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MATRIX SPLITTING AND IDEALS IN $\mathcal{B}(\mathcal{H})$

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Abstract. We investigate the relationship between ideal membership of an operator and its pieces relative to several canonical types of partitions of the entries of its matrix representation with respect to a given orthonormal basis. Our main theorems establish that if T lies in an ideal $\mathcal{I}$, then $\sum P_nTP_n$ (or more generally $\sum Q_nTP_n$) lies in the arithmetic mean closure of $\mathcal{I}$ whenever $\{P_n\}$ (and also $\{Q_n\}$) is a sequence of mutually orthogonal projections; and in any basis for which T is a block band matrix, in particular, when in Patnaik–Petrovic–Weiss universal block tridiagonal form, then all the sub/super/main-block diagonals of T are in $\mathcal{I}$. And in particular, the principal ideal generated by this T is the finite sum of the principal ideals generated by each sub/super/main-block diagonal.

1. Introduction

In the study of infinite matrix representations of operators in $\mathcal{B}(\mathcal{H})$, and especially the structure of commutators, it is common and natural to split up a target operator T into a sum of two (or a finite sum) of its natural parts. For example, every finite matrix is the sum of its upper triangular part and its lower triangular part (including the diagonal in either part as you choose).

Formally this obviously holds also for infinite matrices, but not in $\mathcal{B}(\mathcal{H})$. That is, as is well-known, the upper or lower triangular part of a matrix representation for a bounded operator is not necessarily a bounded operator. The Laurent operator with zero-diagonal matrix representation $(\frac{1}{i-j})_{i\neq j}$ represents a bounded operator but its upper and lower triangular parts represent unbounded operators. From this we can produce a compact operator whose upper triangular part is unbounded.

EXAMPLE 1. Consider the zero-diagonal Laurent matrix $(\frac{1}{i-j})_{i\neq j}$, which corresponds to the Laurent multiplication operator $M_\varphi \in \mathcal{B}(L^2(\mathbb{S}^1))$ where, in $L^2(\mathbb{S}^1)$,

$$\varphi(z) := \sum_{0\neq n\in\mathbb{Z}} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{\overline{z}^n}{n} = \log(1-z) - \log(1-\overline{z}) = \log\left(\frac{1-z}{\overline{1-z}}\right),$$

which is bounded since it is the principle logarithm of a unit modulus function, so $\varphi \in L^\infty(\mathbb{S}^1)$. On the other hand, the upper triangular part $\Delta(M_\varphi)$ of M_φ corresponds to

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multiplication by $\log(1-z) \notin L^\infty(\mathbb{S}^1)$, and is therefore not a bounded operator. Additionally, as is well-known, the same boundedness/unboundedness properties are shared by the corresponding Toeplitz operator T_φ and its $\Delta(T_\varphi)$. Indeed, this follows from the fact that if $P \in \mathcal{B}(L^2(\mathbb{S}^1))$ is the projection onto the Hardy space H^2, then $PM_\varphi P$ and $P^\perp M_\varphi P^\perp$ are unitarily equivalent, and $PM_\varphi P^\perp = P\Delta(M_\varphi)P^\perp$ is bounded.

To produce a compact operator whose upper triangular part is unbounded, start by taking successive corners $P_n T_\varphi P_n$ where P_n is the projection onto the polynomials of degree at most n. Relative to the orthonormal monomials $\{z^k\}_{k=0}^n$, the matrix representation for $P_n\Delta(T_\varphi)P_n$ is an upper triangular Toeplitz matrix whose first row is $(0,1,1/2,\dots,1/n)$. Consider the unit vector $q(z) := (z+\cdots+z^n)/\sqrt{n}$, which is represented as $(0,1,\dots,1)^*/\sqrt{n}$ relative to this basis. Then a straightforward computation produces

$$\|P_n\Delta(T_\varphi)P_n q\|^2 = \frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{j=1}^{n-k}\frac{1}{j}\right)^2 \geqslant \frac{1}{n}\sum_{k=0}^{n-1}\log^2(n-k) \geqslant \frac{\log^2(n/2)}{2}.$$

Therefore, $\|P_n\Delta(T_\varphi)P_n\| \to \infty$.

Then set

$$K := \bigoplus_{n=1}^{\infty} \frac{P_n T_\varphi P_n}{\|P_n\Delta(T_\varphi)P_n\|^{1/2}}, \quad \text{so that} \quad \Delta(K) = \bigoplus_{n=1}^{\infty} \frac{P_n \Delta(T_\varphi) P_n}{\|P_n\Delta(T_\varphi)P_n\|^{1/2}}.$$

Notice that K is compact, being a direct sum of finite rank operators whose norms

$$\frac{\|P_n T_\varphi P_n\|}{\|P_n\Delta(T_\varphi)P_n\|^{1/2}} \leqslant \frac{\|T_\varphi\|}{\|P_n\Delta(T_\varphi)P_n\|^{1/2}} \to 0.$$

Moreover, its upper triangular part $\Delta(K)$ is unbounded, being a direct sum of finite rank operators with norms $\|P_n\Delta(T_\varphi)P_n\|^{1/2}$ approaching infinity. Similarly, the operator

$$K' := \bigoplus_{n=1}^{\infty} \frac{P_n T_\varphi P_n}{\|P_n\Delta(T_\varphi)P_n\|}$$

is compact, but its upper triangular part $\Delta(K')$ is bounded and noncompact.

Focusing attention on $\mathcal{B}(\mathcal{H})$ ideals yields a fruitful area of study: for a Hilbert–Schmidt operator, in any basis, any partition of the entries of its matrix representation has its parts again Hilbert–Schmidt.[1] This leads to a natural question for which the authors are unaware of the answer: is the Hilbert–Schmidt ideal the *only* (nonzero) ideal with this property?

