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INEQUALITIES ON 2×2 BLOCK ACCRETIVE MATRICES

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Abstract. A 2×2 block matrix $\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ is accretive partial transpose (APT) if both $\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ and $\begin{pmatrix} A & Y^* \\ X & B \end{pmatrix}$ are accretive. This article presents some inequalities related to this class of matrices. One of our results refines a recent inequality in [Oper. Matrices, 15 (2021) 581–587].

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n\times n$ complex matrices. If $A\in\mathbb{M}_n$ is positive semidefinite (definite), then we write $A\geqslant 0$ $(A>0)$. For two Hermitian matrices A,B of the same size, $A\geqslant B$ $(A>B)$ means that $A-B\geqslant 0$ $(A-B>0)$. We say that $A\in\mathbb{M}_n$ is accretive if its real part $\mathrm{Re}A := \frac{A+A^*}{2}$ is positive definite, where A^* means the conjugate transpose of A. It is known that for every $A\geqslant 0$, there exists a unique $B\geqslant 0$ such that $B^2=A$ [5, Theorem 7.2.6] and we denote $A^{1/2}=B$. If all eigenvalues of A are real, then they are arranged nonincreasingly $\lambda_1(A)\geqslant\ldots\geqslant\lambda_n(A)$; the singular values of $A\in\mathbb{M}_n$, denoted by $s_j(X)$, are similarly arranged. Note that the singular values of A are the eigenvalues of $|A|$, where $|A|=(A^*A)^{\frac{1}{2}}$, i.e., $s_j(A)=\lambda_j(|A|)$, $j=1,\ldots,n$. The geometric mean of two positive definite matrices $A,C\in\mathbb{M}_n$ is defined by

$$A\sharp C := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}. \tag{1}$$

It is known that the notion of geometric mean could be extended to cover all positive semidefinite matrices; see [2, p. 107]. Recently, Drury [3] defined the geometric mean of two accretive matrices via the following formula

$$A\sharp C = \left(\frac{2}{\pi}\int_0^\infty (tA+t^{-1}C)^{-1}\frac{dt}{t}\right)^{-1},$$

and proved the relationship (1) is also valid for two accretive matrices $A,C\in\mathbb{M}_n$. The readers can consult [3] for more properties.

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For the 2×2 block matrix

$$M=\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\in \mathbb{M}_{2n}$$

with each block in $\mathbb{M}_n$, its partial transpose is defined by

$$M^\tau := \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}.$$

A matrix M is called partial positive transpose (PPT) if M and M^τ are positive semidefinite. We extend the notion to accretive matrices. If

$$M=\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix}\in \mathbb{M}_{2n}$$

and

$$M^\tau := \begin{pmatrix} A & Y^* \\ X & C \end{pmatrix}$$

are both accretive, then we say that M is APT (i.e., accretive partial transpose). Clearly, the class of APT matrices includes the class of PPT matrices. Lee [6] obtained a matrix inequality involving the off-diagonal block of a PPT matrix and the geometric mean of its diagonal blocks.

THEOREM 1.1. [6, Theorem 2.1] *Let* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\in \mathbb{M}_{2n}$ *be PPT. Then, for some unitary matrix* $V\in\mathbb{M}_n$,

$$|B|\leqslant \frac{A\sharp C+V^*(A\sharp C)V}{2}.$$

Recently, Fu et al.[4] presented a stronger result.

THEOREM 1.2. [4, Theorem 2.3] *Let* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\in \mathbb{M}_{2n}$ *be PPT. Then*

$$|B|\leqslant (A\sharp C)\sharp(V^*(A\sharp C)V),\ \ |B^*|\leqslant (A\sharp C)\sharp(V(A\sharp C)V^*),$$

where $V\in\mathbb{M}_n$ *is any unitary matrix such that* $B=V|B|$.

When $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\in \mathbb{M}_{2n}$ is positive semidefinite, Fu et al.[4, Theorem 2.2] also obtained that

$$|B|\leqslant (V^*AV)\sharp C,\ \ |B|\leqslant A\sharp(VCV^*). \tag{2}$$

Liu et al. [8] extended Theorem 1.1 to the case of APT matrices.

THEOREM 1.3. [8, Theorem 3.4] *Let* $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} \in \mathbb{M}_{2n}$ *be APT. Then, for some unitary matrix* $V \in \mathbb{M}_n$,

$$|X+Y| \leqslant \mathrm{Re}\,(A\sharp C + V^*(A\sharp C)V).$$

The main objective of this paper is to offer a refined result of Theorem 1.3.

