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# UNIT VECTORS IN FULL HILBERT $C(Z)$–MODULES

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*Abstract.* In this paper, we show that full Hilbert $C(Z)$-modules, where $Z$ is a compact Hausdorff space may fail to have unit vectors. We also show that while real Hilbert $C_{\mathbb{R}}(Z)$-modules may not have unit vectors, their complexifications as (complex) Hilbert $C(Z)$-modules may have unit vectors. In particular, we prove that: (i) the unit vectors in full Hilbert $C(Z)$-modules are precisely the extreme points of their unit balls; (ii) the extreme and the exposed points of the unit ball of full Hilbert $C(Z)$-modules with unit vectors coincide as $Z$ has a diffuse measure; otherwise, their unit balls have no exposed points.

## 1. Introduction

A Hilbert C$^*$-module $\mathscr{M}$ over a C$^*$-algebra $A$ (or a Hilbert $A$-module $\mathscr{M}$) is a right $A$-module equipped with an $A$-valued inner product $\langle\cdot,\cdot\rangle$ which is $A$-linear in the second variable, fulfills $\langle x,y\rangle = \langle y,x\rangle^*$ and is positive definite in the sense that $\langle x,x\rangle \geqslant 0$ and $\langle x,x\rangle = 0$ if and only if $x=0$. Moreover, it is complete with respect to the norm

$$\|x\| = \sqrt{\|\langle x,x\rangle\|}, \qquad (x \in \mathscr{M}).$$

The range ideal of a Hilbert $A$-module $\mathscr{M}$ is the closed two-sided ideal $\langle \mathscr{M},\mathscr{M}\rangle := \overline{\text{span}}\{\langle x,y\rangle : x,y \in \mathscr{M}\}$ in $A$. A Hilbert $A$-module $\mathscr{M}$ is said to be full if $\langle \mathscr{M},\mathscr{M}\rangle = A$. The reader is referred to [8] and [11] for the general theory of Hilbert C$^*$-modules.

An element $x$ in a Hilbert C$^*$-module over a unital $C^*$-algebra with unit 1 is called a unit vector if $\langle x,x\rangle = 1$. Unit vectors in Hilbert C$^*$-modules play a crucial role in the construction of semigroups of endomorphisms from product systems (see [15]). Only full Hilbert C$^*$-modules may have unit vectors and they do not necessarily exist in full Hilbert modules over arbitrary noncommutative unital $C^*$-algebras (see [15, Example 3.3]). In this paper, we show by an example that the same is true even in full Hilbert modules over commutative unital C$^*$-algebras. Indeed, we give a Hilbert $C(Z)$-module which fails to have any unit vector (Example 1). We show in Example 2 that while real Hilbert $C_{\mathbb{R}}(Z)$-modules (introduced in [5]) may not have unit vectors, their complexifications as (complex) Hilbert $C(Z)$-modules may have unit vectors. We,

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in particular, show that the unit vectors of full Hilbert $C(Z)$-modules are precisely the extreme points of their unit balls (Proposition 1). In Proposition 2, we give a general form of [14, Proposition 2] in the setting of Hilbert $C(Z)$-modules. We show that if $Z$ has a diffuse measure, then the extreme and the exposed points of the unit ball of full Hilbert $C(Z)$-modules with unit vectors coincide; otherwise, their unit balls have no exposed points.

## 2. Main results

Henceforth, we assume that $Z$ is a compact Hausdorff space and $C(Z)$ is the commutative unital C$^*$-algebra consisting of all complex-valued continuous functions on $Z$. Also, we assume that the inner product of Hilbert spaces are linear in the second variable and conjugate linear in the first variable.

A generalization of the Serre-Swan theorem [16] asserts that the category of Hilbert $C(Z)$-modules is equivalent to the category of continuous fields of Hilbert spaces over $Z$ (see [4] and [17]). Let us give some basics about continuous fields of Hilbert spaces over $Z$ which will be needed in this note. For more information on the continuous field of Banach spaces see [2] and [3, Remark 4.4, Proposition 4.8].