For the compact operators $\mathcal{K}(\mathcal{H})$, depending on the shape of the matrix parts for T, the problem of determining when its parts are in $\mathcal{K}(\mathcal{H})$ (i.e., ideal invariant) can be

[1]Of course, for any ideal $\mathcal{I}$ contained within the Hilbert–Schmidt ideal $\mathcal{L}_2$, and any $T \in \mathcal{I}$, the upper triangular part $\Delta(T) \in \mathcal{L}_2$, but one may wonder if anything stronger can be said. In the case of the trace-class ideal $\mathcal{L}_1$, Gohberg–Krein [9, Theorem III.2.1] showed that $\Delta(T)$, in the terminology of [6], lies in the arithmetic mean closure of the principal ideal generated by $\operatorname{diag}(1/n)$.

a little subtler. Indeed, as noted in Example 1, the upper triangular part of a compact operator may not be compact (nay bounded); on the other hand, it is well-known and elementary that the diagonal sequence (d_n) of a compact operator converges to zero (i.e., $\operatorname{diag}(d_n)$ is compact), and the same holds for all the sub/super-diagonals as well. In contrast, this fails for certain matrix representations for a finite rank operator; that is, the diagonal of a finite rank operator may not be finite rank (e.g., $(\frac{1}{ij})_{i,j\geqslant 1}$ is rank-1 but its diagonal $\operatorname{diag}(\frac{1}{j^2}) \notin \mathcal{F}(\mathcal{H})$).

Here we study this question for general $\mathcal{B}(\mathcal{H})$-ideals: For an ideal $\mathcal{I}$ and all pairs $\{P_n\},\{Q_n\}$ of sequences of mutually orthogonal projections, when are the generalized diagonals $\sum Q_nTP_n \in \mathcal{I}$ whenever $T \in \mathcal{I}$? (The canonical block diagonals are $\sum P_{n+k}TP_n$ and $\sum P_nTP_{n+k}$, where $\sum P_n = I$.) We find this especially pertinent in our current search for commutator forms of compact operators [16], growing out of [1]; and, in view of the second author's work with V. Kaftal [11] on diagonal invariance for ideals, useful in recent discoveries by the authors with S. Petrovic and S. Patnaik [17] on their universal finite-block tridiagonalization for arbitrary $\mathcal{B}(\mathcal{H})$ operators and the consequent work on commutators [16].

Evolution of questions:

1. For which $\mathcal{B}(\mathcal{H})$-ideals $\mathcal{I}$ does a tridiagonal operator $T \in \mathcal{I}$ have its three diagonal parts also in $\mathcal{I}$? This question arose from the stronger question: for which tridiagonal operators $T \in \mathcal{K}(\mathcal{H})$ are the diagonals parts in $\langle T \rangle$? Theorem 4 guarantees the latter is always true, even for finite band operators.

2. The same questions but more generally for a block tridiagonal T (see Definition 2) and its three block diagonals (see Definition 3). Again, Theorem 4 guarantees this is always true, and likewise for finite block band operators. That is, if $T = \begin{pmatrix} B & A & 0 & \\ C & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} \in \mathcal{I}$, then $\begin{pmatrix} 0 & A & 0 & \\ 0 & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} \in \mathcal{I}$, and similarly for B,C.

3. A more general context: given two sequences of (separately) mutually orthogonal projections, $\{P_n\}_{n=1}^{\infty},\{Q_n\}_{n=1}^{\infty}$, for $T \in \mathcal{I}$ what can be said about ideal membership for $\sum_{n=1}^{\infty} Q_nTP_n$? In Theorem 9 we establish that $\sum_{n=1}^{\infty} Q_nTP_n$ always lies in the arithmetic mean closure $\overline{\mathcal{I}}^{am}$ defined in [6] (see herein page 363). This follows from a generalization (see Theorem 8) of Fan's famous submajorization theorem [7, Theorem 1] concerning partial sums of diagonals of operators.

Throughout the paper we will prefer bi-infinite sequences (i.e., indexed by $\mathbb{Z}$ instead of $\mathbb{N}$) of projections, but this is only to make the descriptions simpler; we will not, however, use the term *bi-infinite* unless necessary for context. The projections are allowed to be zero, so this is no restriction. We first establish some terminology.

DEFINITION 2. A sequence $\{P_n\}_{n\in\mathbb{Z}}$ of mutually orthogonal projections $P_n \in \mathcal{B}(\mathcal{H})$ for which $\sum P_n = I$ is a *block decomposition* and for $T \in \mathcal{B}(\mathcal{H})$, partitions it into a (bi-)infinite matrix of operators $T_{i,j} := P_iTP_j$.

We say that an operator T is a *block band operator relative to* $\{P_n\}$ if there is some $M \geqslant 0$, called the *block bandwidth*, for which $T_{i,j} = 0$ whenever $|i-j| > M$. If

$M = 0$ (resp. $M = 1$), we say T is *block diagonal (resp. block tridiagonal) relative to* $\{P_n\}$.

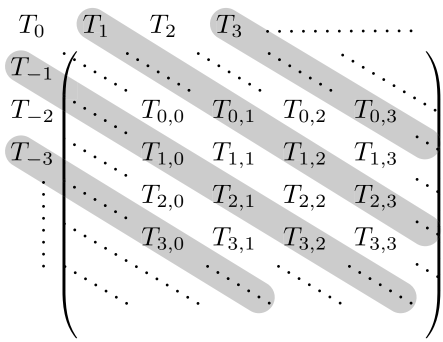
Finally, in all the above definitions, if $\operatorname{Tr} P_n \leqslant 1$ for all $n \in \mathbb{Z}$, which, up to a choice of phase for each range vector, simply corresponds to a choice of orthonormal basis, then we omit the word "block." In this case, the operators $T_{i,j}$ are scalars and $(T_{i,j})$ is the matrix representation (again, up to a choice of phase for each vector) for T relative to this basis.