In Section 2, we first present an inequality on 2×2 block accretive matrices. It will then be applied to obtain a refinement of Theorem 1.3. As a consequence, a singular values inequality is given. At last, we will give an alternative proof of the inequality $A\sharp A^* \geqslant \mathrm{Re}A$ when $A \in \mathbb{M}_n$ is an accretive matrix.

2. Main results

We first summarize some properties of the geometric mean of positive semidefinite matrices; see [2, Chapter 4].

PROPOSITION 2.1. *Let* $A, C \geqslant 0$. *Then*

(i) $A\sharp C = A^{1/2}UC^{1/2}$ *for some unitary matrix* U.

(ii) $(A\sharp C)^{-1} = A^{-1}\sharp C^{-1}$ *when* $A, C > 0$.

(iii) $X^*AX\sharp X^*CX \geqslant X^*(A\sharp C)X$ *with equality holds if* X *is nonsingular.*

(iv) $A\sharp C = \max\left\{X : X = X^*,\ \begin{pmatrix} A & X \\ X & C \end{pmatrix} \geqslant 0\right\}$.

For a general 2×2 block accretive matrix, we give the following two inequalities on its off-diagonal block and the geometric mean of its diagonal blocks.

THEOREM 2.2. *Let* $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} \in \mathbb{M}_{2n}$ *be accretive. Then*

$$\left|\frac{X+Y}{2}\right| \leqslant (U^*(\mathrm{Re}A)U)\sharp \mathrm{Re}C \qquad \textit{and} \qquad \left|\frac{X^*+Y^*}{2}\right| \leqslant \mathrm{Re}A\sharp(U(\mathrm{Re}C)U^*),$$

where $U \in \mathbb{M}_n$ *is any unitary matrix such that* $\frac{X+Y}{2} = U\left|\frac{X+Y}{2}\right|$.

Proof. Since $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix}$ is accretive, $\mathrm{Re}\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} = \begin{pmatrix} \mathrm{Re}A & \frac{X+Y}{2} \\ \frac{X^*+Y^*}{2} & \mathrm{Re}C \end{pmatrix}$ is positive definite. Hence by (2), we have

$$\left|\frac{X+Y}{2}\right| \leqslant (U^*(\mathrm{Re}A)U)\sharp \mathrm{Re}C,$$

and

$$\left|\frac{X^*+Y^*}{2}\right| \leqslant \mathrm{Re}A\sharp(U(\mathrm{Re}C)U^*). \quad \square$$

It is clear that U in Theorem 2.2 is the unitary matrix in the polar decomposition of $\frac{X+Y}{2}$.

Theorem 2.2 leads us to an improvement of Theorem 1.3.

THEOREM 2.3. *Let* $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} \in \mathbb{M}_{2n}$ *be APT. Then*

$$\left|\frac{X+Y}{2}\right| \leqslant (\mathrm{Re}A\sharp\mathrm{Re}C)\sharp(U^*(\mathrm{Re}A\sharp\mathrm{Re}C)U),$$

and

$$\left|\frac{X^*+Y^*}{2}\right| \leqslant (\mathrm{Re}A\sharp\mathrm{Re}C)\sharp(U(\mathrm{Re}A\sharp\mathrm{Re}C)U^*),$$

where $U \in \mathbb{M}_n$ *is any unitary matrix such that* $\frac{X+Y}{2} = U\left|\frac{X+Y}{2}\right|$.

Proof. Since $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix}$ and $\begin{pmatrix} A & Y^* \\ X & C \end{pmatrix}$ are accretive,

$$\mathrm{Re}\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} = \begin{pmatrix} \mathrm{Re}A & \frac{X+Y}{2} \\ \frac{X^*+Y^*}{2} & \mathrm{Re}C \end{pmatrix} \quad \text{and} \quad \mathrm{Re}\begin{pmatrix} A & Y^* \\ X & C \end{pmatrix} = \begin{pmatrix} \mathrm{Re}A & \frac{X^*+Y^*}{2} \\ \frac{X+Y}{2} & \mathrm{Re}C \end{pmatrix}$$

are positive definite. This means that $\mathrm{Re}\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix}$ is PPT.

