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GAPS BETWEEN SOME SPECTRAL CHARACTERISTICS OF DIRECT SUM OF HILBERT SPACE OPERATORS

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Abstract. In the present study, we investigate how the gaps between some spectral characteristics (operator norm, upper and lower bounds of spectrum and numerical range) of finite direct sum of Hilbert space operators are related to the same spectral characteristics of the coordinate operators.

1. Introduction

As is known in the mathematical literature one of the fundamental questions of the spectral theory of linear operators is to obtain its spectrum and numerical range and calculate spectral and numerical radii of the given operator. In many cases, serious theoretical and technical difficulties are encountered in finding the spectrum and numerical range of non-selfadjoint linear bounded operators. Note that there is one formula for the calculation of the spectral radius $r(A)$ of the linear bounded operator in any Banach space

$$r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$$

[9]. On the other hand, it is also known that

$$r(A) \leqslant w(A) \leqslant \|A\|$$

and

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|$$

for $A \in L(H)$.

In addition, for a linear bounded operator A in Hilbert space we have the following relations

$$r(A) \leqslant w(A) \leqslant \|A\|.$$

It is beneficial to recall that for the spectrum $\sigma(A)$ and numerical range $W(A)$ of any linear bounded operator A the following inclusion holds

$$\sigma(A) \subset \overline{W(A)}$$

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(See [7, 9] for more information).

In [11] some spectral radius inequalities for 2×2 block operator matrix, sum, product, and commutators of two linear bounded Hilbert space operators have been examined. In [1] some estimates for numerical and spectral radii of the Frobenius companion matrix have been obtained.

Some upper and lower bounds for the numerical indices in Hilbert space operators have been obtained in [2].

In [3] some estimates for spectral and numerical radii have been obtained for the product, sum, commutator, anticommutator of two Hilbert spaces operators.

In [6] several numerical radius inequalities for $n\times n$ block operator matrices in the direct sum of Hilbert spaces have been proved. Some numerical radius inequalities for $n\times n$ accretive matrices have been obtained in [5].

Several new norms and numerical radius inequalities for 2×2 block operator matrices have been researched in [4].

Recently, several new $\mathbb{A}$-numerical radius inequalities for many type $n\times n$ block operator matrices have been offered in [4] in the direct sum of Hilbert spaces.

Subadditivity of the spectral radius of commutative two operators in Banach spaces has been investigated in [15]. By the same author the subadditivity and submultiplicativity properties of local spectral radius of bounded positive operators have been researched in Banach spaces [18]. The same properties of local spectral radius in partially ordered Banach spaces have been established in [16]. In Banach space ordered by a normal and generating core, several inequalities for the spectral radius of a positive commutator of positive operators have been surveyed in [18]. The numerical range and numerical of some Volterra integral operator in Hilbert Lebesgue spaces at finite intervals has been considered [8, 10].

This paper is organized as follows: We present a necessary auxiliary theorem in Section 2. In the last section we prove our main results.

2. Some auxiliary important results

In this section, we will prove certain auxiliary results which will be used later.

THEOREM 1. *Let $n\in\mathbb{N}$. For the numbers $a_1,a_2,\dots,a_n\in\mathbb{R}$ and $b_1,b_2,\dots,b_n\in\mathbb{R}$*

$$\min_{1\leqslant m\leqslant n}(a_m-b_m)\leqslant \max_{1\leqslant m\leqslant n} a_m-\max_{1\leqslant m\leqslant n} b_m\leqslant \max_{1\leqslant m\leqslant n}(a_m-b_m);$$

$$\begin{aligned}\max_{1\leqslant m\leqslant n}(a_m-b_m)&\leqslant \max_{1\leqslant m\leqslant n} a_m-\min_{1\leqslant m\leqslant n} b_m\\ &\leqslant \max_{1\leqslant m\leqslant n}(a_m-b_m)+\sum_{\substack{k,m=1\\k<m}}^{n}|a_k-a_m|+\sum_{\substack{k,m=1\\k<m}}^{n}|b_k-b_m|;\end{aligned}$$

$$\min_{1\leqslant m\leqslant n}(a_m-b_m)\leqslant \min_{1\leqslant m\leqslant n} a_m-\min_{1\leqslant m\leqslant n} b_m\leqslant \max_{1\leqslant m\leqslant n}(a_m-b_m);$$

are true.

Proof. For all cases, we prove by mathematical induction.