Let $(H_z)_{z\in Z}$ be a family of Hilbert spaces. A vector field over $Z$ is a function $x$ defined on $Z$ such that $x(z)\in H_z$ for each $z\in Z$. Note that each vector field is an element of $\prod_{z\in Z} H_z$.

DEFINITION 1. A pair $((H_z)_{z\in Z},\Gamma)$, where $(H_z)_{z\in Z}$ is a family of Hilbert spaces and $\Gamma$ is a subset of $\prod_{z\in Z} H_z$ is said to be a continuous field of Hilbert spaces if it satisfies the following properties:
(i) $\Gamma$ is a complex linear subspace of

$$C(Z)-\prod_{z\in Z} H_z=\left\{x\in\prod_{z\in Z} H_z : [z\mapsto \|x(z)\|]\in C(Z)\right\};$$

(ii) For every $z\in Z$, the set $\{x(z)|x\in\Gamma\}$ is equal to $H_z$;
(iii) Let $x\in\prod_{z\in Z} H_z$. If for every $z\in Z$ and every $\varepsilon>0$, there is an $x'\in\Gamma$ such that $\|x(z')-x'(z')\|<\varepsilon$, for all $z'$ in some neighborhood of $z$, then $x\in\Gamma$.

For any continuous field of Hilbert spaces $((H_z)_{z\in Z},\Gamma)$, the space $\Gamma$ can be considered as a Hilbert $C(Z)$-module equipped with the point-wise multiplication

$$(x\cdot f)(z)=f(z)x(z),$$

and $C(Z)$-valued inner product

$$\langle x,y\rangle(z)=\langle x(z),y(z)\rangle,$$

for all $f\in C(Z)$, $x,y\in\Gamma$, and $z\in Z$. Note that, by [2, 10.7.1], the function $z\mapsto\langle x,y\rangle(z)$ belongs to $C(Z)$. Moreover, $\Gamma$ is a Banach space with the norm $\|x\|=\sup_{z\in Z}\|x(z)\|$. In general, every Hilbert $C(Z)$-module $\mathscr{M}$ is isomorphic to some continuous field of Hilbert spaces $((H_z)_{z\in Z},\Gamma)$ as Hilbert $C(Z)$-modules (see [3] and [17, Theorem

3.12]). It is worth mentioning that a Hilbert $C(Z)$-module $\mathscr{M}$ is full if and only if each Hilbert space $H_z = \{x(z) : x \in \Gamma\}$ in the corresponding continuous field of Hilbert spaces $(\{H_z\}_{z\in Z}, \Gamma)$ is nontrivial.

Note that $C(Z)$ itself as a Hilbert $C(Z)$-module has unit vectors as well as Hilbert $\mathbb{C}$-modules (Hilbert spaces) have unit vectors while they are not identified with $C(Z)$, for any compact Hausdorff space $Z$. By the following example, we show that not every full Hilbert $C^*$-module over a commutative unital $C^*$-algebra has unit vectors. For $n \geqslant 1$, let $S^n$ denote the $n$-sphere (or $n$-dimensional unit sphere) in euclidean space $\mathbb{R}^{n+1}$.

EXAMPLE 1. Consider the real differentiable manifold $S^2$. According to a well-known result of Borel and Serre, $S^2$ admits an almost complex structure $J$, i.e., a linear bundle morphism $J$ of the tangent bundle $T(S^2)$ satisfying $J^2 = -Id$. It is worth mentioning that an explicit structure $J$ can be constructed on $S^2$. To see this, consider the sphere $S^2$ as embedded into the imaginary part $\mathrm{Im}(\mathbb{H})$ of the quaternions $\mathbb{H}$, i.e.,

$$S^2 \cong \{z \in \mathrm{Im}(\mathbb{H}) : \|z\| = 1\}.$$

A cross product $\times$ is defined on $\mathrm{Im}(\mathbb{H})$ relative to the standard orientation determined by the basic quaternions $i, j$ and $k$ (namely, $i^2 = j^2 = k^2 = ijk = -1$) as

