho_{12} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1m} \\ A_{12}^* & A_{22} & A_{23} & \dots & A_{2m} \\ A_{13}^* & A_{23}^* & A_{33} & \dots & A_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1m}^* & A_{2m}^* & A_{3m}^* & \dots & A_{mm} \end{bmatrix}$$

Then

$$\rho_1 = \begin{bmatrix} \mathrm{Tr} A_{11} & \mathrm{Tr} A_{12} & \mathrm{Tr} A_{13} & \dots & \mathrm{Tr} A_{1m} \\ \mathrm{Tr} A_{12}^* & \mathrm{Tr} A_{22} & \mathrm{Tr} A_{23} & \dots & \mathrm{Tr} A_{2m} \\ \mathrm{Tr} A_{13}^* & \mathrm{Tr} A_{23}^* & \mathrm{Tr} A_{33} & \dots & \mathrm{Tr} A_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr} A_{1m}^* & \mathrm{Tr} A_{2m}^* & \mathrm{Tr} A_{3m}^* & \dots & \mathrm{Tr} A_{mm} \end{bmatrix}, \qquad \rho_2 = A_{11} + A_{22} + \ldots + A_{mm}.$$

Since the inequality (3) is unitarily invariant, we may suppose that $T$ is a diagonal matrix with nonnegative diagonal entries $t_{11}, t_{22}, \ldots, t_{mm}$. After some calculations we obtain the following formulas:

$$\mathrm{Tr}\, T\rho_1 = \sum_{i=1}^{m} t_{ii}\, \mathrm{Tr} A_{ii} \tag{4}$$

$$\mathrm{Tr}(T \otimes I){\rho_{12}}^2 = \sum_{i=1}^{m} t_{ii}\, \mathrm{Tr}(A_{ii}^2) + \sum_{j>i} (t_{ii} + t_{jj})\, \mathrm{Tr} A_{ij}^* A_{ij} \tag{5}$$

$$\mathrm{Tr}\, T\rho_1^2 = \sum_{i=1}^{m} t_{ii} (\mathrm{Tr} A_{ii})^2 + \sum_{j>i} (t_{ii} + t_{jj}) |\, \mathrm{Tr} A_{ij}|^2 \tag{6}$$

$$\mathrm{Tr}(T \otimes \rho_2)\rho_{12} = \sum_{i=1}^{m} t_{ii}\, \mathrm{Tr}(A_{ii}^2) + \sum_{j>i} (t_{ii} + t_{jj})\, \mathrm{Tr} A_{ii} A_{jj} \tag{7}$$

where $i, j = 1, \ldots, m$. By using the formulas (4), (5), (6), (7); the left hand side of the inequality (3) becomes

$$\sum_{j>i} (t_{ii} + t_{jj}) \Big[ \mathrm{Tr} A_{ij}^* A_{ij} - |\, \mathrm{Tr} A_{ij}|^2 - \mathrm{Tr} A_{ii} A_{jj} \Big] + \sum_{i=1}^{m} t_{ii} \Big[ \mathrm{Tr} A_{ii} - (\mathrm{Tr} A_{ii})^2 \Big]$$

Here we will use the formula $\mathrm{Tr} A_{11} + \mathrm{Tr} A_{22} + \ldots + \mathrm{Tr} A_{mm} = 1$ and $\mathrm{Tr} A_{ii} - (\mathrm{Tr} A_{ii})^2 = \sum_{\substack{j=1 \\ i \neq j}}^{m} \mathrm{Tr} A_{ii}\, \mathrm{Tr} A_{jj}\ \forall i$. Then we substitute this formula into the sum

$$\sum_{i=1}^{m} t_{ii} \Big[ \mathrm{Tr} A_{ii} - (\mathrm{Tr} A_{ii})^2 \Big]$$

and we obtain

$$\sum_{i=1}^{m} t_{ii}\Big[\sum_{\substack{j=1\\ i\neq j}}^{m} \mathrm{Tr}A_{ii}\,\mathrm{Tr}A_{jj}\Big] = \sum_{j>i}(t_{ii}+t_{jj})\,\mathrm{Tr}A_{ii}\,\mathrm{Tr}A_{jj}$$

Then,

$$\mathrm{Tr}\,T\rho_1 + \mathrm{Tr}(T\otimes I_2)\rho_{12}^2 - \mathrm{Tr}\,T\rho_1^2 - \mathrm{Tr}(T\otimes\rho_2)\rho_{12}$$

is equal to

$$\sum_{j>i}(t_{ii}+t_{jj})\Big[\mathrm{Tr}A_{ij}^*A_{ij} - |\mathrm{Tr}A_{ij}|^2 - \mathrm{Tr}A_{ii}A_{jj} + \mathrm{Tr}A_{ii}\,\mathrm{Tr}A_{jj}\Big].$$