For $n=2$, it is clear that

$$\begin{aligned}\max\{a_1,a_2\}-\max\{b_1,b_2\} &= \frac{1}{2}[(a_1+a_2+|a_1-a_2|)-(b_1+b_2+|b_1-b_2|)]\\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2))+(|a_1-a_2|-|b_1-b_2|)]\\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-(|b_1-b_2|-|a_1-a_2|)]\\ &\geqslant \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-||b_1-b_2|-|a_1-a_2||]\\ &\geqslant \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-|(b_1-b_2)-(a_1-a_2)|]\\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-|(a_1-b_1)-(a_2-b_2)|]\\ &= \min\{a_1-b_1,a_2-b_2\}.\end{aligned}$$

Now assume that

$$\max_{1\leqslant m\leqslant n-1} a_m - \max_{1\leqslant m\leqslant n-1} b_m \geqslant \min_{1\leqslant m\leqslant n-1}(a_m-b_m)$$

for any $n\in\mathbb{N}$, $n>2$.

Then one can easily see that

$$\begin{aligned}\max\{a_1,\dots,a_n\}-\max\{b_1,\dots,b_n\} &= \max\{\max_{1\leqslant m\leqslant n-1} a_m, a_n\} - \max\{\max_{1\leqslant m\leqslant n-1} b_m, b_n\}\\ &\geqslant \min\{\max_{1\leqslant m\leqslant n-1} a_m - \max_{1\leqslant m\leqslant n-1} b_m, a_n-b_n\}\\ &\geqslant \min\{\min_{1\leqslant m\leqslant n-1}(a_m-b_m), a_n-b_n\}\\ &= \min_{1\leqslant m\leqslant n}(a_m-b_m).\end{aligned}$$

From this and by mathematical induction, for any $n\in\mathbb{N}$

$$\min_{1\leqslant m\leqslant n}(a_m-b_m)\leqslant \max_{1\leqslant m\leqslant n} a_m - \max_{1\leqslant m\leqslant n} b_m.$$

holds.

Similarly, for $n=2$ by simple calculations we again have that

$$\begin{aligned}\max\{a_1,a_2\}-\max\{b_1,b_2\} &= \frac{1}{2}[(a_1+a_2+|a_1-a_2|)-(b_1+b_2+|b_1-b_2|)]\\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2))+(|a_1-a_2|-|b_1-b_2|)]\\ &= \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-(|a_1-a_2|-|b_1-b_2|)]\\ &\leqslant \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-||a_1-a_2|-|b_1-b_2||]\end{aligned}$$

$$\leqslant \frac{1}{2}[((a_1-b_1)+(a_2-b_2))-|(a_1-b_1)-(a_2-b_2)|]$$
$$=\max\{a_1-b_1,a_2-b_2\}.$$

Now assume that for $n\in\mathbb{N}$, $n>2$

$$\max_{1\leqslant m\leqslant n-1} a_m - \max_{1\leqslant m\leqslant n-1} b_m \leqslant \max_{1\leqslant m\leqslant n-1}(a_m-b_m).$$

From this assumption one obtains that

$$\begin{aligned}\max_{1\leqslant m\leqslant n} a_m - \max_{1\leqslant m\leqslant n} b_m &= \max\{\max_{1\leqslant m\leqslant n-1} a_m, a_n\} - \max\{\max_{1\leqslant m\leqslant n-1} b_m, b_n\}\\ &\leqslant \max\{\max_{1\leqslant m\leqslant n-1} a_m - \max_{1\leqslant m\leqslant n-1} b_m, a_n-b_n\}\\ &\leqslant \max\{\min_{1\leqslant m\leqslant n-1}(a_m-b_m), a_n-b_n\}\\ &= \max_{1\leqslant m\leqslant n}(a_m-b_m).\end{aligned}$$

Consequently, by mathematical induction we obtain that,

$$\max_{1\leqslant m\leqslant n} a_m - \max_{1\leqslant m\leqslant n} b_m \leqslant \max_{1\leqslant m\leqslant n}(a_m-b_m)$$

holds for any $n\in\mathbb{N}$.

Now we prove the second part of the theorem.