$$u \times v \mapsto \mathrm{Im}(uv) = \frac{1}{2}(uv - vu),$$

where $uv$ and $vu$ are the quaternion products of $u$ and $v$. Now, define $J \in \mathrm{End}(T(S^2))$ as

$$J_z(v) := z \times v,$$

where $z \in S^2$, $v \in T_z(S^2) \subset \mathrm{Im}(\mathbb{H})$. Then, $J$ is an almost complex structure on $S^2$, that is,

$$J_z^2 = -Id_{T_z(S^2)},$$

for all $z \in S^2$ (see [7, Proposition 2.1], and see also [6, Comments, p. 112] for $S^6$). Note that $z \times v = zv$ as $z \perp v$ and an easy application of three dimensional case of the Binet-Cauchy identity yields

$$\langle J_z(u), J_z(v)\rangle_{\mathbb{R}^3} = \langle u, v\rangle_{\mathbb{R}^3},$$

for all $z \in S^2$, and $u, v \in T_z(S^2)$. Moreover, each tangent space $T_z(S^2)$ can be considered as a complex vector space if we define complex scalar multiplication as

$$(a + ib)v \mapsto av + bJ_z(v).$$

Now, for each $z \in S^2$, define a functional $\langle \cdot, \cdot \rangle_z : T_z(S^2) \times T_z(S^2) \to \mathbb{C}$ as

$$\langle u, (a+ib)v\rangle_z = \langle u, av + bJ_z(v)\rangle_z := (a+ib)\langle u, v\rangle_{\mathbb{R}^3},$$

for all $u, v \in T_z(S^2)$ and $a, b \in \mathbb{R}$. We have

$$\langle J_z(u), J_z(v)\rangle_z = \langle u, v\rangle_{\mathbb{R}^3},$$

for all $z \in S^2$, and $u, v \in T_z(S^2)$. In particular, $\langle \cdot, \cdot \rangle_z$ is a complex inner product on $T_z(S^2)$.

Let $\mathscr{M}$ consist of all continuous sections of the tangent bundle $T(S^2) \to S^2$, i.e, all continuous vector fields $x : S^2 \to \mathbb{R}^3$ such that $\langle x(z), z \rangle_{\mathbb{R}^3} = 0$, for all $z \in S^2$. Now, considering the complex scalar multiplication and the complex inner product $\langle \cdot, \cdot \rangle_z$ defined as above on each tangent space $T_z(S^2)$, $\mathscr{M}$ equipped with the module action

$$(x \cdot f)(z) := f(z)x(z)$$

and $C(S^2)$-valued inner product

$$\langle x, y \rangle(z) := \langle x(z), y(z) \rangle_z,$$

for all $f \in C(S^2)$, $x, y \in \mathscr{M}$ and $z \in S^2$, is a full (complex) Hilbert $C(S^2)$-module. But, $\mathscr{M}$ has no unit vector. Indeed, suppose on the contrary that $\mathscr{M}$ has a unit vector $x_0$. Then,

$$\langle x_0, x_0 \rangle(z) = \langle x_0(z), x_0(z) \rangle_z = \langle x_0(z), x_0(z) \rangle_{\mathbb{R}^3} = 1,$$

for all $z \in S^2$. This implies that $x_0(z) \neq 0$, for all $z \in S^2$ which is a contradiction by the hairy ball theorem which states that there is no non-vanishing continuous tangent vector field on even-dimensional unit spheres.

Real Hilbert C$^*$-modules are the same as complex Hilbert C$^*$-modules except that the underlying field is $\mathbb{R}$ (For the definition of real C$^*$-algebras and real Hilbert C$^*$-modules see [9] and [5], respectively). The next example gives a class of full real Hilbert $C_{\mathbb{R}}(Z)$-modules without unit vectors, where $C_{\mathbb{R}}(Z)$ denotes the real C$^*$-algebra consisting of all continuous real-valued functions on $Z$. It also shows that while real Hilbert $C_{\mathbb{R}}(Z)$-modules may not have unit vectors, their complexification as (complex) Hilbert $C(Z)$-modules may have unit vectors. In particular, unlike the full Hilbert $C(S^2)$-module given in Example 1 which lacks any unit vector, the Hilbert $C(S^2)$-module given below possesses unit vectors.