If $\{Q_n\}_{n\in\mathbb{Z}}$ is an (unrelated) block decomposition, the pair $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$ still determines a (bi-)infinite matrix of operators $T_{i,j} = Q_i T P_j$, but this time there is an inherent asymmetry in that $(T^*)_{i,j} \neq (T_{j,i})^*$. In this case, all the terms defined just above may be modified with the adjective "asymmetric."

DEFINITION 3. Suppose that $\{P_n\}_{n\in\mathbb{Z}}$ is a block decomposition for an operator $T \in \mathcal{B}(\mathcal{H})$. For each $k \in \mathbb{Z}$, we call

$$T_k := \sum_{n\in\mathbb{Z}} T_{n,n+k} = \sum_{n\in\mathbb{Z}} P_n T P_{n+k}$$

the k^{th} *block diagonal* of T, which converges in the strong operator topology. Visually, these operators may be described with the following diagram[2]



We call the collection $\{T_k\}_{k\in\mathbb{Z}}$ the *shift decomposition* of T (relative to the block decomposition $\{P_n\}_{n\in\mathbb{Z}}$). The *asymmetric shift decomposition* $\{T_k\}_{k\in\mathbb{Z}}$ relative to *different* block decompositions $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$ is given by

$$T_k := \sum_{n\in\mathbb{Z}} Q_n T P_{n+k}.$$

We note for future reference that sums of the above form don't require the sequences of projections to sum to the identity in order to converge in the strong operator topology, only that each sequence consists of mutually orthogonal projections. Moreover, it is elementary to show that when T is compact, so is T_k for all $k \in \mathbb{Z}$.

[2]For the case when the projections $P_n = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$, the matrix below is uni-infinite. This recovers uni-infinite matrix results from the bi-infinite approach we described in the paragraph preceding Definition 2.

REMARK 1. Although one has the formal equality $T = \sum_{k\in\mathbb{Z}} T_k$ in the sense that T is uniquely determined by $\{T_k\}_{k\in\mathbb{Z}}$, this sum doesn't necessarily converge even in the weak operator topology [13], hence it doesn't necessarily converge in any of the usual operator topologies. If $\operatorname{rank} P_n = 1$ (and $Q_n = P_n$) for all $n \in \mathbb{Z}$ then $\sum_{k\in\mathbb{Z}} T_k$ does converge to T in the *Bures topology*[3] [2, 13]. On the other hand, if T is a block band operator relative to this block decomposition, then convergence is irrelevant: $T = \sum_{k=-M}^{M} T_k$.

The reason for our "shift" terminology in Definition 3 is that if the block decomposition $\{P_n\}_{n\in\mathbb{Z}}$ consists of rank-1 projections, then the operators T_k have the form $T_k = U^k D_k$ where D_k are diagonal operators and U is the bilateral shift relative to any orthonormal basis corresponding to $\{P_n\}_{n\in\mathbb{Z}}$.

REMARK 2. All compact selfadjoint operators are diagonalizable via the spectral theorem. However, this is certainly not the case for arbitrary selfadjoint operators, the selfadjoint approximation theorem of Weyl–von Neumann notwithstanding. Nevertheless, every selfadjoint operator with a cyclic vector is *tri*diagonalizable; for $T = T^*$ with cyclic vector v, apply Gram–Schmidt to the linearly independent spanning collection $\{T^n v\}_{n=0}^{\infty}$ and then T is tridiagonal in the resulting orthonormal basis. Consequently, every selfadjoint operator is block diagonal with each nonzero block in the direct sum itself tridiagonal.

The second author, along with Patnaik and Petrovic [17, 16], recently established that every bounded operator is *block* tridiagonalizable, meaning $T = T_{-1} + T_0 + T_1$, hence block banded (with block bandwidth 1) and with finite block sizes growing no faster than exponentially.

Our first main theorem is an algebraic equality of ideals for block band operators relative to some block decomposition.

THEOREM 4. *Let $T \in \mathcal{B}(\mathcal{H})$ be an asymmetric block band operator of bandwidth M relative to the block decompositions $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$, and let $\{T_k\}_{k=-M}^{M}$ be the asymmetric shift decomposition of T. Then the following ideal equality holds:*

$$\langle T \rangle = \sum_{k=-M}^{M} \langle T_k \rangle.$$

Proof. The ideal inclusion $\langle T \rangle \subseteq \sum_{k=-M}^{M} \langle T_k \rangle$ is obvious since $T = \sum_{k=-M}^{M} T_k$. Therefore it suffices to prove $T_k \in \langle T \rangle$ for each $-M \leqslant k \leqslant M$. Of course, when $T \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$, then $\langle T \rangle = \mathcal{B}(\mathcal{H})$ and so $T_k \in \langle T \rangle$ is trivial. Therefore we only need to address the case when $T \in \mathcal{K}(\mathcal{H})$.

[3]The Bures topology on $B(\mathcal{H})$ is a locally convex topology constructed from the (rank-1) projections P_n as follows. Let $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} P_n B(\mathcal{H}) P_n$ be the algebra of diagonal matrices and $E : B(\mathcal{H}) \to \mathcal{D}$ the conditional expectation given by $T \mapsto T_0 := \sum_{n\in\mathbb{Z}} P_n T P_n$. Then to each $\omega \in \ell_1 \cong \mathcal{D}_*$, associate the seminorm $T \mapsto \operatorname{Tr}(\operatorname{diag}(\omega) E(T^*T)^{1/2})$, where $\operatorname{diag} : \ell_\infty \to \mathcal{D}$ is the natural *-isomorphism. These seminorms generate the Bures topology.

The remainder of the proof is essentially due to the following observation: if you zoom out and squint, then a band matrix looks diagonal. That is, we exploit the relative thinness of the diagonal strip of support entries.