For $n=2$ it is clear that

$$\begin{aligned}2\max\{a_1,a_2\}-2\min\{b_1,b_2\} &= (a_1+a_2+|a_1-a_2|)-(b_1+b_2-|b_1-b_2|)\\ &= ((a_1-b_1)+(a_2-b_2))+(|a_1-a_2|+|b_1-b_2|)\\ &\geqslant ((a_1-b_1)+(a_2-b_2))+|(a_1-a_2)-(b_1-b_2)|\\ &= ((a_1-b_1)+(a_2-b_2))+|(a_1-b_1)-(a_2-b_2)|\\ &= 2\max\{(a_1-b_1),(a_2-b_2)\}.\end{aligned}$$

Then we have

$$\max\{a_1-b_1,a_2-b_2\}\leqslant\max\{a_1,a_2\}-\min\{b_1,b_2\}.$$

On the other hand we have that

$$\begin{aligned}2\max\{a_1,a_2\}-2\min\{b_1,b_2\} &= (a_1+a_2+|a_1-a_2|)-(b_1+b_2-|b_1-b_2|)\\ &= ((a_1-b_1)+(a_2-b_2))+|(a_1-b_1)-(a_2-b_2)|\\ &\quad +[|a_1-a_2|+|b_1-b_2|-|(a_1-b_1)-(a_2-b_2)|]\\ &\leqslant [(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|]\\ &\quad +||a_1-a_2|+|b_1-b_2|-|(a_1-b_1)-(a_2-b_2)||\\ &= [(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|]\\ &\quad +|||a_1-a_2|+|b_1-b_2||-|(a_1-b_1)-(a_2-b_2)||\end{aligned}$$

$$\begin{aligned}
&\leqslant [(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|] \\
&\quad +||a_1-a_2|+|b_1-b_2|-(a_1-b_1)-(a_2-b_2)| \\
&= [(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|] \\
&\quad +|(|a_1-a_2|+(a_1-a_2))+(|b_1-b_2|-(b_1-b_2))| \\
&= [(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|] \\
&\quad +2|a_1-a_2|+2|b_1-b_2|.
\end{aligned}$$

Then, for $n=2$, we have

$$\max\{a_1,a_2\}-\min\{b_1,b_2\} \leqslant \max\{a_1-b_1,a_2-b_2\}+|a_1-a_2|+|b_1-b_2|.$$

Now assume that the mentioned inequalities hold for $k=n-1$. Then for $k=n$, it is clear that

$$\begin{aligned}
\max_{1\leqslant m\leqslant n} a_m-\min_{1\leqslant m\leqslant n} b_m &= \max\{\max_{1\leqslant m\leqslant n-1} a_m,a_n\}-\min\{\min_{1\leqslant m\leqslant n-1} b_m,b_n\} \\
&\leqslant \max\{\max_{1\leqslant m\leqslant n-1} a_m-\min_{1\leqslant m\leqslant n-1} b_m,a_n-b_n\} \\
&\quad +|\max_{1\leqslant m\leqslant n-1} a_m-a_n|+|\min_{1\leqslant m\leqslant n-1} b_m-b_n| \\
&\leqslant \max\{\max_{1\leqslant m\leqslant n-1}(a_m-b_m)+\sum_{\substack{k,m=1\\ k<m}}^{n-1}|a_k-a_m|+\sum_{\substack{k,m=1\\ k<m}}^{n-1}|b_k-b_m|,|a_n-b_n|\} \\
&\quad +|\max_{1\leqslant m\leqslant n-1} a_m-a_n|+|\min_{1\leqslant m\leqslant n-1} b_m-b_n| \\
&\leqslant \max\{\max_{1\leqslant m\leqslant n-1}(a_m-b_m),|a_n-b_n|\} \\
&\quad +\sum_{\substack{k,m=1\\ k<m}}^{n-1}|a_k-a_m|+\sum_{\substack{k,m=1\\ k<m}}^{n-1}|b_k-b_m| \\
&\quad +\sum_{k=1}^{n-1}|a_k-a_n|+\sum_{k=1}^{n-1}|b_k-b_n| \\
&= \max_{1\leqslant m\leqslant n}(a_m-b_m)+\sum_{\substack{k,m=1\\ k<m}}^{n}|a_k-a_m|+\sum_{\substack{k,m=1\\ k<m}}^{n}|b_k-b_m|.
\end{aligned}$$