By Theorem 1.2, we have

$$\left|\frac{X+Y}{2}\right| \leqslant (\mathrm{Re}A\sharp\mathrm{Re}C)\sharp(U^*(\mathrm{Re}A\sharp\mathrm{Re}C)U),$$

and

$$\left|\frac{X^*+Y^*}{2}\right| \leqslant (\mathrm{Re}A\sharp\mathrm{Re}C)\sharp(U(\mathrm{Re}A\sharp\mathrm{Re}C)U^*). \quad \square$$

REMARK 1. It is apparent that if $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix}$ is PPT (i.e., $X = Y$), Theorem 2.3 becomes Theorem 1.2.

COROLLARY 2.4. *Let* $\begin{pmatrix} A & X \\ Y^* & C \end{pmatrix} \in \mathbb{M}_{2n}$ *be APT. Then*

$$\prod_{j=1}^{k} s_j\left(\frac{X+Y}{2}\right) \leqslant \prod_{j=1}^{k} s_j(A\sharp C), \;\; k = 1, \ldots, n.$$

Proof. By Theorem 2.3 and Proposition 2.1 (i), it is easy to obtain that

$$\begin{aligned}\prod_{j=1}^{k} s_j\left(\frac{X+Y}{2}\right) &\leqslant \prod_{j=1}^{k} s_j((\mathrm{Re}A\sharp\mathrm{Re}C)\sharp(U^*(\mathrm{Re}A\sharp\mathrm{Re}C)U))\\ &\leqslant \prod_{j=1}^{k} s_j((\mathrm{Re}A\sharp\mathrm{Re}C)^{\frac{1}{2}}W(U^*(\mathrm{Re}A\sharp\mathrm{Re}C)U)^{\frac{1}{2}}),\end{aligned}$$

where W is any unitary matrix such that

$$(\mathrm{Re}A\sharp\mathrm{Re}C)\sharp(U^*(\mathrm{Re}A\sharp\mathrm{Re}C)U) = (\mathrm{Re}A\sharp\mathrm{Re}C)^{\frac{1}{2}}W(U^*(\mathrm{Re}A\sharp\mathrm{Re}C)U)^{\frac{1}{2}}.$$

Applying Horn inequality [9, p. 80] here, we have

$$\begin{aligned}\prod_{j=1}^{k} s_j\left(\frac{X+Y}{2}\right) &\leqslant \prod_{j=1}^{k} s_j((\mathrm{Re}A\sharp\mathrm{Re}C)^{\frac{1}{2}})s_j((\mathrm{Re}A\sharp\mathrm{Re}C)^{\frac{1}{2}})\\ &= \prod_{j=1}^{k} s_j((\mathrm{Re}A\sharp\mathrm{Re}C)).\end{aligned}$$

The result follows from inequality $\mathrm{Re}A\sharp\mathrm{Re}C \leqslant \mathrm{Re}(A\sharp C)$ [7, Theorem 1.1] and the Fan-Hoffman inequality [1, p. 73]. □

Note that Corollary 2.4 is first given by Liu et al. [8, Theorem 2.1].

Next, we give an alternative proof of the inequality due to Liu et al. [8].

THEOREM 2.5. [8, Proposition 4.1] *If* $A \in \mathbb{M}_n$ *is accretive, then* $A\sharp A^* \geqslant \mathrm{Re}A$.

Proof. It is clear that $A\sharp A^*$ is Hermitian and accretive. Thus, $A\sharp A^*$ is positive definite.

Using Proposition 2.1 (ii) and (iii),

$$A\sharp A^* - A^*(A\sharp A^*)^{-1}A = A\sharp A^* - (A^*A^{-1}A)\sharp(A^*(A^*)^{-1}A) = 0.$$

So $M = \begin{pmatrix} A\sharp A^* & A \\ A^* & A\sharp A^* \end{pmatrix}$ is positive semidefinite. Similarly, $M^\tau = \begin{pmatrix} A\sharp A^* & A^* \\ A & A\sharp A^* \end{pmatrix}$ is also positive semidefinite. This means that M is PPT. Therefore,

$$\frac{M+M^\tau}{2} = \begin{pmatrix} A\sharp A^* & \mathrm{Re}A \\ \mathrm{Re}A & A\sharp A^* \end{pmatrix} \geqslant 0.$$

By Proposition 2.1 (iv), $\mathrm{Re}A \leqslant A\sharp A^*$. □

REMARK 2. We give a concise proof of Theorem 2.5 without using inner product, which is different from that in [8]. It is more concise.

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