Indeed, for $-M \leqslant j,k \leqslant M$ define projections $R_{k,j} := \sum_{n\in\mathbb{Z}} P_{n(2M+1)+j+k}$ and $S_j := \sum_{n\in\mathbb{Z}} Q_{n(2M+1)+j}$ Then whenever $n \neq m$, $Q_{n(2M+1)+j}TP_{m(2M+1)+j+k} = 0$ since the bandwidth of T is M and

$$\begin{aligned} |(n(2M+1)+j)-(m(2M+1)+j+k)| &\geqslant |n-m|(2M+1)-k \\ &\geqslant (2M+1)-M > M. \end{aligned}$$

Therefore, for each k,j,

$$S_jTR_{k,j} = \sum_{n\in\mathbb{Z}} Q_{n(2M+1)+j}TP_{n(2M+1)+j+k}$$

converges in the strong operator topology, and summing over j yields

$$\sum_{j=-M}^{M} S_jTR_{k,j} = \sum_{j=-M}^{M}\sum_{n\in\mathbb{Z}} Q_{n(2M+1)+j}TP_{n(2M+1)+j+k} = \sum_{n\in\mathbb{Z}} Q_nTP_{n+k} = T_k.$$

As a finite sum, the left-hand side is trivially in $\langle T\rangle$ and therefore so is each k^{th} generalized block diagonal T_k. □

Before establishing our second main theorem (Theorem 9), we acquaint the reader with the prerequisite ideas concerning Fan's theorem [7, Theorem 1], Hardy–Littlewood submajorization, fundamentals of the theory of operator ideals and arithmetic mean closed ideals, all of which are intimately related.

For a single operator, Fan's submajorization theorem [7, Theorem 1] states that if the matrix representation for a compact operator $T \in \mathcal{K}(\mathcal{H})$ has diagonal sequence $(d_j)_{j\in J}$ (with any index set J), then

$$\sum_{n=1}^{m} |d_n|^* \leqslant \sum_{n=1}^{m} s_n(T) \quad \text{for all } m \in \mathbb{N}, \tag{1}$$

where $s(T) := (s_n(T))_{n\in\mathbb{N}}$ denotes the (monotone) *singular value sequence* of T, and where $(|d_n|^*)_{n\in\mathbb{N}}$ denotes the *monotonization*[4] of the (possibly unordered) sequence $(|d_j|)_{j\in J}$; the monotonization is always an element of the convex cone c_0^* of nonnegative nonincreasing sequences (indexed by $\mathbb{N}$) converging to zero, even when $(|d_j|)_{j\in J}$ is indexed by another set J different from $\mathbb{N}$. The set of inequalities (1) may be encapsulated, for pairs of sequences in c_0^*, by saying that $(|d_n|^*)$ is *submajorized* by $s(T)$, which is often denoted $(|d_n|^*) \prec\!\!\prec s(T)$, although the precise notation for submajorization varies throughout the literature. If, in addition, the infinite sums are equal (allowing also for the case $\infty = \infty$), then we say that $(|d_n|^*)$ is *majorized* by $s(T)$, which is often

[4]This is the measure-theoretic *nonincreasing rearrangement* relative to the counting measure on the index set, say J, of $(|d_n|)$. Associated to this, there is a injection (not necessarily a bijection) $\pi : \mathbb{N} \to J$ with $d^{-1}(\mathbb{C}\setminus\{0\}) \subseteq \pi(\mathbb{N})$ such that $|d_n|^* = |d_{\pi(n)}|$. This of course requires $0 \notin (d\circ\pi)(\mathbb{N})$ when $d^{-1}(\mathbb{C}\setminus\{0\})$ is infinite since $(|d_n|^*)$ is nonincreasing.

denoted $(|d_n|^*) \prec s(T)$. We remark the trivial fact that the submajorization order is finer than the usual pointwise order on c_0^*; that is, $(a_n) \leqslant (b_n)$ implies $(a_n) \prec\!\!\!\prec (b_n)$ for any $(a_n), (b_n) \in c_0^*$.

However, we view Fan's theorem in a slightly different way which is more amenable to our purposes. In particular, consider the canonical trace-preserving conditional expectation[5] $E : \mathcal{B}(\mathcal{H}) \to \mathcal{D}$ onto the masa (maximal abelian selfadjoint algebra) of diagonal operators relative to a fixed, but arbitrary, orthonormal basis. Then the sequence $(|d_n|^*)$ is simply $s(E(T))$, and in this language:

THEOREM 5. ([7, Theorem 1]) *If $T \in \mathcal{K}(\mathcal{H})$ and $E : \mathcal{B}(\mathcal{H}) \to \mathcal{D}$ is the canonical conditional expectation onto a masa of diagonal operators, then*

$$s(E(T)) \prec\!\!\!\prec s(T),$$

that is, $s(E(T))$ is submajorized by $s(T)$.

The submajorization order features prominently in operator theory, but especially in the theory of diagonals of operators and in the related theory of operator ideals in $\mathcal{B}(\mathcal{H})$.

For the reader's convenience we briefly review the basics of ideal theory. Let c_0^* denote the convex cone of nonnegative nonincreasing sequences converging to zero. To an ideal $\mathcal{I}$, Schatten [19], in a manner quite similar to Calkin [3], associated the convex subcone $\Sigma(\mathcal{I}) := \{s(T) \in c_0^* \mid T \in \mathcal{I}\}$, called the *characteristic set* of $\mathcal{I}$, which satisfies the properties:

1. If $(a_n) \leqslant (b_n)$ (pointwise) and $(b_n) \in \Sigma(\mathcal{I})$, then $(a_n) \in \Sigma(\mathcal{I})$; that is, $\Sigma(\mathcal{I})$ is a *hereditary subcone* of c_0^* with respect to the usual pointwise ordering.

2. If $(a_n) \in \Sigma(\mathcal{I})$, then $(a_{\lceil \frac{n}{2} \rceil}) \in \Sigma(\mathcal{I})$; that is, $\Sigma(\mathcal{I})$ is closed under 2-*ampliations*.

