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DETERMINANTAL POLYNOMIALS OF A WEIGHTED SHIFT MATRIX WITH PALINDROMIC GEOMETRIC WEIGHTS

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Dedicated to Professor Vandanjav Adiyasuren on the occasion of his retirement from National University of Mongolia.

(*Communicated by I. M. Spitkovsky*)

Abstract. We find an explicit expression of the determinantal polynomials of a weighted shift matrix with palindromic geometric weights.

1. Introduction

Let A be an $n\times n$ complex matrix. The numerical radius $w(A)$ of A is the maximum of the modulus of its numerical range defined by

$$W(A)=\{\langle Ax,x\rangle : x\in\mathbb{C}^n, \quad \|x\|=1\}.$$

The numerical range of A is a nonempty compact convex subset of the complex plane $\mathbb{C}$, which contains all the eigenvalues of A and therefore its convex hull [9]. For references on the theory of numerical range, see, for instance, [8, 10].

We consider a weighted shift matrix with weights $a_1,a_2,\ldots,a_{n-1}$ is an $n\times n$ matrix of the following form

$$S=S(a_1,a_2,\ldots,a_{n-1})=\begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & 0\\ 0 & 0 & a_2 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & & \ddots & \ddots & 0\\ 0 & 0 & \cdots & \cdots & \ddots & a_{n-1}\\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

The numerical range $W(S(a_1,a_2,\ldots,a_{n-1}))$ of this weighted shift matrix is a circular disc with centered at the origin and the radius

$$w(S(a_1,a_2,\ldots,a_{n-1}))=\max\{z\in\mathbb{R} : \det(zI_n-\operatorname{Re}\,(S(a_1,\ldots,a_{n-1})))=0\}.$$

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A general formula of the characteristic polynomial

$$P_n(z:a_1,\dots,a_{n-1}) = \det(zI_n - \text{ Re }(S(a_1,\dots,a_{n-1})))$$

given in [13, Lemma 1] in terms of circularly symmetric functions of $|a_1|^2, |a_2|^2, \dots, |a_{n-1}|^2$. Namely,

$$P_n(z:a_1,\dots,a_{n-1}) = z^n + \sum_{1\leqslant k\leqslant \lfloor \frac{n}{2} \rfloor} \left(-\frac{1}{4}\right)^k S_k(a_1,\dots,a_{n-1}) z^{n-2k},$$

where the circularly symmetric functions

$$S_k = \sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant n-1} a_{i_1}^2 a_{i_2}^2 \cdots a_{i_k}^2, \tag{1}$$

the sum is taken over

$$1 \leqslant i_1 < i_2 < \ldots < i_k < n-1, \quad i_2 - i_1 \geqslant 2, i_3 - i_2 \geqslant 2, \dots, i_k - i_{k-1} \geqslant 2.$$

The subject of weighted shift matrices has attracted many authors, and they produce number of interesting papers [5, 7, 12, 13, 17]. For instance, in the case $a_1 = a_2 = \ldots = a_{n-1} = 1$, we have $w(S(a_1,a_2,\dots,a_{n-1})) = \cos(\pi/(n+1))$. The numerical radius of the weighted shift matrices $S(1,\dots,1,a,1,\dots,1)$ and $S(1,\dots,a,a,1,\dots,1)$ were computed respectively in [5] and [15]. From (1) it is clear that the polynomial $P_n(z: a_1,\dots,a_{n-1})$ satisfies the equation

$$P_n(z:a_1,\dots,a_{n-1}) = P_n(z:|a_1|,\dots,|a_{n-1}|).$$

Hence we may assume the weights a_j are non-negative real numbers. We call the characteristic polynomial $P_n(z:a_1,\dots,a_{n-1})$ the *determinantal polynomial* of the weighted shift matrix $S(a_1,\dots,a_{n-1})$. A weighted shift matrix $S(a_1,a_2,\dots,a_{n-1})$ is called a *palindromic geometric weighted shift matrix* if the following hold:

(i) the weights $a_1,a_2,\dots,a_{n-1}$ satisfy the palindromic property: $a_j = a_{n-j}$ for $j = 1,2,\dots n-1$, that is, $a_1 = a_{n-1}, a_2 = a_{n-2}, \dots$ and

(ii) the sequence $(a_1,a_2,\dots,a_m)$ is geometric, in the sense that $a_1 = 1, a_2 = r, a_3 = r^2, \dots, a_m = r^{m-1}$ for odd $n = 2m+1$ (for even $n = 2m$).