Now we will show that

$$(t_{ii}+t_{jj})\Big[\mathrm{Tr}A_{ij}^*A_{ij} - |\mathrm{Tr}A_{ij}|^2 - \mathrm{Tr}A_{ii}A_{jj} + \mathrm{Tr}A_{ii}\,\mathrm{Tr}A_{jj}\Big] \geqslant 0, \quad j>i.$$

Since $t_{11}, t_{22}, \ldots, t_{mm} \geqslant 0$ then $t_{ii}+t_{jj} \geqslant 0$ for any $i,j$. Also since $\rho_{12}\geqslant 0$, the principal submatrix $\begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij}^* & A_{jj}\end{bmatrix} \geqslant 0$. The proof of the trace inequality

$$\mathrm{Tr}A_{ij}^*A_{ij} - |\mathrm{Tr}A_{ij}|^2 - \mathrm{Tr}A_{ii}A_{jj} + \mathrm{Tr}A_{ii}\,\mathrm{Tr}A_{jj} \geqslant 0, \quad j>i$$

follows from the theorem in [2]. Hence,

$$\sum_{j>i}(t_{ii}+t_{jj})\Big[\mathrm{Tr}A_{ij}^*A_{ij} - |\mathrm{Tr}A_{ij}|^2 - \mathrm{Tr}A_{ii}A_{jj} + \mathrm{Tr}A_{ii}\,\mathrm{Tr}A_{jj}\Big] \geqslant 0$$

This completes the proof. □

COROLLARY 1. *In the above theorem putting* $T = |\psi\rangle\langle\psi|$, *where* $|\psi\rangle \in \mathbb{C}^m$ *is any vector, we obtain the following inequality*

$$\langle\psi|\,\mathrm{Tr}_2\,\rho_{12}(\log_2\rho_{12} - \log_2\rho_1\otimes I - I\otimes\log_2\rho_2)|\psi\rangle \geqslant 0$$

*which means that the operator*

$$\mathrm{Tr}_2\,\rho_{12}(\log_2\rho_{12} - \log_2\rho_1\otimes I - I\otimes\log_2\rho_2)$$

*is positive semidefinite on* $\mathbb{C}^m$.

This corollary shows that the partial trace of the operator $\rho_{12}(\log_2\rho_{12} - \log_2\rho_1\otimes I - I\otimes\log_2\rho_2)$ is positive semidefinite on $\mathbb{C}^m$. In fact this operator has some other properties not only for $q=2$ but also for $q\in(1,\infty)$. Hence in the rest of the paper we concentrate on the operator

$$\rho_{12}(\log_q\rho_{12} - \log_q\rho_1\otimes I - I\otimes\log_q\rho_2), \qquad (q>1) \tag{8}$$

which is equaivalent to

$$\frac{1}{q-1}[\rho_{12}^q - \rho_{12}(I_1\otimes\rho_2^{q-1}) - \rho_{12}(\rho_1^{q-1}\otimes I_2) + \rho_{12}] \tag{9}$$

by (1).

LEMMA 1. *Partial traces*

$$\operatorname{Tr}_2 \rho_{12}(\log_q \rho_{12} - \log_q \rho_1 \otimes I - I \otimes \log_q \rho_2) \in \mathbb{M}_m(\mathbb{C}) \tag{10}$$

$$\operatorname{Tr}_1 \rho_{12}(\log_q \rho_{12} - \log_q \rho_1 \otimes I - I \otimes \log_q \rho_2) \in \mathbb{M}_n(\mathbb{C}) \tag{11}$$

*of the operator (8) are Hermitian.*

We need the following proposition to prove the lemma:

PROPOSITION 1. *Let $\rho_{12} \in \mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})$ be a density matrix, $\rho_1 \in \mathbb{M}_m(\mathbb{C})$, $\rho_2 \in \mathbb{M}_n(\mathbb{C})$ be its reduced densities and $q \in (1,\infty)$. Then the operators $\operatorname{Tr}_2 \rho_{12}(I_1 \otimes \rho_2^{q-1})$, $\operatorname{Tr}_1 \rho_{12}(\rho_1^{q-1} \otimes I_2)$ are positive semidefinite.*