EXAMPLE 2. Let $Z = S^n$ and $\mathscr{M}$ be the (real) vector space consisting of all continuous tangent vector fields $x : Z \to \mathbb{R}^{n+1}$, i.e., all continuous vector fields $x : Z \to \mathbb{R}^{n+1}$ such that $\langle x(z), z \rangle_{\mathbb{R}^{n+1}} = 0$, for all $z \in Z$. Given $x \in \mathscr{M}$ and $f \in C_{\mathbb{R}}(Z)$, $x \cdot f$ defined as $z \mapsto f(z)x(z)$ is an element of $\mathscr{M}$. Moreover, $\mathscr{M}$ equipped with a $C_{\mathbb{R}}(Z)$-valued inner product $(x, y) \mapsto \langle x, y \rangle$ defined as $z \mapsto \langle x(z), y(z) \rangle_{\mathbb{R}^{n+1}}$ has the structure of a full real Hilbert $C_{\mathbb{R}}(Z)$-module (for real Hilbert C$^*$-modules, see [5]).

(i) Let $n$ be even. Then, $\mathscr{M}$ does not have any unit vector. In fact, suppose on the contrary that $\mathscr{M}$ has a unit vector $x$. Then, $x(z) \neq 0$ for all $z \in Z$, which is a contradiction by the hairy ball theorem. Consequently, $\mathscr{M}$ as a real Hilbert $C_{\mathbb{R}}(Z)$-module has no unit vector. Now, consider the complex Hilbert module $\mathscr{M}_c := \mathscr{M} + i\mathscr{M}$ over complex C$^*$-algebra $C(Z) = C_{\mathbb{R}}(Z) + iC_{\mathbb{R}}(Z)$ in which the module action and $C(Z)$-valued inner product is defined as

$$(x + iy)(f + ig) := (x \cdot f - y \cdot g) + i(x \cdot g + y \cdot f)$$

and

$$\langle x+iy, u+iv\rangle := \langle x,u\rangle + \langle y,v\rangle + i(\langle y,u\rangle - \langle x,v\rangle),$$

respectively, for all $f,g \in C_{\mathbb{R}}(Z)$ and $x,y,u,v \in \mathscr{M}$ (see [5, Proposition 2.5]). Choose $x,y \in \mathscr{M}$ with no common zero on $Z$, for example,

$$x(z_1,z_2,\cdots,z_n,z_{n+1}) = (-z_2,z_1,-z_4,z_3,\cdots,-z_n,z_{n-1},0)$$

and

$$y(z_1,z_2,\cdots,z_n,z_{n+1}) = (0,-z_3,z_2,-z_5,z_4,\cdots,-z_{n+1},z_n).$$

Define

$$x_0(z) := \frac{1}{\sqrt{\|x(z)\|^2+\|y(z)\|^2}}\, x(z)$$

and

$$y_0(z) := \frac{1}{\sqrt{\|x(z)\|^2+\|y(z)\|^2}}\, y(z),$$

where $z \in Z$. We have

$$\langle x_0+iy_0, x_0+iy_0\rangle(z) = \langle x_0,x_0\rangle(z) + \langle y_0,y_0\rangle(z) = 1,$$

for all $z \in Z$. That is, $x_0+iy_0$ is a unit vector of the complexified Hilbert $C(Z)$-module $\mathscr{M}_c$ of $\mathscr{M}$.