In a previous joint work, [14] the author of the current paper and Adiyasuren provided the explicit expression of the determinantal polynomial

$$P_n(z:1,r,r^2,\dots r^{n-2})$$

for the weighted shift matrix with geometric weights $S(1,r,r^2,\dots r^{n-2})$. The expression is given as the following

$$\begin{aligned} P_n(\zeta,r) &= \det\left(\zeta I_n - 2\text{Re}\left(S\left(1,r,\dots,r^{n-2}\right)\right)\right) \\ &= \zeta^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j (r^2)^{j(j-1)} \prod_{i=1}^{j} \frac{1-(r^2)^{n-2j+i}}{1-(r^2)^i} \zeta^{n-2j}, \end{aligned} \tag{2}$$

for $r \neq 1$. The determinantal polynomial $P_n(\zeta : 1, r, \ldots, r^{n-2})$ is abbreviated to $P_n(\zeta, r)$. We take $P_2(\zeta, r) = \zeta^2 - 1, P_1(\zeta, r) = \zeta, P_0(\zeta, r) = 1$. The polynomial in (2) satisfy the following recurrence relation

$$P_n(\zeta, r) = \zeta P_{n-1}(\zeta, r) - (r^2)^{n-2} P_{n-2}(\zeta, r). \tag{3}$$

In this paper, we obtain the explicit expression of the determinantal polynomial $Q_n(z, r)$ of the weighted shift matrix $S(1, r, r^2, \ldots, r^2, r, 1)$ with palindromic geometric weights in Theorem 1.2 and Corollary 1.3 in terms of the polynomial $P_n(\zeta, r)$ in (2).

PROPOSITION 1.1. *Let r be a real number and n be a positive integer. Then*

$$r^{n^2-2n} P_n\left(\frac{z}{r^{n-2}}, \frac{1}{r}\right) = P_n(z, r). \tag{4}$$

Proof. By a straightforward computation, we have

$$\begin{aligned}
& r^{n^2-2n} P_n\left(\frac{z}{r^{n-2}}, \frac{1}{r}\right) \\
&= r^{n^2-2n}\left(\left(\frac{z}{r^{n-2}}\right)^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \left(\left(\frac{1}{r}\right)^2\right)^{j(j-1)} \prod_{i=1}^{j} \frac{1-\left(\left(\frac{1}{r}\right)^2\right)^{n-2j+i}}{1-\left(\left(\frac{1}{r}\right)^2\right)^i} \left(\frac{z}{r^{n-2}}\right)^{n-2j}\right) \\
&= z^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j (r)^{-2j(j-1)} \prod_{i=1}^{j} \frac{1-\left(\left(\frac{1}{r}\right)^2\right)^{n-2j+i}}{1-\left(\left(\frac{1}{r}\right)^2\right)^i} z^{n-2j} \cdot r^{n^2-2n-(n-2)(n-2j)} \\
&= z^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j (r)^{-2j(j-1)} \frac{(-1)^j}{(-1)^j} \prod_{i=1}^{j} \frac{1-\left(r^2\right)^{n-2j+i}}{r^{2(n-2j+i)}} \frac{r^{2i}}{1-\left(r^2\right)^i} z^{n-2j} \cdot r^{2nj-4j} \\
&= z^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(r)^{2nj-2j^2-2j}}{r^{2(n-2j)j}} \prod_{i=1}^{j} \frac{1-\left(r^2\right)^{n-2j+i}}{1-\left(r^2\right)^i} z^{n-2j} \\
&= z^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j (r)^{2j^2-2j} \prod_{i=1}^{j} \frac{1-\left(r^2\right)^{n-2j+i}}{1-\left(r^2\right)^i} z^{n-2j} \\
&= P_n(z, r).
\end{aligned}$$