On the other hand we have that

$$\begin{aligned}
\max_{1\leqslant m\leqslant n} a_m-\min_{1\leqslant m\leqslant n} b_m &= \max\{\max_{1\leqslant m\leqslant n-1} a_m,a_n\}-\min\{\min_{1\leqslant m\leqslant n-1} b_m,b_n\} \\
&\geqslant \max\{\max_{1\leqslant m\leqslant n-1} a_m-\min_{1\leqslant m\leqslant n-1} b_m,a_n-b_n\} \\
&\geqslant \max\{\max_{1\leqslant m\leqslant n-1}(a_m-b_m),a_n-b_n\} \\
&= \max_{1\leqslant m\leqslant n}(a_m-b_m).
\end{aligned}$$

Now we prove the third part of the theorem.
For $n=2$, we have

$$\begin{aligned} 2\min\{a_1,a_2\}-2\min\{b_1,b_2\} &= (a_1+a_2-|a_1-a_2|)-(b_1+b_2-|b_1-b_2|) \\ &= ((a_1-b_1)+(a_2-b_2))+(|b_1-b_2|-|a_1-a_2|) \\ &\leqslant ((a_1-b_1)+(a_2-b_2))+||b_1-b_2|-|a_1-a_2|| \\ &\leqslant ((a_1-b_1)+(a_2-b_2))+|(b_1-b_2)-(a_1-a_2)| \\ &= ((a_1-b_1)+(a_2-b_2))+|(a_1-b_1)-(a_2-b_2)| \\ &= 2\max\{a_1-b_1,a_2-b_2\}. \end{aligned}$$

On the other hand we have that

$$\begin{aligned} 2\min\{a_1,a_2\}-2\min\{b_1,b_2\} &= (a_1+a_2-|a_1-a_2|)-(b_1+b_2-|b_1-b_2|) \\ &= ((a_1-b_1)+(a_2-b_2))-(|a_1-a_2|-|b_1-b_2|) \\ &\geqslant ((a_1-b_1)+(a_2-b_2))-||a_1-a_2|-|b_1-b_2|| \\ &\geqslant ((a_1-b_1)+(a_2-b_2))-|(a_1-a_2)-(b_1-b_2)| \\ &= ((a_1-b_1)+(a_2-b_2))-|(a_1-b_1)-(a_2-b_2)| \\ &= 2\min\{a_1-b_1,a_2-b_2\}. \end{aligned}$$

By the above,it is clear that

$$\begin{aligned} \min_{1\leqslant m\leqslant n} a_m - \min_{1\leqslant m\leqslant n} b_m &= \min\{\min_{1\leqslant m\leqslant n-1} a_m, a_n\} - \min\{\min_{1\leqslant m\leqslant n-1} b_m, b_n\} \\ &\leqslant \max\{\min_{1\leqslant m\leqslant n-1} a_m - \min_{1\leqslant m\leqslant n-1} b_m, a_n-b_n\} \\ &\leqslant \max\{\max_{1\leqslant m\leqslant n-1}(a_m-b_m), a_n-b_n\} \\ &= \max_{1\leqslant m\leqslant n}(a_m-b_m) \end{aligned}$$

holds for any $n\in\mathbb{N}$. Similarly, we also have that

$$\begin{aligned} \min_{1\leqslant m\leqslant n} a_m - \min_{1\leqslant m\leqslant n} b_m &= \min\{\min_{1\leqslant m\leqslant n-1} a_m, a_n\} - \min\{\min_{1\leqslant m\leqslant n-1} b_m, b_n\} \\ &\geqslant \min\{\min_{1\leqslant m\leqslant n-1} a_m - \min_{1\leqslant m\leqslant n-1} b_m, a_n-b_n\} \\ &\geqslant \min\{\min_{1\leqslant m\leqslant n-1}(a_m-b_m), a_n-b_n\} \\ &= \min_{1\leqslant m\leqslant n}(a_m-b_m). \end{aligned}$$

This completes the proof. □

3. Some spectral characteristics numbers of finite direct sum of operators

Let $\sigma(A)$ and $W(A)$ be spectrum and numerical range sets of the linear bounded operator A in any Hilbert space respectively [7, 9]. And also assume that

$$gap(A;w,r)=w(A)-r(A),$$

$$
\begin{aligned}
gap(A;w,c) &= w(A) - c(A),\\
gap(A;r,c) &= r(A) - c(A),\\
gap(A;r,\tau) &= r(A) - \tau(A),\\
gap(A;\tau,c) &= \tau(A) - c(A),
\end{aligned}
$$

where

$$
\begin{aligned}
w(A) &= \sup|\lambda| : \lambda \in W(A),\\
r(A) &= \sup|\lambda| : \lambda \in \sigma(A),\\
\tau(A) &= \inf|\lambda| : \lambda \in \sigma(A),\\
c(A) &= \inf|\lambda| : \lambda \in W(A)
\end{aligned}
$$

(the number $c(A)$ sometimes is called the Crawford number of A).