(ii) Let $n$ be odd, say $n = 2k-1$. Then, $\tilde{x} : Z \to \mathbb{R}^{n+1}$ defined as

$$(z_1,z_2,\cdots,z_{2k-1},z_{2k}) \mapsto (-z_2,z_1,\cdots,-z_{2k},z_{2k-1})$$

is a nowhere vanishing continuous tangent vector field on $S^n$ which is also a unit vector in $\mathscr{M}$. In particular, $\frac{1}{\sqrt{2}}\tilde{x} + i\frac{1}{\sqrt{2}}\tilde{x}$ is a unit vector in the complexified Hilbert $C(Z)$-module $\mathscr{M}_c$ of $\mathscr{M}$.

Our next two results reveal the role that unit vectors in full Hilbert $C(Z)$-modules may play in determining the extremal structure of their unit balls. Let us recall two concepts: A point $x$ in a convex set $\mathscr{C}$ of a normed space $\mathscr{E}$ is called an extreme point if for every $y,z \in \mathscr{C}$, the equation $x = \lambda y + (1-\lambda)z$ with $\lambda \in [0,1]$ implies that $x = y = z$. A point $x \in \mathscr{C}$ is called an exposed point if there exists a bounded $\mathbb{R}$-linear functional $f : \mathscr{E} \to \mathbb{R}$, called exposing functional of $\mathscr{C}$ at $x$, such that $f(x) > f(y)$, for all $y \in \mathscr{C} \setminus \{x\}$). Any exposed point of $\mathscr{C}$ is an extreme point of $\mathscr{C}$ but in general the converse need not be true.

The following result relates the existence of unit vectors in full Hilbert $C(Z)$-modules to the extreme points in their unit balls. It shows that full Hilbert $C(Z)$-modules without extreme points of their unit ball are exactly those without unit vectors. For the special case of the C$^*$-algebra $C(Z)$ see, e.g., [13, Lemma] and [1, Corollary].

PROPOSITION 1. *Let $\mathscr{M}$ be a full Hilbert $C(Z)$-module. Then, $x \in \mathscr{M}$ is a unit vector if and only if $x$ is an extreme point of the unit ball of $\mathscr{M}$.*

*Proof.* Suppose that $((H_z)_{z\in Z},\Gamma)$ is the continuous field of Hilbert spaces corresponding to $\mathscr{M}$. We show that an element $x$ is an extreme point of the unit ball $\Gamma_1$ of the $C(Z)$-module $\Gamma$ if and only if $x$ is a unit vector. Suppose that $x\in\Gamma_1$ is a unit vector, i.e., $\langle x,x\rangle = 1$. It is enough to look only at $x=\frac{1}{2}y+\frac{1}{2}w$, for some vectors $y,w\in\Gamma_1$. Choose $z_0\in Z$ arbitrarily. We have $x(z_0)=\frac{1}{2}y(z_0)+\frac{1}{2}w(z_0)$, and therefore $\|y(z_0)+w(z_0)\| = 2\|x(z_0)\| = 2$. This implies that $\|y(z_0)\| = \|w(z_0)\| = \|x(z_0)\| = 1$. Since the vector $x(z_0)$ is an extreme point of the unit ball of the Hilbert space $H_{z_0}$, we have $x(z_0)=y(z_0)=w(z_0)$. That $z_0$ was arbitrary yields that $x=y=w$.

Conversely, suppose that $x\in\Gamma_1$ is not a unit vector. Hence, $\|x(z_0)\|<1$ for some $z_0\in Z$. We have two cases: (i) $x(z_0)=0$. Let $\varepsilon\in(0,\frac{1}{2})$. By the continuity of the function $z\mapsto\|x(z)\|$ at $z_0$, there is some open subset $U_{z_0}$ containing $z_0$ such that $\|x(z)\|<\varepsilon$, for all $z\in U_{z_0}$. And, by Urysohn's lemma, there is $f\in C(Z)$ such that

$$\|f\|=1,\quad f(z_0)=1,\quad f|_{U^C_{z_0}}=0.$$

Moreover, since $\mathscr{M}$ is full, there is some $y_0\in\Gamma$ such that $y_0(z_0)\neq 0$, and $\|y_0\|<\frac{1}{2}$. Now, consider the equation