The proof is completed. $\square$

The main result is an explicit expression of the determinantal polynomials of the palindromic geometric weighted shift matrix. If we replace the shift matrix

$$S\left(1, r, r^2, \ldots, r^{m-2}, r^{m-1}, r^{m-2}, \ldots, r^2, r, 1\right)$$

by its scalar multiple

$$\begin{aligned}&\frac{1}{r^{m-1}}S\left(1,r,r^2,\ldots,r^{m-2},r^{m-1},r^{m-2},\ldots,r^2,r,1\right)\\&=S\left(\frac{1}{r^{m-1}},\frac{1}{r^{m-2}},\ldots,\frac{1}{r},1,\frac{1}{r},\frac{1}{r^{m-2}},\frac{1}{r^{m-1}}\right)\end{aligned}$$

for $r>1$, $n=2m+1$ is odd. So we could take its limit as $n\to\infty$. The limit S would be a weighted shift operator on $\ell^2(\mathbb{Z})$ and the operator S satisfy $\operatorname{tr}(|S|)<\infty$. We believe that the results of this paper helps to develop for the further study of related topics.

THEOREM 1.2. *Let m be a positive greater than 1. Then*

a) *If $n=2m$ is even with $S_n=S(1,r,r^2,\ldots,r^{m-2},r^{m-1},r^{m-2},\ldots,r^2,r,1)$, then the determinantal polynomial of S_n is given by*

$$\begin{aligned}Q_n(z,r)&=P_m(z,r)^2-r^{2m-2}P_{m-1}(z,r)^2\\&=\left(P_m(z,r)-r^{m-1}P_{m-1}(z,r)\right)\left(P_m(z,r)+r^{m-1}P_{m-1}(z,r)\right).\end{aligned}\tag{5}$$

b) *If $n=2m+1$ is odd with $S_n=S(1,r,r^2,\ldots,r^{m-1},r^{m-1},\ldots,r^2,r,1)$, then the determinantal polynomial of S_n is given by*

$$Q_n(z,r)=P_m(z,r)\left(P_{m+1}(z,r)-r^{2m-2}P_{m-1}(z,r)\right).\tag{6}$$

Proof. First of all, for $n=2m$ we expand the following $2m\times 2m$ determinant by $(m+1)$-th column

$$\det\left(zI_n-(S_n+S_n^*)\right)$$

$$=\begin{vmatrix}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots\cdots\cdots & 0\\
-1 & z & -r & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots\cdots\cdots & 0\\
0 & -r & z & -r^2 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots\cdots\cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots\\
\vdots & \vdots & \ddots & -r^{m-3} & z & -r^{m-2} & 0 & 0 & \cdots & \cdots & \cdots\cdots\cdots & 0\\
\vdots & \vdots & \ddots & 0 & -r^{m-2} & z & -r^{m-1} & 0 & 0 & \cdots & \cdots\cdots\cdots & 0\\
\vdots & \vdots & \ddots & 0 & 0 & -r^{m-1} & z & -r^{m-2} & 0 & 0 & \cdots\cdots\cdots & 0\\
\vdots & \vdots & \ddots & 0 & 0 & 0 & -r^{m-2} & z & -r^{m-3} & 0 & \cdots\cdots\cdots & 0\\
\vdots & \vdots & \ddots & 0 & 0 & 0 & 0 & -r^{m-3} & z & -r^{m-4} & 0\ \ 0\ \cdots & 0\\
\vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots\ \cdots\ \vdots & \vdots\\
\vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots\ \ddots\ \vdots & \vdots\\
0 & \vdots & \ddots & & & 0 & 0 & 0 & 0 & 0 & -r^2\ \ z\ \ -r & 0\\
0 & \vdots & \ddots & & & 0 & 0 & 0 & 0 & 0 & 0\ \ -r\ \ z & -1\\
0 & \vdots & \ddots & & & 0 & 0 & 0 & 0 & 0 & 0\ \ 0\ \ -1 & z
\end{vmatrix}$$

$$
= (-1)^{2(m+1)} z \cdot \left|\begin{array}{cccccc|ccccccc}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & -r & z & -r^2 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -r^{m-3} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & -r^{m-2} & z & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & \cdots & 0 & z & -r^{m-3} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \cdots & 0 & -r^{m-3} & z & -r^{m-4} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & 0 & -r^{m-4} & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & z
\end{array}\right| \tag{7}
$$

$$
+ (-r^{m-1})(-1)^{2m+1} \cdot \left|\begin{array}{cccccc|ccccccc}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & -r & z & -r^2 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -r^{m-3} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & -r^{m-1} & -r^{m-2} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
0 & \cdots & \cdots & \cdots & \cdots & 0 & z & -r^{m-3} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -r^{m-3} & z & -r^{m-4} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & z
\end{array}\right| \tag{8}
$$

$$
+ (-r^{m-2})(-1)^{2m+3} \cdot \left|\begin{array}{cccccc|ccccccc}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -r & z & -r^2 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -r^{m-3} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & -r^{m-2} & z & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 & 0 & -r^{m-1} & -r^{m-2} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & 0 & -r^{m-3} & z & -r^{m-4} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & -r^{m-4} & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & z
\end{array}\right|. \tag{9}
$$