Likewise, if S is a hereditary (with respect to the pointwise order) convex subcone of c_0^* which is closed under 2-ampliations, then $\mathcal{I}_S := \{T \in \mathcal{K}(\mathcal{H}) \mid s(T) \in S\}$ is an ideal of $\mathcal{B}(\mathcal{H})$. Finally, the maps $S \mapsto \mathcal{I}_S$ and $\mathcal{I} \mapsto \Sigma(\mathcal{I})$ are inclusion-preserving inverses of each other between the classes of $\mathcal{B}(\mathcal{H})$-ideals and characteristic subsets of c_0^*.

Ideals whose characteristic sets are also hereditary subcones with respect to the submajorization order (i.e., $B \in \mathcal{I}$ and $s(A) \prec\!\!\!\prec s(B)$ implies $A \in \mathcal{I}$) were introduced by

[5]For an inclusion of unital C*-algebras $\mathcal{B} \subseteq \mathcal{A}$ (with $1_{\mathcal{B}} = 1_{\mathcal{A}}$), a *conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$* is a unital positive linear map $E : \mathcal{A} \to \mathcal{B}$ such that $E(bab') = bE(a)b'$ for all $a \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. A conditional expectation is called *faithful* if $a \geqslant 0$ and $E(a) = 0$ imply $a = 0$. If $\mathcal{A}$ is a semifinite von Neumann algebra with a faithful normal semifinite trace τ, then the expectation is said to be *trace-preserving* if $\tau(a) = \tau(E(a))$ for all $a \in \mathcal{A}_+$.

Dykema, Figiel, Weiss and Wodzicki[6] in [6] and are said to be *arithmetic mean closed*[7] (abbreviated as *am-closed*). Given an ideal $\mathcal{I}$, the smallest am-closed ideal containing $\mathcal{I}$ is called the am-closure, denoted $\overline{\mathcal{I}}^{am}$, and its characteristic set consists simply of the hereditary closure (with respect to the submajorization order) of $\Sigma(\mathcal{I})$. That is,

$$\Sigma(\overline{\mathcal{I}}^{am}) = \left\{(a_n) \in c_0^* \mid \exists (b_n) \in \Sigma(\mathcal{I}), (a_n) \prec\!\!\!\prec (b_n)\right\}.$$

In general, ideals are not am-closed. Indeed, the sequence $(1,0,0,\dots)$ corresponding to a rank-1 projection P submajorizes any (nonnegative) sequence (a_n) whose sum is at most 1. Consequently, if $T \in \mathcal{L}_1$, the trace class, then $s(T) \prec\!\!\!\prec s(\mathrm{Tr}(|T|)P)$. Therefore, since any nonzero ideal $\mathcal{I}$ contains the finite rank operators, if $\mathcal{I}$ is am-closed it must also contain the trace class $\mathcal{L}_1$. Additionally, it is immediate that $\mathcal{L}_1$ is am-closed, making it the minimum am-closed ideal and the am-closure of the ideal of finite rank operators.

Arithmetic mean closed ideals are important within the lattice of operator ideals not least for their connection to Fan's theorem, but also because of the following sort of converse due to the second author with Kaftal, a characterization of diagonally invariant ideals.

THEOREM 6. ([11, Corollary 4.4 and Theorem 4.5]) *For an operator ideal $\mathcal{I}$, and the canonical conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \mathcal{D}$ onto a masa of diagonal operators,*

$$E(\mathcal{I}) = \overline{\mathcal{I}}^{am} \cap \mathcal{D}.$$

Consequently, $\mathcal{I}$ is am-closed if and only if $E(\mathcal{I}) \subseteq \mathcal{I}$.

They used the term *diagonal invariance* to refer to $E(\mathcal{I}) \subseteq \mathcal{I}$, and so $\mathcal{I}$ is am-closed if and only if it is diagonally invariant. The reader should note that the inclusion

[6] The description given in [6] is not in terms of the submajorization order, but these two definitions are easily shown to be equivalent. Instead, for an ideal $\mathcal{I}$, [6] defines the *arithmetic mean ideal* $\mathcal{I}_a$ and *pre-arithmetic mean ideal* ${}_a\mathcal{I}$ whose characteristic sets are given by

$$\Sigma(\mathcal{I}_a) := \left\{(a_n) \in c_0^* \middle| \exists (b_n) \in \Sigma(\mathcal{I}), a_n \leqslant \frac{1}{n}\sum_{k=1}^{n} b_k\right\}, \quad \Sigma({}_a\mathcal{I}) := \left\{(a_n) \in c_0^* \middle| \exists (b_n) \in \Sigma(\mathcal{I}), \frac{1}{n}\sum_{k=1}^{n} a_k \leqslant b_n\right\}$$

Then the *arithmetic mean closure* of $\mathcal{I}$ is $\overline{\mathcal{I}}^{am} := {}_a(\mathcal{I}_a)$, and $\mathcal{I}$ is called *am-closed* if $\mathcal{I} = \overline{\mathcal{I}}^{am}$ This viewpoint also allows one to define the *arithmetic mean interior* $({}_a\mathcal{I})_a$, and one always has the inclusions ${}_a\mathcal{I} \subseteq ({}_a\mathcal{I})_a \subseteq \mathcal{I} \subseteq {}_a(\mathcal{I}_a) \subseteq \mathcal{I}_a$.

[7] Although am-closed ideals were introduced in this generality by [6], they had been studied at least as early as [8, 18], but only in the context of *symmetrically normed ideals*. In the study of symmetrically normed ideals by Gohberg and Krein [8], they only considered those which were already am-closed, but they did not have any terminology associated to this concept.

Around the same time, both Mityagin [14] and Russu [18] concerned themselves with the existence of so-called *intermediate* symmetrically normed ideals, which are necessarily not am-closed, or in the language of Russu, do not possess the *majorant property*. In [18], Russu also established that the majorant property is equivalent to the *interpolation property* studied by Mityagin [15] and Calderón [4].

