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GENERALIZING THE ANDO–HIAI INEQUALITY FOR SECTORIAL MATRICES

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Abstract. In this paper, we extend a remarkable norm inequality of Ando and Hiai in 1994 about comparing the power of geometric mean and the geometric mean of powers of two positive semidefinite matrices to the case of sectorial matrices. To this end, we develop several new matrix inequalities that compare the real part of sectorial matrices.

1. Introduction

Comparing norms of matrices is a fundamental problem in matrix analysis. Recent studies about the applications of matrix norms in various scenarios can be found in, for example, [1, 20, 21, 22]. We denote by $\mathbf{M_n}$ the set of all complex matrices of order n. If $A \in \mathbf{M_n}$ is (Hermitian) positive semidefinite, then A^r, where $r > 0$, is well defined via the usual functional calculus. When $0 \leqslant \alpha \leqslant 1$, the α-power mean of positive definite $A, B \in \mathbf{M_n}$ is defined and denoted by

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{1/2})^\alpha A^{1/2}.$$

Furthermore, $A \sharp_\alpha B$ for positive semidefinite $A, B \in \mathbf{M_n}$ is defined by

$$A \sharp_\alpha B = \lim_{\varepsilon \to 0^+} (A + \varepsilon I) \sharp_\alpha (B + \varepsilon I),$$

where the limit process is in the strong operator topology. If $\alpha = 1/2$, we simply write $A \sharp B$ for $A \sharp_{1/2} B$. The norm we consider in this paper is unitarily invariant, that is, $\|UAV\| = \|A\|$ for any $A, U, V \in \mathbf{M_n}$ with U, V being unitary. In particular, the frequently used spectral/operator norm, Hilbert-Schmidx/Frobenius norm, trace/nuclear norm belong to the class of unitarily invariant norms.

In 1994, Ando and Hiai [3] proved the following remarkable norm inequality.

THEOREM 1.1. *Let $A, B \in \mathbf{M_n}$ be positive semidefinite and let $0 \leqslant \alpha \leqslant 1$. Then*

$$\|(A \sharp_\alpha B)^r\| \leqslant \|A^r \sharp_\alpha B^r\|, \quad 0 \leqslant r \leqslant 1.$$

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Ando and Hiai stated their result in the form of weakly log majorization between eigenvalues, but the above statement is of no loss of generality to their result [3, Theorem 2.3]. This can be seen by using a standard argument in matrix analysis via the anti-symmetric product; see [4, p. 18] or [9] for details. The main result of the paper is an extension of Theorem 1.1 to a larger class of matrices, namely, sectorial matrices to be introduced below.

Recall that the field of values (or numerical range) of $A \in \mathbf{M_n}$ is defined as the set on the complex plane

$$W(A) = \{u^*Au|\ u^*u = 1,\ u \in \mathbb{C}^n\}.$$

Also, we define the set on the complex plane

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leqslant (\Re z)\tan\theta\}$$

for a fixed $\theta \in [0,\pi/2)$. It is easy to observe that the shape of S_θ is a sector on the complex plane. The larger class of matrices we focus in the paper is matrices A with $W(A) \subset S_\theta$. This class of matrices has attracted quite a number of researchers recently [2, 5, 8, 10, 13, 14, 16, 17, 18, 19, 6, 7, 15]. Part of the reason is that sectorial matrices are considered as a very natural generalization of positive definite matrices. One obvious fact is that $W(A) \subset S_0$ if and only if A is positive definite. Therefore, by adjusting the angle θ, one considerably relax the restrictive positive definiteness requirement. For any $A \in \mathbf{M_n}$, its real (or Hermitian) part is denoted by $\Re A := (A + A^*)/2$, where A^* means the conjugate transpoes of A. It is easy to oberve that if $W(A) \subset S_\theta$, then $\Re A$ is positive definite. For two Hermitian matrices $A,B \in \mathbf{M_n}$, if $A - B$ is positive semidefinite then we write $A \geqslant B$ (or $B \leqslant A$).