The determinant (7) is a block diagonal, so which is equal to

$$(-1)^{2(m+1)} z P_m(z,r) r^{(m-3)(m-1)} P_{m-1}\left(\frac{z}{r^{m-3}}, \frac{1}{r}\right) \tag{10}$$

The (8) determinant is block upper triangular which is equal to

$$(-r^{m-1})(-1)^{2m+1} \begin{vmatrix} z & -1 & 0 & 0 & \cdots & 0 \\ -1 & z & -r & 0 & \cdots & 0 \\ 0 & -r & z & -r^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & -r^{m-3} & z & -r^{m-2} \\ 0 & 0 & \cdots & 0 & 0 & -r^{m-1} \end{vmatrix} \cdot \begin{vmatrix} z & -r^{m-3} & 0 & \cdots & \cdots\cdots & 0 \\ -r^{m-3} & z & -r^{m-4} & 0 & 0 \ldots & 0 \\ \cdots & \ddots & \ddots & \ddots & \ddots \cdots & \vdots \\ \cdots & \cdots & \ddots & \ddots & \ddots \cdots & \vdots \\ 0 & 0 & \cdots & -r^2 & z \;\; -r & 0 \\ 0 & 0 & \cdots & 0 & -r \;\; z & -1 \\ 0 & 0 & \cdots & 0 & 0 \;\; -1 & z \end{vmatrix}.$$

Expanding the last row of first determinant the last product is equal to

$$\begin{aligned} &(-r^{m-1})(-1)^{2m+1}(-1)^{m+m}(-r^{m-1}) P_{m-1}(z,r) r^{(m-3)(m-1)} P_{m-1}\left(\frac{z}{r^{m-3}}, \frac{1}{r}\right) \\ &= -r^{(m-1)^2} P_{m-1}(z,r) P_{m-1}\left(\frac{z}{r^{m-3}}, \frac{1}{r}\right). \end{aligned} \tag{11}$$

The (9) determinant is block lower triangular which is equal to

$$(-r^{m-2})(-1)^{2m+3} \cdot \begin{vmatrix} z & -1 & 0 & 0 & \cdots & 0 \\ -1 & z & -r & 0 & \cdots & 0 \\ 0 & -r & z & -r^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & -r^{m-3} & z & -r^{m-2} \\ 0 & 0 & \cdots & 0 & -r^{m-2} & z \end{vmatrix} \cdot \begin{vmatrix} -r^{m-2} & 0 & 0 & \cdots & \cdots\cdots & 0 \\ -r^{m-3} & z & -r^{m-4} & 0 & 0 \cdots & 0 \\ 0 & -r^{m-4} & z & -r^{m-5} & \cdots\cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \cdots & \vdots \\ 0 & 0 & \cdots & -r^2 & z \;\; -r & 0 \\ 0 & 0 & \cdots & 0 & -r \;\; z & -1 \\ 0 & 0 & \cdots & 0 & 0 \;\; -1 & z \end{vmatrix}.$$

Expanding the last row of first determinant the last product is equal to

$$\begin{aligned} &(-r^{m-2})(-1)^{2m+3} \cdot (-1)^{1+1}(-r^{m-2}) P_m(z,r) r^{(m-4)(m-2)} P_{m-2}\left(\frac{z}{r^{m-4}}, \frac{1}{r}\right) \\ &= -r^{(m-2)^2} P_m(z,r) P_{m-2}\left(\frac{z}{r^{m-4}}, \frac{1}{r}\right). \end{aligned} \tag{12}$$

Thus by (10), (11) and (12) we obtain the following

$$\begin{aligned} Q_n(z,r) = \; & r^{m^2-4m+3} z P_m(z,r) P_{m-1}\left(\frac{z}{r^{m-3}}, \frac{1}{r}\right) \\ & -r^{(m-1)^2} P_{m-1}(z,r) P_{m-1}\left(\frac{z}{r^{m-3}}, \frac{1}{r}\right) \\ & -r^{(m-2)^2} P_m(z,r) P_{m-2}\left(\frac{z}{r^{m-4}}, \frac{1}{r}\right). \end{aligned}$$