In the modern theory of symmetrically normed ideals, those which are am-closed (equivalently, have the majorant or interpolation properties), are said to be *fully symmetric*, but this term also implies the norm preserves the submajorization order. For more information on fully symmetrically normed ideals and related topics, we refer the reader to [12].

$E(\mathcal{I}) \subseteq \overline{\mathcal{I}}^{am} \cap \mathcal{D}$ is a direct consequence of Fan's theorem, when viewed through the lens of Theorem 5, so the new content of Theorem 6 lies primarily in the reverse inclusion.

At this point, we note an important contrapositive consequence of Theorem 4 and Theorem 6, as well as the Schur–Horn theorem for positive compact operators [10, Proposition 6.4 and Proposition 6.6].

COROLLARY 7. *Let $E : B(\mathcal{H}) \to \mathcal{D}$ onto a masa of diagonal operators.*

1. *If $\mathcal{I} \subsetneq \overline{\mathcal{I}}^{am}$ then $E(\mathcal{I}) \setminus \mathcal{I}$ is non-empty, and no operator in this set is a band operator with respect to the basis corresponding $\mathcal{D}$.*

2. *If T is a positive compact operator for which its principal ideal is not am-closed, i.e. $\langle T \rangle \subsetneq \overline{\langle T \rangle}^{am}$ then there is some unitary U for which $E(UTU^*) \notin \langle T \rangle$, and consequently UTU^* is not a band operator with respect to the basis corresponding to $\mathcal{D}$.*

Proof.

1. That $E(\mathcal{I})$ is nonempty follows immediately from Theorem 6. Then take any $E(T) \in E(\mathcal{I}) \setminus \mathcal{I}$. Since $T \in \mathcal{I}$, $E(T) \notin \langle T \rangle$, and so by the contrapositive of Theorem 4 (note $E(T) = T_0$), it follows that T is not a band operator with respect to the basis corresponding to $\mathcal{D}$.

2. It suffices to prove the contrapositive, namely, if $E(UTU^*) \in \langle T \rangle$ for all unitary operators U, then $\langle T \rangle$ is am-closed; indeed, then the portion of 2 after "consequently" follows from 1. Take any sequence $(a_n) \in \Sigma(\overline{\langle T \rangle}^{am})$. Then there is a sequence $(b_n) \in \Sigma(\langle T \rangle)$ with $(a_n) \preccurlyeq (b_n)$. Moreover, there is some $c > 0$ and some $m \in \mathbb{N}$ such that $(b_n) \leqslant cD_m(s(T))$, where D_m denotes the m-ampliation operator (i.e., $D_m(s_n) = (s_{\lceil \frac{n}{m} \rceil})$). Therefore,

$$(a_n) \preccurlyeq (b_n) \leqslant cD_m(s(T)) \preccurlyeq (cm)s(T).$$

Then by [11, Theorem 3.4][8] there is some intermediate sequence (d_n) for which

$$\frac{1}{cm}(a_n) \leqslant (d_n) \prec s(T).$$

Then, by the Schur–Horn theorem for positive compact operators [10, Proposition 6.4 and Proposition 6.6], there is some unitary operator U such that $s(E(UTU^*)) = (d_n)$. By hypothesis, $E(UTU^*) \in \langle T \rangle$ and hence $(d_n) \in \Sigma(\langle T \rangle)$. Therefore, $\frac{1}{cm}(a_n) \in \Sigma(\langle T \rangle)$ and so $(a_n) \in \Sigma(\langle T \rangle)$. Hence $\Sigma(\overline{\langle T \rangle}^{am}) \subseteq \Sigma(\langle T \rangle)$ and thus $\langle T \rangle = \overline{\langle T \rangle}^{am}$ is am-closed. □

[8]reader be aware: the notation differs slightly

The next theorem, due originally to Gohberg–Krein [8, Theorems II.5.1 and III.4.2], bootstraps Theorem 5 to apply to conditional expectations onto block diagonal algebras instead of simply diagonal masas. We include this more modern proof both for completeness and to make the statement accord with that of Theorem 5.

THEOREM 8. *Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{Z}}$ be a block decomposition and consider the associated conditional expectation $E_{\mathcal{P}} : \mathcal{B}(\mathcal{H}) \to \bigoplus_{n\in\mathbb{Z}} P_n\mathcal{B}(\mathcal{H})P_n$ defined by $E_{\mathcal{P}}(T) := T_0 = \sum_{n\in\mathbb{Z}} P_nTP_n$. If $T \in \mathcal{K}(\mathcal{H})$, then $s(E_{\mathcal{P}}(T))$ is submajorized by $s(T)$, i.e.,*

$$s(E_{\mathcal{P}}(T)) \prec\!\!\prec s(T).$$

Moreover, if $T \in \mathcal{I}$, then $E_{\mathcal{P}}(T) \in \overline{\mathcal{I}}^{am}$ In addition, if $s(E_{\mathcal{P}}(T)) = s(T)$, then $E_{\mathcal{P}}(T) = T$.

Proof. Suppose that $\mathcal{D}$ is a diagonal masa contained in the algebra $\bigoplus_{n\in\mathbb{Z}} P_n\mathcal{B}(\mathcal{H})P_n$, and let $E : \mathcal{B}(\mathcal{H}) \to \mathcal{D}$ be the associated canonical trace-preserving conditional expectation. Because of the algebra inclusion $\mathcal{D} \subseteq \bigoplus_{n\in\mathbb{Z}} P_n\mathcal{B}(\mathcal{H})P_n$, we see that $E \circ E_{\mathcal{P}} = E$.