Now we introduce the geometric mean for two sectorial matrices and to be consistent we keep using the notation $A\sharp B$ and $A\sharp_\alpha B$. In his study of principal powers of matrices with positive definite real part, Drury [6] first brought in the following definition: Let $A,B \in \mathbf{M_n}$ with $\Re A, \Re B$ being positive definite. Then

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty (xA + x^{-1}B^{-1})^{-1}\frac{dx}{x}\right)^{-1}.$$

A weighted version was then considered by Raissouli, Moslehian and Furuichi [12]: For $A,B \in \mathbf{M_n}$ with $\Re A, \Re B$ being positive definite,

$$A\sharp_\alpha B = \frac{\sin\alpha\pi}{\pi}\int_0^\infty x^{\alpha-1}(A^{-1} + xB^{-1})^{-1}dx. \tag{1}$$

It is worth mentioning that when $\alpha = 1/2$, Raissouli, Moslehian and Furuichi's definition (1) coincides with the aforementioned Drury's definition; see [12, Proposition 2.1].

One of the remarkable properties about the geometric mean is the following inequality [12, Theorem 2.4]: Let $A,B \in \mathbf{M_n}$ with $\Re A, \Re B$ being positive definite. Then it holds

$$(\Re A)\sharp_\alpha(\Re B) \leqslant \Re(A\sharp_\alpha B), \tag{2}$$

while when $\alpha = 1/2$, this was previously obtained in [11].

2. Auxiliary results

As Theorem 1.1 involves fractional power of matrices, we need to record a formula to facilitate our derivations in the sequel. It follows from (1) that if $A \in \mathbf{M_n}$ with $\Re A$ being positive definite, then for any $0 \leqslant r \leqslant 1$, it holds

$$A^r = I \sharp_r A = \frac{\sin \alpha\pi}{\pi} \int_0^\infty x^{\alpha-1}(I + xA^{-1})^{-1} dx. \tag{3}$$

Clearly, the formula was known for positive definite matrices.

We present several lemmas for later development.

LEMMA 2.1. [8, Lemma 2.4] *Let $A \in \mathbf{M_n}$. If $\Re A$ is positive definite, then*

$$\Re A^{-1} \leqslant (\Re A)^{-1}.$$

We also have a reverse inequality, as stated below.

LEMMA 2.2. [10, Lemma 3] *Let $A \in \mathbf{M_n}$. If $W(A) \subset S_\theta$, then*

$$(\sec \theta)^2 \Re A^{-1} \geqslant (\Re A)^{-1}.$$

LEMMA 2.3. [6, Corollary 2.4] *Let $A \in \mathbf{M_n}$. If $W(A) \subset S_\theta$, then*

$$W(A^r) \subset S_{r\theta}$$

for any $0 \leqslant r \leqslant 1$.

LEMMA 2.4. [15, Lemma 3.1] *Let $A \in \mathbf{M_n}$. If $W(A) \subset S_\theta$, then*

$$\cos \theta \|A\| \leqslant \|\Re A\|.$$

A reverse inequality corresponding to Lemma 2.4 is well known.

LEMMA 2.5. [4, p. 74] *Let $A \in \mathbf{M_n}$. Then*

$$\|A\| \geqslant \|\Re A\|.$$

PROPOSITION 2.6. *Let $A \in \mathbf{M_n}$. If $W(A) \subset S_\theta$, then for any $0 \leqslant r \leqslant 1$ it holds*

$$\Re A^r \leqslant (\sec \theta)^2 (\Re A)^r.$$

Proof. First of all, by Lemma 2.1,

$$\Re(I + xA^{-1})^{-1} \leqslant (I + x\Re A^{-1})^{-1}. \tag{4}$$

On the other hand, by Lemma 2.2,

$$\begin{aligned} \Re I + x\Re A^{-1} &\geqslant I + (\cos\theta)^2 (\Re A)^{-1} \\ &\geqslant (\cos\theta)^2 (I + x(\Re A)^{-1}), \end{aligned}$$

and hence

$$\Re(I+xA^{-1})^{-1} \leqslant (\sec\theta)^2(I+x(\Re A)^{-1})^{-1}. \tag{5}$$

(4) and (5) together imply

$$\Re(A^{-1}+x^2A)^{-1} \leqslant (\sec\theta)^2(I+x(\Re A)^{-1})^{-1}.$$

Now by (3),

$$\begin{aligned}\Re A^r &= \frac{\sin r\pi}{\pi}\int_0^\infty \Re(I+xA^{-1})^{-1}x^{-r}\,dx\\ &\leqslant \frac{\sin r\pi}{\pi}\int_0^\infty (\sec\theta)^2(I+x(\Re A)^{-1})^{-1}x^{-r}\,dx\\ &= (\sec\theta)^2(\Re A)^r.\end{aligned}$$

The proof is complete. □

The next result is a complement of Proposition 2.6.