It is well known that $\sigma(A) \subset \overline{W(A)}$ for any $A \in L(H)$ (see [7]).

Let H_m be a Hilbert space, $A_m \in L(H_m)$, for $1 \leqslant m \leqslant n < \infty$, and $H = \bigoplus_{m=1}^{n} H_m$, $A = \bigoplus_{m=1}^{n} A_m$.

Note that the relations between the numbers $\sigma(A), W(A), w(A), r(A), \tau(A), c(A)$ of the direct sum Hilbert space operators with the same sets and numbers of coordinate operators have been investigated in [13, 14]. For example, in [14] the following has been proved:

$$c(A) \leqslant \min_{1 \leqslant m \leqslant n} c(A_m).$$

In general, the above inequality is not an equality. For instance, taking the two operators

$$
\begin{aligned}
A_1 &= iI, \quad A_1 : \mathbb{C} \to \mathbb{C},\\
A_2 &= -iI, \quad A_2 : \mathbb{C} \to \mathbb{C}
\end{aligned}
$$

we have

$$c(A_1) = c(A_2) = 1.$$

But

$$c(A_1 \oplus A_2) = 0 < \min\{c(A_1), c(A_2)\}.$$

In the case

$$
\begin{aligned}
A_1 &= iI, \quad A_1 : \mathbb{C} \to \mathbb{C},\\
A_2 &= I, \quad A_2 : \mathbb{C} \to \mathbb{C}
\end{aligned}
$$

it is clear that

$$c(A_1) = c(A_2) = 1,$$

but

$$c(A_1 \oplus A_2) = \frac{\sqrt{2}}{2} < \min\{c(A_1), c(A_2)\}.$$

Using Theorem 1 one can prove the following.

THEOREM 2. *Let* $H = \bigoplus_{m=1}^{n} H_m$ *and* $A = \bigoplus_{m=1}^{n} A_m$*.Then the following hold*

$$\min_{1 \leqslant m \leqslant n} gap(A_m; w_m, r_m) \leqslant gap(A; w, r) \leqslant \max_{1 \leqslant m \leqslant n} gap(A_m; w_m, r_m);$$

$$\max_{1 \leqslant m \leqslant n} gap(A_m; r_m, \tau_m) \leqslant gap(A; r, \tau) \leqslant \max_{1 \leqslant m \leqslant n} gap(A_m; r_m, \tau_m) + \sum_{\substack{k,m=1\\ k<m}}^{n} |r_k - r_m| + \sum_{\substack{k,m=1\\ k<m}}^{n} |\tau_k - \tau_m|.$$

If $c(A) = \inf_{1 \leqslant m \leqslant n} c(A_m)$*, then we have*

$$\max_{1 \leqslant m \leqslant n} gap(A_m; w_m, c_m) \leqslant gap(A; w, c) \leqslant \max_{1 \leqslant m \leqslant n} gap(A_m; w_m, c_m) + \sum_{\substack{k,m=1\\ k<m}}^{n} |w_k - w_m| + \sum_{\substack{k,m=1\\ k<m}}^{n} |c_k - c_m|;$$

$$\max_{1 \leqslant m \leqslant n} gap(A_m; r_m, c_m) \leqslant gap(A; r, c) \leqslant \max_{1 \leqslant m \leqslant n} gap(A_m; r_m, c_m) + \sum_{\substack{k,m=1\\ k<m}}^{n} |r_k - r_m| + \sum_{\substack{k,m=1\\ k<m}}^{n} |c_k - c_m|;$$

$$\max_{1 \leqslant m \leqslant n} gap(A_m; \tau_m, c_m) \leqslant gap(A; \tau, c) \leqslant \max_{1 \leqslant m \leqslant n} gap(A_m; \tau_m, c_m),$$

where

$$\begin{aligned} w_m &= w(A_m), \\ r_m &= r(A_m), \\ \tau_m &= \tau(A_m), \\ c_m &= c(A_m), \quad 1 \leqslant m \leqslant n. \end{aligned}$$