Let $T \in \mathcal{K}(\mathcal{H})$ and consider $E_{\mathcal{P}}(T)$. By applying the Schmidt decomposition to each P_nTP_n one obtains partial isometries U_n, V_n (the latter may even be chosen unitary) in $P_n\mathcal{B}(\mathcal{H})P_n$ so that $U_nP_nTP_nV_n$ is a positive operator in $\mathcal{D}$. Then $U := \bigoplus_{n\in\mathbb{Z}} U_n, V := \bigoplus_{n\in\mathbb{Z}} V_n$ are partial isometries for which $s(E(U(E_{\mathcal{P}}(T)V)$. Then since $U,V \in \bigoplus_{n\in\mathbb{Z}} P_n\mathcal{B}(\mathcal{H})P_n$ they commute with the conditional expectation $E_{\mathcal{P}}$ and hence

$$s(E_{\mathcal{P}}(T)) = s(E(U(E_{\mathcal{P}}(T)V) = s(E(E_{\mathcal{P}}(UTV))) = s(E(UTV)).$$

By Fan's theorem (Theorem 5), $s(E(UTV)) \prec\!\!\prec s(UTV) \leqslant \|U\|s(T)\|V\| = s(T)$, and therefore $s(E_{\mathcal{P}}(T)) \prec\!\!\prec s(T)$. Finally, this fact along with the definition of the arithmetic mean closure guarantees $T \in \mathcal{I}$ implies $E_{\mathcal{P}}(T) \in \overline{\mathcal{I}}^{am}$

For the case of equality, now suppose that $s(E_{\mathcal{P}}(T)) = s(T)$. By diagonalizing each block of $E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)$ independently, let $\{e_n\}_{n\in\mathbb{N}}$ be its orthonormal sequence of eigenvectors, each of which is in one of the subspaces $P_j\mathcal{H}$, satisfying $E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)e_n = s_n(T)^2e_n$. Then the projections Q_n onto $\operatorname{span}\{e_1,\dots,e_n\}$ commute with each P_j, and hence also with the expectation $E_{\mathcal{P}}$. We note for later reference that

$$\|E_{\mathcal{P}}(T)Q_n^{\perp}\|^2 = \|Q_n^{\perp}E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)Q_n^{\perp}\| \leqslant s_{n+1}(E_{\mathcal{P}}(T))^2. \tag{2}$$

Observe that for any operator X, because $P_jX^*P_jXP_j \leqslant P_jX^*XP_j$

$$E_{\mathcal{P}}(X)^*E_{\mathcal{P}}(X) = \sum_{j\in\mathbb{Z}} P_jX^*P_jXP_j \leqslant \sum_{j\in\mathbb{Z}} P_jX^*XP_j = E_{\mathcal{P}}(X^*X), \tag{3}$$

with equality if and only if $P_jX^*P_j^{\perp}XP_j = 0$ for all $j \in \mathbb{Z}$ if and only if $P_j^{\perp}XP_j = 0$ for all $j \in \mathbb{Z}$ if and only if $X = E_{\mathcal{P}}(X)$.

Applying (3) to $X = TQ_n$,

$$\sum_{j=1}^{n} s_j(E_{\mathcal{P}}(T))^2 = \operatorname{Tr}(Q_nE_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)Q_n) = \operatorname{Tr}(E_{\mathcal{P}}(TQ_n)^*E_{\mathcal{P}}(TQ_n))$$

$$\leqslant \operatorname{Tr}(E_{\mathcal{P}}(Q_nT^*TQ_n)) = \operatorname{Tr}(Q_nT^*TQ_n) \leqslant \sum_{j=1}^{n} s_j(T)^2,$$

where the last inequality follows from Theorem 5. We must have equality throughout since $s(E_{\mathcal{P}}(T)) = s(T)$. Consequently, $TQ_n = E_{\mathcal{P}}(TQ_n) = E_{\mathcal{P}}(T)Q_n$ for all $n \in \mathbb{N}$ by the equality case of (3) for $X = TQ_n$.

By (2) and $E_{\mathcal{P}}(T) \in \mathcal{K}(\mathcal{H})$, $\|E_{\mathcal{P}}(T)Q_n^{\perp}\| \to 0$ as $n \to \infty$, but we also claim

$$\|TQ_n^{\perp}\| \leqslant s_{n+1}(T). \tag{4}$$

Suppose not. Then we could find some unit vector $x \in Q_n^{\perp}\mathcal{H}$ with $\langle T^*Tx, x\rangle = \|Tx\|^2 > s_{n+1}(T)^2$, and therefore, for the projection $R = Q_n + (x \otimes x)$,

$$\operatorname{Tr}(RT^*TR) = \operatorname{Tr}(Q_nT^*TQ_n) + \langle T^*Tx, x\rangle > \sum_{j=1}^{n+1} s_j(T)^2,$$

contradicting the fact that, because R is a projection of rank $n+1$, by Theorem 5

$$\operatorname{Tr}(RT^*TR) \leqslant \sum_{j=1}^{n+1} s_j(RT^*TR) \leqslant \sum_{j=1}^{n+1} s_j(T)^2.$$

Finally, again noting that $TQ_n = E_{\mathcal{P}}(T)Q_n$,

$$0 \leqslant \|T - E_{\mathcal{P}}(T)\| \leqslant \|T - TQ_n\| + \|E_{\mathcal{P}}(T)Q_n - E_{\mathcal{P}}(T)\| = \|TQ_n^{\perp}\| + \|E_{\mathcal{P}}(T)Q_n^{\perp}\|.$$

Since $\|TQ_n^{\perp}\| \leqslant s_{n+1}(T)$ by (4) and $\|E_{\mathcal{P}}(T)Q_n^{\perp}\| \leqslant s_{n+1}(E_{\mathcal{P}}(T))$ by (2), the right-hand side converges to zero as $n \to \infty$. Therefore, $\|T - E_{\mathcal{P}}(T)\| = 0$ and hence $T = E_{\mathcal{P}}(T)$. □

REMARK 3. When T is Hilbert–Schmidt, the proof that $s(E_{\mathcal{P}}(T)) = s(T)$ implies $E_{\mathcal{P}}(T) = T$ may be shortened considerably. Indeed, $T^*T, E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)$ are trace-class with $s(T^*T) = s(E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T))$ and so $\operatorname{Tr}(T^*T) = \operatorname{Tr}(E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T))$. Since the expectation $E_{\mathcal{P}}(T)$ is trace-preserving,

$$\begin{aligned}\operatorname{Tr}(E_{\mathcal{P}}(T^*T) - E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)) &= \operatorname{Tr}(E_{\mathcal{P}}(T^*T - E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)))\\ &= \operatorname{Tr}(T^*T - E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)) = 0.\end{aligned}$$

Since $E_{\mathcal{P}}(T^*T) - E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)$ is a positive operator by (3) and the trace is faithful, we must have $E_{\mathcal{P}}(T^*T) = E_{\mathcal{P}}(T)^*E_{\mathcal{P}}(T)$, and hence $T = E_{\mathcal{P}}(T)$ by the equality case of (3).