*Proof.* We will show that

$$\langle x, \operatorname{Tr}_2 \rho_{12}(I_1 \otimes \rho_2^{q-1})x\rangle \geqslant 0, \qquad \forall x \in \mathbb{C}^m$$

Let $\{f_j\}_{j=1}^n$ be an orthonormal basis of $\mathbb{C}^n$. By the definiton of the partial trace [5] we have

$$\langle x, \operatorname{Tr}_2 \rho_{12}(I_1 \otimes \rho_2^{q-1})x\rangle = \sum_{j=1}^n \langle (x \otimes f_j), \rho_{12}(I_1 \otimes \rho_2^{q-1})(x \otimes f_j)\rangle$$

This definition is independent of the choice of the orthonormal basis. Thus we may assume that the basis $\{f_j\}_{j=1}^n$ consists of the eigenvectors of the density operator $\rho_2$.

Writing $\rho_2 = \sum_j \lambda_j |f_j\rangle\langle f_j|$ we have

$$\begin{aligned}
\langle x, \operatorname{Tr}_2 \rho_{12}(I_1 \otimes \rho_2^{q-1})x\rangle &= \sum_{j=1}^n \langle (x \otimes f_j), \rho_{12}(I_1 \otimes \rho_2^{q-1})(x \otimes f_j)\rangle \\
&= \sum_{j=1}^n \langle (x \otimes f_j), \rho_{12}(x \otimes \rho_2^{q-1} f_j)\rangle \\
&= \sum_{j=1}^n \lambda_j^{q-1} \langle (x \otimes f_j), \rho_{12}(x \otimes f_j)\rangle
\end{aligned}$$

where $0 \leqslant \lambda_j \in sp(\rho_2)\ \forall j$. And $\langle (x \otimes f_j), \rho_{12}(x \otimes f_j)\rangle \geqslant 0$ for all $j$ since $\rho_{12}$ is positive. Hence

$$\langle x, \operatorname{Tr}_2 \rho_{12}(I_1 \otimes \rho_2^{q-1})x\rangle \geqslant 0, \qquad \forall x \in \mathbb{C}^m$$

The positivity of the operator $\operatorname{Tr}_1 \rho_{12}(\rho_1^{q-1} \otimes I_2)$ can be proved in an analoguos way. □

*Proof (of Lemma 1).* In order to show the assertion, we will prove that the partial traces of the operator (9) are Hermitian. Hence we will show that the operators

$$\operatorname{Tr}_1(\rho_{12}^q - \rho_{12}(I_1 \otimes \rho_2^{q-1}) - \rho_{12}(\rho_1^{q-1} \otimes I_2) + \rho_{12}) \tag{12}$$

$$\mathrm{Tr}_2(\rho_{12}^q - \rho_{12}(I_1 \otimes \rho_2^{q-1}) - \rho_{12}(\rho_1^{q-1} \otimes I_2) + \rho_{12}) \tag{13}$$

are Hermitian.
We will only prove that the operator (13) is Hermitian, the Hermitianness of (12) can be proved in an anologous way. By the linearity of partial trace, the operator (13) is equal to

$$\mathrm{Tr}_2\,\rho_{12}^q - \mathrm{Tr}_2\,\rho_{12}(I_1 \otimes \rho_2^{q-1}) - \rho_1^q + \rho_1 \tag{14}$$

In (14) the operators $\mathrm{Tr}_2\,\rho_{12}^q$, $\rho_1^q$ and $\rho_1$ are all Hermitian. So, we only have to show that $\mathrm{Tr}_2\,\rho_{12}(I_1 \otimes \rho_2^{q-1})$ is Hermitian. But in the above proposition we proved that $\mathrm{Tr}_2\,\rho_{12}(I_1 \otimes \rho_2^{q-1}) \geqslant 0$. Hence the operator (13) is Hermitian. $\square$

The most important problem that remains is to understand the case of $q > 1$ with $q \neq 2$. It seems that some new ideas are needed for a general solution. Having this in mind we performed some numerical computations in Wolfram Mathematica 12. These examples suggest that the operators $\mathrm{Tr}_2\,\rho_{12}(\log_q \rho_{12} - \log_q \rho_1 \otimes I - I \otimes \log_q \rho_2)$ and $\mathrm{Tr}_1\,\rho_{12}(\log_q \rho_{12} - \log_q \rho_1 \otimes I - I \otimes \log_q \rho_2)$ are not only Hermitian but also positive. If it is true, this would imply the partial subadditivity of the Tsallis entropy. Hence our future work will be to investigate this claim.

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