$$x=\frac{1}{2}(x+y_0\cdot f)+\frac{1}{2}(x-y_0\cdot f).$$

It is straightforward to see that $x+y_0\cdot f\neq x-y_0\cdot f$ and both $x+y_0\cdot f$ and $x-y_0\cdot f$ belong to $\Gamma_1$. This implies that $x$ is not an extreme point of $\Gamma_1$. (ii) $x(z_0)\neq 0$. Choose $\varepsilon\in(0,1)$ such that $\|x(z_0)\|<\frac{1}{1+\varepsilon}$. Again, by the continuity of the map $z\mapsto\|x(z)\|$ there is some open subset $U_{z_0}$ containing $z_0$ such that

$$\|x(z)\|<\frac{1}{1+\varepsilon}\qquad(z\in U_{z_0}).$$

Also, let $V$ be an open subset of $Z$ containing $z_0$ such that $V\subset\overline{V}\subset U_{z_0}$. By Urysohn's lemma there exists $f_0\in C(Z)$ such that $\|f_0\|=1$, $f_0|_{\overline{V}}=1$, and $f_0|_{U^c_{z_0}}=0$. Putting $f=\varepsilon f_0+1$, we have $\|f\|\leqslant 1+\varepsilon$ and $f|_{\overline{V}}=1+\varepsilon$, $f|_{U^c_{z_0}}=1$. Consider

$$x=x\cdot f+x\cdot(1-f)=(1-\frac{1}{1+\varepsilon})x\cdot f+\frac{1}{1+\varepsilon}(x\cdot f+(1+\varepsilon)x\cdot(1-f)).$$

Let $y_1=x\cdot f$ and $y_2=x\cdot f+(1+\varepsilon)x\cdot(1-f)$. Since $x(z_0)\neq 0$, we have $y_1\neq y_2$. In addition, $y_1,y_2\in\Gamma_1$. In fact, $\|y_1(z)\|=\|f(z)x(z)\|\leqslant 1$ as $z\in U_{z_0}$. Also, if $z\in U^c_{z_0}$, we have $\|y_1(z)\|\leqslant 1$. Hence, $\|y_1\|\leqslant 1$. Similarly, if $z\in U_{z_0}$, then

$$\|y_2(z)\|\leqslant|1+\varepsilon(1-f(z))|\,\|x(z)\|=|1-\varepsilon^2 f_0(z)|\,\|x(z)\|<\frac{1+\varepsilon^2}{1+\varepsilon}<1.$$

And, if $z\in U^c_{z_0}$, $\|y_2(z)\|\leqslant\|x(z)\|\leqslant 1$. Therefore, $\|y_2\|\leqslant 1$. Consequently, $x$ is not an extreme point of $\Gamma_1$. $\square$

In the following, we give a general form of [14, Proposition 2] (and [12, Corollary 1]) in the setting of Hilbert $C(Z)$-modules. By a diffuse measure $\mu$ on $Z$ we mean a nonnegative measure $\mu$ such that $\mu(V)>0$, for all nonempty open subset $V$ of $Z$ (see [14] and examples therein).

PROPOSITION 2. *Let $\mathscr{M}$ be a full Hilbert $C(Z)$-module with unit vectors. If there is no diffuse measure on $Z$, then the unit ball $\mathscr{M}_1$ of $\mathscr{M}$ has no exposed points. Otherwise, $Ext(\mathscr{M}_1) = Exp(\mathscr{M}_1)$.*

*Proof.* Suppose that there is no diffuse measure on $Z$. Also, suppose on the contrary that $y \in Exp(\mathscr{M}_1)$, i.e., $y$ is an exposed point of $\mathscr{M}_1$. Hence, there is a bounded linear functional $L : \mathscr{M} \to \mathbb{C}$ such that

$$\operatorname{Re} L(y) > \operatorname{Re} L(x) \qquad (x \in \mathscr{M}_1 \setminus \{y\}).$$