REMARK 4. Fan's theorem (Theorem 5) is a special case of Theorem 8 by selecting the projections P_n to have rank one, and therefore $E = E_{\mathcal{P}}$. As we need Theorem 5 to prove Theorem 8, the latter doesn't provide an independent proof of Fan's theorem.

Our second main theorem says that there is nothing special about the main diagonal T_0: for all $k \in \mathbb{Z}$, $s(T_k) \prec\!\!\prec_w s(T)$; Moreover, this holds even for *asymmetric* shift decompositions.

THEOREM 9. *Suppose that* $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$ *are block decompositions and let* $T \in \mathcal{K}(\mathcal{H})$ *with asymmetric shift decomposition* $\{T_k\}_{k\in\mathbb{Z}}$. *Then* $s(T_k) \prec\!\!\prec s(T)$. *Consequently, if* T *lies in some ideal* $\mathcal{I}$, *then* $T_k \in \overline{\mathcal{I}}^{am}$*; in particular,* $T_k \in \overline{\langle T\rangle}^{am}$.

Proof. It suffices to prove the theorem for T_0 since T_k is simply T_0 relative to the translated block decomposition pair $\{P_{n+k}\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$.

Each Q_nTP_n has the polar decomposition $Q_nTP_n = U_n|Q_nTP_n|$ where U_n is a partial isometry[9] with $Q_nU_n = U_n = U_nP_n$. Then $U := \sum_{n\in\mathbb{Z}} U_n$ converges in the strong operator topology since the collections $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$ are each mutually orthogonal and hence also U is a partial isometry. Moreover,

$$T_0^*T_0 = \left(\sum_{n\in\mathbb{Z}} P_nT^*Q_n\right)\left(\sum_{m\in\mathbb{Z}} Q_mTP_m\right) = \sum_{n\in\mathbb{Z}} |Q_nTP_n|^2.$$

Since the operators $|Q_nTP_n|^2$ are orthogonal (i.e., their products are zero), $|T_0| = (T_0^*T_0)^{1/2} = \sum_{n\in\mathbb{Z}}|Q_nTP_n|$. Thus,

$$\begin{aligned} E_{\mathcal{P}}(U^*T) &= \sum_{n\in\mathbb{Z}} P_nU^*TP_n = \sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}} P_nU_m^*TP_n\right) \\ &= \sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}} P_nP_mU_m^*Q_mTP_n\right) = \sum_{n\in\mathbb{Z}} U_n^*Q_nTP_n \\ &= \sum_{n\in\mathbb{Z}} |Q_nTP_n| = |T_0|. \end{aligned}$$

Finally, by Theorem 8 and since U^* is a contraction,

$$s(T_0) = s(|T_0|) = s(E_{\mathcal{P}}(U^*T)) \prec\!\!\prec s(U^*T) \leqslant s(T).$$

Therefore, if $T \in \mathcal{I}$, then $T_0 \in \overline{\mathcal{I}}^{am}$ by definition. □

REMARK 5. In the previous theorem we assumed that $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$ were block decompositions, but the condition that they sum to the identity is not actually necessary, only that the sequences of projections were (separately) mutually orthogonal. Indeed, suppose that $\{P_n\}_{n\in\mathbb{Z}}, \{Q_n\}_{n\in\mathbb{Z}}$ are sequences of mutually orthogonal projections. Set $P_0' := I - \sum_{n\in\mathbb{Z}} P_n$, and for $n < 0$ set $P_n' := P_n$, while for $n > 0$, set $P_n' := P_{n-1}$. Define Q_n' for $n \in \mathbb{Z}$ analogously. Then $\{P_n'\}_{n\in\mathbb{Z}}$ and $\{Q_n'\}_{n\in\mathbb{Z}}$ are block decompositions. Let $\{T_k'\}_{k\in\mathbb{Z}}$ denote the asymmetric shift decomposition of T relative to these block decompositions, and let $T_k := \sum_{n\in\mathbb{Z}} Q_nTP_{n+k}$. Then

$$s(T_k) \leqslant s(T_k') \prec\!\!\prec s(T),$$

[9] That $Q_nU_n = U_n = U_nP_n$ follows from well-known facts (e.g., see [5, Theorem I.8.1]) when U_n is taken to be the canonical unique partial isometry on $\mathcal{H}$ mapping $\overline{\operatorname{ran}(|Q_nTP_n|)} \to \overline{\operatorname{ran}(Q_nTP_n)}$ and noting also the range projection of Q_nTP_n is dominated by Q_n and the projection onto $\overline{\operatorname{ran}(|Q_nTP_n|)} = \ker^{\perp}(Q_nTP_n)$ is dominated by P_n.

where the submajorization follows from Theorem 9, and the inequality is due to the fact that $s(T_k')$ is the "union" (over n) of the singular values of the blocks $Q_n' T P_{n+k}'$ (reordered so as to be monotonic). This union contains the singular values of $Q_n T P_{n+k}$ for each $n \in \mathbb{Z}$, and hence contains $s(T_k)$. Because of the monotonic reordering, $s(T_k) \leqslant s(T_k')$. Hence $s(T_k) \prec\!\!\prec s(T)$, and so $T_k \in \overline{\langle T \rangle}^{am}$.

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