Applying the recurrence relation (3) and the identity (4) we can simplify the last equation further to

$$
\begin{aligned}
Q_n(z,r) &= zP_m(z,r)\left[r^{m^2-4m+3}P_{m-1}\left(\frac{z}{r^{m-3}},\frac{1}{r}\right)\right] \\
&\quad -r^{2m-2}P_{m-1}(z,r)\left[r^{(m-1)(m-3)}P_{m-1}\left(\frac{z}{r^{m-3}},\frac{1}{r}\right)\right] \\
&\quad -r^{2m-4}P_m(z,r)\left[r^{(m-2)(m-4)}P_{m-2}\left(\frac{z}{r^{m-4}},\frac{1}{r}\right)\right] \\
&= zP_m(z,r)\left[P_{m-1}(z,r)\right]-r^{2m-2}P_{m-1}(z,r)\left[P_{m-1}(z,r)\right] \\
&\quad -r^{2m-4}P_m(z,r)\left[P_{m-2}(z,r)\right] \\
&= zP_m(z,r)P_{m-1}(z,r)-r^{2m-2}\left[P_{m-1}(z,r)\right]^2-r^{2m-4}P_m(z,r)P_{m-2}(z,r) \\
&= P_m(z,r)\left(zP_{m-1}(z,r)-r^{2m-4}P_{m-2}\right)-r^{2m-2}\left(P_{m-1}(z,r)\right)^2 \\
&= \left(P_m(z,r)\right)^2-r^{2m-2}\left(P_{m-1}(z,r)\right)^2 \quad (\text{by } (3)).
\end{aligned}
$$

This proves (5).

Second, for $n=2m+1$ we expand the following $(2m+1)\times(2m+1)$ determinant by $(m+1)$-th column

$$
\det\left(zI_n-(S_n+S_n^*)\right)
$$

$$
=\begin{vmatrix}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -r & z & -r^2 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & -r^{m-3} & z & -r^{m-2} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & 0 & -r^{m-2} & z & -r^{m-1} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & 0 & 0 & -r^{m-1} & z & -r^{m-1} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & 0 & 0 & 0 & -r^{m-1} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & -r^{m-2} & z & -r^{m-3} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & z
\end{vmatrix}
$$

$$= (-1)^{2(m+1)} z \cdot \left|\begin{array}{cccccc|ccccccc}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -r & z & -r^2 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & -r^{m-3} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & -r^{m-2} & z & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
\hline
0 & 0 & \cdots & \cdots & \cdots & 0 & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & -r^{m-2} & z & -r^{m-3} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & 0 & -r^{m-3} & \ddots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & z
\end{array}\right| \tag{13}$$

$$+ (-1)^{2m+1}(-r^{m-1}) \cdot \left|\begin{array}{cccccc|ccccccc}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -r & z & -r^2 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -r^{m-3} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & -r^{m-1} & -r^{m-1} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -r^{m-2} & z & -r^{m-3} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & 0 & -r^{m-3} & \ddots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & z
\end{array}\right| \tag{14}$$

$$+ (-1)^{(2m+3)}(-r^{m-1}) \cdot \left|\begin{array}{cccccc|ccccccc}
z & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & z & -r & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -r & z & -r^2 & \cdots & 0 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & -r^{m-3} & z & -r^{m-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & -r^{m-2} & z & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & -r^{m-1} & -r^{m-1} & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -r^{m-2} & z & -r^{m-3} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & 0 & -r^{m-3} & \ddots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & -r^2 & z & -r & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & -r & z & -1 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & z
\end{array}\right|. \tag{15}$$

The determinant (13) is a block diagonal, so which is equal to

$$(-1)^{2(m+1)} z P_m(z,r)(r^{m-2})^m P_m\left(\frac{z}{r^{m-2}}, \frac{1}{r}\right). \tag{16}$$

The determinant (14) is a upper block diagonal, so which is equal to

$$(-1)^{(m+1)+m}(-r^{m-1})(-1)^{m+m}(-r^{m-1})P_{m-1}(z,r)(r^{m-2})^m P_m\left(\frac{z}{r^{m-2}},\frac{1}{r}\right)$$

$$= -r^{m^2-2}P_{m-1}(z,r)P_m\left(\frac{z}{r^{m-2}},\frac{1}{r}\right). \tag{17}$$

