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PROPER NONNEGATIVE SPLITTINGS OVER PROPER CONES OF RECTANGULAR MATRICES

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Abstract. In this paper, we further investigate the proper nonnegative splittings of rectangular matrices. The concept of proper nonnegative splittings over proper cones of rectangular matrices is proposed. Convergence results for the proper double nonnegative splitting over proper cones of a rectangular matrix are established, and comparison theorems for the spectral radii of matrices arising from proper nonnegative splittings over proper cones of the same rectangular matrix or different rectangular matrices are presented. The results obtained in this paper extend the results of proper nonnegative splittings over field to ones over proper cones of rectangular matrices. For ill-posed linear systems, the regularized iterative method based on splittings over proper cones is introduced, and the application of research results of proper nonnegative splittings over proper cones in the ill-posed linear system is given.

1. Introduction

The rectangular linear system of the form

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, appears in many areas of applications, for example, finite difference discretization of partial differential equations with suitable boundary conditions. There are two forms of splitting iteration methods for solving the rectangular linear system (1):

(1). If A has the single splitting [5]

$$A = U - V \tag{2}$$

with U, $V \in \mathbb{R}^{m\times n}$, then the approximate solution x_{k+1} of (1) is generated by

$$x_{k+1} = U^\dagger V x_k + U^\dagger b, \tag{3}$$

where $U^\dagger$ is the Moore-Penrose inverse of U [2, 33], the matrix $U^\dagger V$ is called the iteration matrix of (3). The splitting (2) is called a proper single splitting

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if $R(A) = R(U)$ and $N(A) = N(U)$ [5], where $R(\cdot)$ and $N(\cdot)$ denote the range space and the null space of a given matrix, respectively. The uniqueness of proper single splittings was provided in [21]. Let $\rho(C)$ be the spectral radius of the real square matrix C, then for the proper single splitting $A = U - V$, the iteration scheme (3) converges to $A^\dagger b$, the least squares solution of minimum norm for any initial vector x_0 if and only if $\rho(U^\dagger V) < 1$ [5, Corollary 1]. Notice that if $A = U - V$ is not a proper single splitting, the iteration scheme (3) may not converge to $A^\dagger b$ for any initial vector x_0 even for $\rho(U^\dagger V) < 1$, see [5, 19]. If the iteration scheme (3) is convergent, then we say that the proper single splitting $A = U - V$ is a convergent splitting. The convergence of the iteration scheme (3) for proper single splittings over field of A has been studied extensively in [5, 10, 16, 17, 19, 20, 23]. The convergence of the iteration scheme (3) for proper single splittings over proper cones of A has been studied in [5, 8].

(2). If A has the double splitting

$$A = P - R - S \tag{4}$$

with $P,\ R,\ S \in \mathbb{R}^{m\times n}$, then the approximate solution x_{k+1} of (1) is generated by [17]

$$x_{k+1} = P^\dagger R x_k + P^\dagger S x_{k-1} + P^\dagger b,\ \ k = 1, 2, \cdots. \tag{5}$$

It should be noted that the concept of double splittings was first introduced by Woźnicki in [32] for a nonsingular matrix, and was extended to rectangular matrices in [17, 19]. The iteration scheme (5) can be rewritten in the following equivalent form

$$\begin{pmatrix} x_{k+1} \\ x_k \end{pmatrix} = \begin{pmatrix} P^\dagger R & P^\dagger S \\ I & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix} + \begin{pmatrix} P^\dagger b \\ 0 \end{pmatrix},\ \ k = 1, 2, \cdots, \tag{6}$$

here I is the identity matrix with a compatible size, and

$$W = \begin{pmatrix} P^\dagger R & P^\dagger S \\ I & 0 \end{pmatrix}$$

is the iteration matrix of (6). The splitting (4) is called a proper double splitting if $R(A) = R(P)$ and $N(A) = N(P)$ [17]. For the proper double splitting $A = P - R - S$, the iterative method (5) or (6) converges to the unique least squares solution of minimum norm of (1) if and only if $\rho(W) < 1$. The convergence of the iteration scheme (6) for proper double splittings over field of A has been studied in [17, 19, 27].