REMARK 1. Note that similar relations can be obtained for the following gaps

$$\begin{aligned} gap(A; \|A\|, w) &:= \|A\| - w(A), \\ gap(A; \|A\|, r) &:= \|A\| - r(A), \\ gap(A; \|A\|, \tau) &:= \|A\| - \tau(A), \\ gap(A; \|A\|, c) &:= \|A\| - c(A). \end{aligned}$$

EXAMPLE 1. Let

$$A_1 = \alpha, \quad \alpha \in \mathbb{R}, \quad \alpha \geqslant 0, A_1 : \mathbb{C} \to \mathbb{C},$$
$$A_2 f(x) = \int_{-x}^{x} f(t)dt, \quad f \in L^2(-1,1), \quad A_2 : L^2(-1,1) \to L^2(-1,1)$$

and $A = A_1 \oplus A_2 : \mathbb{C} \oplus L^2(-1,1) \to \mathbb{C} \oplus L^2(-1,1)$.

In this case, $\|A_1\| = \alpha$, $w(A_1) = r(A_1) = c(A_1) = \tau(A_1) = \alpha$, $\sigma(A_1) = \{\alpha\}$, $W(A_1) = \{\alpha\}$,

$$\begin{aligned}
&gap(A_1; w_1, r_1) = 0,\\
&gap(A_1; r_1, \tau_1) = 0,\\
&gap(A_1; \|A_1\|, w_1) = 0,\\
&gap(A_1; \|A_1\|, r_1) = 0,\\
&gap(A_1; \|A_1\|, \tau_1) = 0,\\
&gap(A_1; \|A_1\|, c_1) = 0
\end{aligned}$$

and $\|A_2\| = \frac{4}{\pi}$, $\sigma(A_2) = \{0\}$, $W(A_2) = \overline{D(0, \frac{2}{\pi})}$, $w(A_2) = \frac{2}{\pi}$, $r(A_2) = \tau(A_2) = c(A_2) = 0$ [10],

$$\begin{aligned}
&gap(A_2; w_2, r_2) = \frac{2}{\pi},\\
&gap(A_2; r_2, \tau_2) = 0,\\
&gap(A_2; \|A_2\|, w_2) = \frac{2}{\pi},\\
&gap(A_2; \|A_2\|, r_2) = \frac{4}{\pi},\\
&gap(A_2; \|A_2\|, \tau_2) = \frac{4}{\pi},\\
&gap(A_2; \|A_2\|, c_2) = \frac{4}{\pi}.
\end{aligned}$$

Then, by Theorem 2, we have

$$\begin{aligned}
\frac{2}{\pi} &\leqslant gap(A; w, r) \leqslant \frac{2}{\pi} + 2\alpha,\\
0 &\leqslant gap(A; r, \tau) \leqslant 2\alpha
\end{aligned}$$

and

$$\begin{aligned}
0 &\leqslant gap(A; \|A\|, w) \leqslant \frac{2}{\pi},\\
0 &\leqslant gap(A; \|A\|, r) \leqslant \frac{4}{\pi},\\
\frac{4}{\pi} &\leqslant gap(A; \|A\|, \tau) \leqslant \frac{4}{\pi} + 2\alpha,\\
\frac{4}{\pi} &\leqslant gap(A; \|A\|, c) \leqslant \frac{4}{\pi} + |\frac{4}{\pi} - \alpha| + \alpha.
\end{aligned}$$

On the other hand, from the equality

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \int_{-x}^{x} \cdot dt \end{pmatrix}$$

and from the relation (see [7])

$$W(A) = W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}\right) + W\left(\begin{pmatrix} 0 & 0 \\ 0 & \int_{-x}^{x} \cdot dt \end{pmatrix}\right)$$

we have

$$W(A) = \alpha + \overline{D\left(0, \frac{2}{\pi}\right)}.$$

Then

$$c(A) = \alpha - \frac{2}{\pi}.$$

If $\alpha = \frac{4}{\pi}$, then we have

$$c(A) = \min\left\{\alpha, \frac{2}{\pi}\right\}.$$

For $\alpha = \frac{4}{\pi}$, Theorem 2 yields that

$$\frac{2}{\pi} \leqslant gap(A; w, c) \leqslant \frac{2}{\pi} + \frac{2}{\pi} + \alpha = \frac{8}{\pi},$$
$$0 \leqslant gap(A; r, c) \leqslant 2\alpha = \frac{8}{\pi},$$
$$0 \leqslant gap(A; \tau, c) \leqslant 2\alpha = \frac{8}{\pi}$$

hold.

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