The determinant (15) is a lower block diagonal, so which is equal to

$$(-1)^{2m+3}P_m(z,r)(-r^{m-1})(-1)^{1+1}(-r^{m-1})(r^{m-3})^{m-1}P_{m-1}\left(\frac{z}{r^{m-3}},\frac{1}{r}\right)$$

$$= -r^{(m-1)^2}P_m(z,r)P_{m-1}\left(\frac{z}{r^{m-3}},\frac{1}{r}\right). \tag{18}$$

Combining (16), (17) and (18) we obtain

$$\begin{aligned} Q_n(z,r) &= r^{m^2-2m}zP_m(z,r)P_m\left(\frac{z}{r^{m-2}},\frac{1}{r}\right) \\ &\quad -r^{m^2-2}P_{m-1}(z,r)P_m\left(\frac{z}{r^{m-2}},\frac{1}{r}\right) - r^{(m-1)^2}P_m(z,r)P_{m-1}\left(\frac{z}{r^{m-3}},\frac{1}{r}\right). \end{aligned}$$

Further, by the recurrence relation (3) and the identity (4) we can simplify the last equation as follows

$$\begin{aligned} Q_n(z,r) &= zP_m(z,r)\left[r^{m^2-2m}P_m\left(\frac{z}{r^{m-2}},\frac{1}{r}\right)\right] - r^{2m-2}P_{m-1}(z,r)\left[r^{m^2-2m}P_m\left(\frac{z}{r^{m-2}},\frac{1}{r}\right)\right] \\ &\quad -r^{2m-2}P_m(z,r)\left[r^{(m-1)(m-3)}P_{m-1}\left(\frac{z}{r^{m-3}},\frac{1}{r}\right)\right] \\ &= zP_m(z,r)\left[P_m(z,r)\right] - r^{2m-2}P_{m-1}(z,r)\left[P_m(z,r)\right] - r^{2m-2}P_m(z,r)\left[P_{m-1}(z,r)\right] \\ &= P_m(z,r)\left(zP_m(z,r) - 2r^{2m-2}P_{m-1}(z,r)\right) \\ &= P_m(z,r)\left(P_{m+1}(z,r) - r^{2m-2}P_{m-1}(z,r)\right) \quad (\text{by } (3)). \end{aligned}$$

This completes the proof. □

In a consequence of Theorem 1.2 we have the following result.

COROLLARY 1.3. *Let $n = 2m+1$. If λ be the largest positive root of the polynomial $P_{m+1}(z,r) - (r^2)^{m-1}P_{m-1}(z,r)$, then*

$$w\left(S(1,r,r^2,\ldots,r^{m-1},r^{m-1},\ldots,r^2,r,1)\right) = \lambda.$$

Proof. Recall the fact that the numerical radius of a weighted shift matrix S is the maximum of modulus the determinantal polynomial $\det\left(zI_n - (S+S^*)/2\right)$. It is clear that the matrix $S(1,r,r^2,\ldots,r^{m-2})$ is the compression of the matrix $S(1,r,r^2,\ldots,r^{m-1}, r^{m-1},\ldots,r^2,r,1)$. So we must have that

$$w(S(1,r,r^2,\ldots,r^{m-2})) < w(S(1,r,r^2,\ldots,r^{m-1},r^{m-1},\ldots,r^2,r,1)).$$

Combining this with the formula (6) of Theorem 1.2 we conclude that the largest positive root of the polynomial $P_m(z,r)$ is less than the largest positive root of the polynomial $P_{m+1}(z,r)-(r^2)^{m-1}P_{m-1}(z,r)$. Now the assertion is immediate from the assumption. $\square$

To illustrate Corollary 1.3, by Theorem 1.2 we find the numerical radius of $S(1,r,r,1)$, $S(1,r,r^2,r^2,r,1)$ and $S(1,r,r^2,r^3,r^3,r^2,r,1)$.