Define a functional $\tilde{L} : C(Z) \to \mathbb{C}$ as

$$\tilde{L}(f) = L(y \cdot f) \qquad (f \in C(Z)).$$

It is clear that $\tilde{L}$ is a linear functional. Also, $\tilde{L}$ is bounded. In fact,

$$|\tilde{L}(f)| = |L(y \cdot f)| \leqslant \|L\| \|y \cdot f\| \leqslant \|L\| \|f\| \|y\|,$$

for all $f \in C(Z)$. Now, let $g \in C(Z)_1 \setminus \{1\}$. Since $y$ is an extreme point of $\mathscr{M}_1$, by Proposition 1, $y(z) \neq 0$ for all $z \in Z$. This implies that $y \cdot g \neq y$. Moreover, since $\|y \cdot g\| \leqslant 1$, we have

$$\operatorname{Re} \tilde{L}(g) = \operatorname{Re} L(y \cdot g) < \operatorname{Re} L(y) = \operatorname{Re} \tilde{L}(1).$$

That is, $1$ is an exposed point of $C(Z)_1$ which contradicts [14, Proposition 2].

On the other hand, suppose that there is a diffuse measure $\mu$ on $Z$ with $\mu(Z) = 1$. Let $\Gamma$ be the continuous field of Hilbert spaces on $Z$ which is isomorphic to $\mathscr{M}$. Again, by Proposition 1, $Ext(\Gamma_1) \neq \emptyset$. Let $y \in Ext(\Gamma_1)$ and define a functional $f_y : \Gamma \to \mathbb{C}$ as

$$f_y(x) = \int_Z \langle y, x \rangle d\mu \qquad (x \in \Gamma).$$

It is clear that $f_y$ is linear. Note that $\|\langle y, x \rangle\| = \sup_{z \in Z} |\langle y(z), x(z) \rangle| \leqslant 1$, for all $x \in \Gamma_1$. We have

$$|f_y(x)| = \left| \int_Z \langle y, x \rangle d\mu \right| \leqslant \int_Z \|\langle y, x \rangle\| \, d\mu \leqslant 1 \qquad (x \in \Gamma_1).$$

Hence, $f_y$ is a bounded linear functional. Since $y$ is a unit vector, we have $f_y(y) = 1$. We show that $f_y$ is an exposing functional, i.e.,

$$\operatorname{Re} f_y(x) < \operatorname{Re} f_y(y) = 1, \qquad (x \in \Gamma_1 \setminus \{y\}).$$

Let $x \in \Gamma_1 \setminus \{y\}$. There is some $z_0 \in Z$ such that $x(z_0) \neq y(z_0)$. We show that $\langle y(z_0), x(z_0) \rangle \neq 1$. Suppose on the contrary that

$$\langle y(z_0), x(z_0) \rangle = \langle y(z_0), y(z_0) \rangle = 1. \tag{1}$$

By the Cauchy-Schwartz inequality,

$$1 = \langle y(z_0), x(z_0) \rangle \leqslant \|y(z_0)\| \|x(z_0)\| \leqslant 1. \tag{2}$$

Consequently, there is some $\lambda \in \mathbb{C}$ such that $x(z_0) = \lambda y(z_0)$. Moreover, by the equation 1, since $\langle y(z_0), x(z_0)\rangle = \langle y(z_0), \lambda y(z_0)\rangle = 1$, we have $\lambda = 1$. This contradicts $y(z_0) \neq x(z_0)$. Therefore, $\mathrm{Re}\,\langle \mathrm{y(z_0), x(z_0)}\rangle < 1$. Now, by the continuity of the map

$$z \mapsto \mathrm{Re}(\langle y(z), x(z)\rangle)$$

there is some open set $U_{z_0}$ containing $z_0$ such that $\mathrm{Re}\,\langle \mathrm{y(z), x(z)}\rangle < 1$, for all $z \in U_{z_0}$. Moreover, since $\mu$ is a diffuse measure, $\mu(U_{z_0}) > 0$. Thus, $\mathrm{Re}\, f_y(x) = \int_Z \mathrm{Re}\,\langle y, x\rangle d\mu < 1$. $\square$

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