PROPOSITION 2.7. *Let $A \in \mathbf{M_n}$. If $W(A) \subset S_\theta$, then for any $0 \leqslant r \leqslant 1$, it holds*

$$\Re A^r \geqslant (\Re A)^r.$$

Proof. By (2),

$$\Re(I\sharp_r A) \geqslant I\sharp_r(\Re A),$$

which is equivalent to the claimed inequality. □

The next result gives a reverse of (2).

PROPOSITION 2.8. *Let $A, B \in \mathbf{M_n}$. If $W(A), W(B) \subset S_\theta$, then for any $0 \leqslant \alpha$ it holds*

$$\Re(A\sharp_\alpha B) \leqslant (\sec\theta)^2((\Re A)\sharp_\alpha(\Re B)).$$

Proof. First of all, by Lemma 2.1,

$$\Re(A^{-1}+xB^{-1})^{-1} \leqslant (\Re A^{-1}+x\Re B^{-1})^{-1}.$$

By Lemma 2.2,

$$\Re A^{-1}+x\Re B^{-1} \geqslant (\cos\theta)^2((\Re A)^{-1}+x(\Re B)^{-1}).$$

Thus we have

$$\Re(A^{-1}+xB^{-1})^{-1} \leqslant (\sec\theta)^2((\Re A)^{-1}+x(\Re B)^{-1})^{-1}.$$

Hence,

$$\begin{aligned}\Re(A\sharp_\alpha B) &= \frac{\sin\alpha\pi}{\pi}\int_0^\infty \Re(A^{-1}+xB^{-1})^{-1}x^{\alpha-1}dx\\ &\leqslant \frac{\sin\alpha\pi}{\pi}\int_0^\infty (\sec\theta)^2((\Re A)^{-1}+x(\Re B)^{-1})^{-1}x^{\alpha-1}dx\\ &= (\sec\theta)^2((\Re A)\sharp_\alpha(\Re B)).\end{aligned}$$

The proof is complete. $\square$

Proposition 2.8 could be regarded as a generalization of Proposition 2.6. The reason we present Proposition 2.6 instead of viewing it as a corollary of Proposition 2.8 is that it reflects the true exploring path order of the authors and it would be of independent interest.

3. Main Theorem

We are in a position to state and prove the main result.

THEOREM 3.1. *Let $A,B\in\mathbf{M_n}$. If $W(A),W(B)\subset S_\theta$, then for any $0\leqslant\alpha\leqslant 1$, it holds*

$$\|(A\sharp_\alpha B)^r\| \leqslant (\sec\theta)^{4+2r}\sec(r\theta)\|A^r\sharp_\alpha B^r\|,\ 0\leqslant r\leqslant 1.$$

Proof. First of all, note that there is closure property by taking inverse and summation of sectorial matrices [8, 14], one observes from (1) that

$$W(A\sharp_\alpha B)\subset S_\theta.$$

With this and Lemma 2.3, we have

$$W((A\sharp_\alpha B)^r)\subset S_{r\theta}.$$

Now by Lemma 2.4, we get

$$\|(A\sharp_\alpha B)^r\| \leqslant \sec(r\theta)\|\Re(A\sharp_\alpha B)^r\|. \tag{6}$$

We estimate

$$\begin{aligned}\|\Re(A\sharp_\alpha B)^r\| &\leqslant (\sec\theta)^2\|(\Re(A\sharp_\alpha B))^r\| && \text{by Proposition 2.6}\\ &\leqslant (\sec\theta)^2\Big\|\Big((\sec\theta)^2((\Re A)\sharp_\alpha(\Re B))\Big)^r\Big\| && \text{by Proposition 2.8}\\ &= (\sec\theta)^{2+2r}\Big\|\Big((\Re A)\sharp_\alpha(\Re B)\Big)^r\Big\|\\ &\leqslant (\sec\theta)^{2+2r}\|(\Re A)^r\sharp_\alpha(\Re B)^r\| && \text{by Theorem 1.1}\\ &\leqslant (\sec\theta)^{2+2r}\Big\|\Big((\sec\theta)^2\Re A^r\Big)\sharp_\alpha\Big((\sec\theta)^2\Re B^r\Big)\Big\| && \text{by Proposition 2.7}\\ &= (\sec\theta)^{4+2r}\|(\Re A^r)\sharp_\alpha(\Re B^r)\|\\ &\leqslant (\sec\theta)^{4+2r}\|\Re(A^r\sharp_\alpha B^r)\| && \text{by (2)}\\ &\leqslant (\sec\theta)^{4+2r}\|A^r\sharp_\alpha B^r\|.\end{aligned}$$