EXAMPLE 1.4. For $n=5$, we apply the formula (6) for $m=2$. Then

$$\begin{aligned} Q_5(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 & 0 \\ -1 & z & -r & 0 & 0 \\ 0 & -r & z & -r & 0 \\ 0 & 0 & -r & z & -1 \\ 0 & 0 & 0 & -1 & z \end{pmatrix} \\ &= P_2(z,r)\left(P_3(z,r)-r^2P_1(z,r)\right) \\ &= (z^2-1)(z^3-(1+2r^2)z). \end{aligned}$$

Hence the largest root of $Q_5(z,r)$ is $\dfrac{\sqrt{1+2r^2}}{2}$. On the other hand, Corollary 1.3 confirms that

$$w(S(1,r,r,1)) = \frac{\sqrt{1+2r^2}}{2}.$$

EXAMPLE 1.5. For $n=7$, we find the numerical radius of $S(1,r,r^2,r^2,r,1)$ as the following:

$$\begin{aligned} Q_7(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & z & -r & 0 & 0 & 0 & 0 \\ 0 & -r & z & -r^2 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & z & -r^2 & 0 & 0 \\ 0 & 0 & 0 & -r^2 & z & -r & 0 \\ 0 & 0 & 0 & 0 & -r & z & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & z \end{pmatrix} \\ &= P_3(z,r)\left[P_4(z,r)-r^4P_2(z,r)\right] \\ &= (z^3-(1+r^2)z)(z^4-(1+r^2+2r^4)z^2+2r^4). \end{aligned}$$

In the above product, the largest positive root of the first term $P_3(z,r)$ is $\sqrt{1+r^2}$. And the largest postive root of the second term $z^4-(1+r^2+2r^4)z^2+2r^4$ is

$$\sqrt{\frac{1+r^2+2r^4+\sqrt{(1+r^2+2r^4)^2-8r^4}}{2}}.$$

A direct computation gives that

$$\sqrt{1+r^2} < \sqrt{\frac{1+r^2+2r^4+\sqrt{(1+r^2+2r^4)^2-8r^4}}{2}}.$$

So the latter is the largest positive root of $Q_7(z,r)$. On the other hand, Corollary 1.3 confirms that

$$w\left(S(1,r,r^2,r^2,r,1)\right) = \frac{1}{2}\sqrt{\frac{1+r^2+2r^4+\sqrt{(1+r^2+2r^4)^2-8r^4}}{2}}.$$

EXAMPLE 1.6. For $n=9$, we find the numerical radius of $S(1,r,r^2,r^3,r^3,r^2,r,1)$ as the following:

$$\begin{aligned} Q_9(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & z & -r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r & z & -r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & z & -r^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^3 & z & -r^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r^3 & z & -r^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r^2 & z & -r & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -r & z & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & z \end{pmatrix} \\ &= P_4(z,r)\left[P_5(z,r) - r^6P_3(z,r)\right] \\ &= (z^4-(1+r^2+r^4)z^2+r^4)\left[z^5-(1+r^2+r^4+2r^6)z^3+(r^4+2r^6+2r^8)z\right] \end{aligned}$$

Hence by Corollary 1.3 we have

$$\begin{aligned} &w\left(S(1,r,r^2,r^3,r^3,r^2,r,1)\right) \\ &= \frac{1}{2}\sqrt{\frac{1+r^2+r^4+2r^6+\sqrt{1+r^2+r^4+2r^6-4(r^4+r^6+2r^8)^2}}{2}}. \end{aligned}$$

There is an another interesting consequence of Theorem 1.2. That is, for $n=2m$, the polynomial $Q_n(z,r)$ has two factors

$$Q_{n\,1}(z,r) = P_m(z,r) + r^{m-1}P_{m-1}(z,r) \text{ and } Q_{n\,2}(z,r) = P_m(z,r) - r^{m-1}P_{m-1}(z,r).$$

According to $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$, these two factors related as the following.

COROLLARY 1.7. *Let $n=2m$. If $n \equiv 0 \pmod 4$, then the polynomials $Q_{n\,1}(z,r)$ and $Q_{n\,2}(z,r)$ satisfies*

$$Q_{n\,1}(z,r) = Q_{n\,2}(-z,r). \tag{19}$$

If $n \equiv 2 \pmod 4$, then the polynomials $Q_{n\,1}(z,r)$ and $Q_{n\,2}(z,r)$ satisfies

$$Q_{n\,1}(z,r) = -Q_{n\,2}(-z,r). \tag{20}$$

Proof. The proof is immediate from the expression (2). □

We assume that $r > 0, r \neq 1$. It would be natural to ask the question:

PROBLEM. Which factor $Q_{n\,1}(z,r)$ and $Q_{n\,2}(z,r)$ has the greatest positive root of the polynomial $Q_n(z,r)$?

The author unable to give a answer to this problem. We are able to find the largest positive root of the determinantal polynomial of the palindromic geometric weighted shift matrix $S(1,r,1)$.

EXAMPLE 1.8. For $n = 4$, by the formula (5) we have,

$$\begin{aligned}
Q_4(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 \\ -1 & z & -r & 0 \\ 0 & -r & z & -1 \\ 0 & 0 & -1 & z \end{pmatrix} \\
&= (P_2(z,r))^2 - r^2 (P_1(z,r))^2 \\
&= (P_2(z,r) - rP_1(z,r))(P_2(z,r) + rP_1(z,r)) \\
&= \left(z^2 - rz - 1\right)\left(z^2 + rz - 1\right).
\end{aligned}$$

Hence the largest root of $Q_4(z,r)$ is $\dfrac{r+\sqrt{r^2+4}}{2}$. This implies that

$$w(S(1,r,1)) = \frac{r+\sqrt{r^2+4}}{4}.$$

REMARK 1.9. It is interesting to note that if we calculate directly the determinant $Q_4(z,r)$, then

$$\begin{aligned}
Q_4(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 \\ -1 & z & -r & 0 \\ 0 & -r & z & -1 \\ 0 & 0 & -1 & z \end{pmatrix} \\
&= z^4 - (r^2+2)z^2 + 1.
\end{aligned}$$

Then its the largest root is $\dfrac{1}{2}\sqrt{\dfrac{r^2+2+\sqrt{r^4+4r^2}}{2}}$. But we can observe that

$$\begin{aligned}
\frac{1}{2}\sqrt{\frac{r^2+2+\sqrt{r^4+4r^2}}{2}} &= \frac{1}{2}\sqrt{\frac{r^2+2r\sqrt{r^2+4}+(r^2+4)}{4}} \\
&= \frac{r+\sqrt{r^2+4}}{4}
\end{aligned}$$

To illustrate Corollary 1.7 and the proposed problem, by Theorem 1.2 we find the determinantal polynomials of $S(1,r,r^2,r,1)$ and $S(1,r,r^2,r^3,r^2,r,1)$.

EXAMPLE 1.10. For $n=6$, we find the determinantal polynomial of $S(1,r,r^2,r,1)$.

$$
\begin{aligned}
Q_6(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 & 0 & 0 \\ -1 & z & -r & 0 & 0 & 0 \\ 0 & -r & z & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 & z & -r & 0 \\ 0 & 0 & 0 & -r & z & -1 \\ 0 & 0 & 0 & 0 & -1 & z \end{pmatrix} \\
&= (P_3(z,r))^2 - r^4 (P_2(z,r))^2 \\
&= \left(P_3(z,r) - r^2 P_2(z,r)\right)\left(P_3(z,r) + r^2 P_2(z,r)\right) \\
&= \left(z^3 - r^2 z^2 - (1+r^2)z + r^2\right)\left(z^3 + r^2 z^2 - (1+r^2)z - r^2\right).
\end{aligned}
$$

EXAMPLE 1.11. For $n=8$, we find the determinantal polynomial of $S(1,r,r^2,r^3,r^2,r,1)$.

$$
\begin{aligned}
Q_8(z,r) &= \det\begin{pmatrix} z & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & z & -r & 0 & 0 & 0 & 0 & 0 \\ 0 & -r & z & -r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & z & -r^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^3 & z & -r^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r^2 & z & -r & 0 \\ 0 & 0 & 0 & 0 & 0 & -r & z & -1 \\ 0 & 0 & 0 & 0 & 0 & -0 & -1 & z \end{pmatrix} \\
&= (P_4(z,r))^2 - r^6 (P_3(z,r))^2 \\
&= (P_4(z,r) - r^3 P_3(z,r))(P_4(z,r) + r^3 (P_3(z,r)) \\
&= (z^4 - r^3 z^3 - (1+r^2+r^4)z^2 + r^3(1+r^2)z + r^4) \\
&\quad \cdot (z^4 + r^3 z^3 - (1+r^2+r^4)z^2 - r^3(1+r^2)z + r^4).
\end{aligned}
$$

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