That is,

$$\|\Re(A\sharp_\alpha B)^r\| \leqslant (\sec\theta)^{4+2r}\|A^r\sharp_\alpha B^r\|. \tag{7}$$

The desired result follows from (6) and (7). □

Just like that in [10], we remark that the question of the optimality of the coefficient, i.e., $(\sec\theta)^{4+2r}\sec(r\theta)$, in the theorem deserves further investigtion.

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REFERENCES

[1] B. HUANG, C. MA, *An iterative algorithm for the least Frobenius norm least squares solution of a class of generalized coupled Sylvester-transpose linear matrix equations*, Applied Math. Comput. 328 (2018) 58–74.
[2] M. ALAKHRASS, M. SABABHEH, *Lieb functions and sectorial matrices*, Linear Algebra Appl. 586 (2020) 308–324.
[3] T. ANDO, F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl. 197–198 (1994) 113–131.
[4] R. BHATIA, *Matrix Analysis*, GTM 169, Springer-Verlag, New York, 1997.
[5] S. DONG, L. HOU, *A complement of the Hadamard-Fischer inequality*, Journal of Intelligent & Fuzzy Systems, 35 (2018) 4011–4015.
[6] S. DRURY, *Principal powers of matrices with positive definite real part*, Linear Multilinear Algebra 63 (2015) 296–301.
[7] S. DRURY, M. LIN, *Singular value inequalities for matrices with numerical ranges in a sector*, Oper. Matrices, 8 (2014) 1143–1148.
[8] M. LIN, *Extension of a result of Hanynsworth and Hartfiel*, Arch. Math. 1 (2015), 93–100.
[9] M. LIN, *Remarks on two recent results of Audenaert*, Linear Algebra Appl. 489 (2016) 24–29.
[10] M. LIN, *Some inequalities for sector matrices*, Oper. Matrices, 10 (2016), 915–921.
[11] M. LIN, F. SUN, *A property of the geometric mean of accretive operator*, Linear Multilinear Algebra 65 (2017) 433–437.
[12] M. RAISSOULI, M. S. MOSLEHIAN, S. FURUICHI, *Relative entropy and Tsallis entropy of two accretive operators*, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 687–693.
[13] C. YANG, F. LU, *Some generalizations of inequalities for sector matrices*, J. Inequal. Appl. (2018) 2018: 183.
[14] D. ZHANG, L. HOU AND L. MA, *Properties of matrices with numerical ranges in a sector*, Bull. Iranian Math. Soc. 43 (2017) 1699–1707.
[15] F. ZHANG, *A matrix decomposition and its applications*, Linear Multilinear Algebra 63 (2015) 2033–2042.
[16] X. JIANG, Y. ZHENG, X. CHEN, *Extending a refinement of Koteljanskiĭ's inequality*, Linear Algebra Appl. 574 (2019) 252–261.
[17] Y. ZHENG, X. JIANG, X. CHEN, et al., *More extensions of a determinant inequality of Hartfiel*, Appl. Math. Comput. 369 (2020) 124827.
[18] Y. ZHENG, X. JIANG, X. CHEN, et al., *On some generalizations of the Brunn-Minkowski inequality*, Linear Algebra Appl. 586 (2020) 103–110.
[19] Y. ZHENG, X. JIANG, X. CHEN, et al., *Means and the Schur complement of sector matrices*, Linear Multilinear Algebra, 2020, doi:10.1080/03081087.2020.1809617.

[20] H. WANG, *Least squares solutions to the rank-constrained matrix approximation problem in the Frobenius norm*, Calcolo 56 (2019) Art. 47, 18 pp.

[21] H. ORERA, J. M. PENA, *Infinity norm bounds for the inverse of Nekrasov matrices using scaling matrices*, Applied Math. Comput. 358 (2019) 119–127.

[22] S. SOLAK, *On the spectral norm of the matrix with integer sequences*, Applied Math. Comput. 232 (2014) 919–921.

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