Comparison theorems between the spectral radii of iteration matrices are useful tools to analyze the convergent rate of iteration methods or to judge the effectiveness of preconditioners [11, 17, 18, 20, 25]. Some comparison theorems for proper single splittings of a semimonotone matrix are established recently in [4, 17, 19], and that for proper single splittings of different semimonotone matrices are proposed in [4, 16, 20]. Comparison theorems for proper double splittings of a rectangular matrix can be found in [1, 4], and that for proper double splittings of different rectangular matrices can be found in [4, 16, 17, 19].

The motivation of this paper is based on the following three facts: (i) most results of proper nonnegative splittings over field of rectangular matrices are still valid even if some conditions are not satisfied; (ii) the results of splittings of nonsingular matrices can be extended to the splittings over proper cones of nonsingular matrices [14, 33]. Moreover, we know from [14, 15, 33] that the results that do not satisfy the conditions of splittings over field may satisfy the conditions of the splittings over proper cones for nonsingular matrices. So we mainly consider the proper nonnegative splittings over proper cones of rectangular matrices in this paper. There are a few results for proper single splittings over proper cones of $A \in \mathbb{R}^{m\times n}$ in [5, 8], but few results for proper double splittings over proper cones of $A \in \mathbb{R}^{m\times n}$.

The remainder of the paper is organized as follows. In Section 2, we give some relevant definitions and notations, which are used in the sequence of this paper. In Section 3, we present comparison theorems of proper single nonnegative splittings over proper cones of rectangular matrices. In Section 4, some convergence and comparison results for proper double nonnegative splittings over proper cones of rectangular matrices are presented. In Section 5, the application of research results of proper nonnegative splittings over proper cones in the ill-posed linear system is given. We give some conclusions in Section 6.

2. Preliminaries

In this section, we will provide some definitions and notations which are useful in the later analysis.

Firstly, let us recall that a nonempty subset K of $\mathbb{R}^n$ is a convex cone if $K+K \subseteq K$ and $\alpha K \subseteq K$ for all $0 \leqslant \alpha$. The convex cone K is said to be proper if it is closed, pointed ($K \cap -K = \{0\}$) and has nonempty interior (usually denotes by int$K$) [6]. It should be noted that both the nonnegative cone $\mathbb{R}^n_+$ and the ice cream cone $\{x \in \mathbb{R}^n | (x_2^2 + x_3^2 + \cdots + x_n^2)^{\frac{1}{2}} \leqslant x_1\}$ are particular proper cones.

Secondly, we will review some concepts for the nonnegativity of squares matrices. An $n\times n$ real matrix A is called nonnegative (respectively, positive) if $A \geqslant 0$ (respectively, $A > 0$). For $n\times n$ real matrices A and B we denote $A - B \geqslant 0$ (respectively, $A - B > 0$) by $A \geqslant B$ (respectively, $A > B$). These can be immediately applied to vectors by identifying them with $n \times 1$ matrices. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be a monotone matrix, if A^{-1} exists and $A^{-1} \geqslant 0$ [1]. These concepts have been extended to proper cones in [6, 8, 9, 14, 15, 33].

Let K be a proper cone in $\mathbb{R}^n$, a vector $x \in \mathbb{R}^n$ is called nonnegative (respectively, positive) over the proper cone K if x belongs to K (respectively, x belongs to int$K$) and is denoted as $x \geqslant_K 0$ (respectively, $x >_K 0$). If $x,\ y \in \mathbb{R}^n$ satisfy $x - y \geqslant_K 0$ (respectively, $x - y >_K 0$), which is denoted as $x \geqslant_K y$ (respectively, $x >_K y$). A matrix $A \in \mathbb{R}^{n\times n}$ is called nonnegative (respectively, positive) over the proper cone K if $AK \subseteq K$ (respectively, $A(K - \{0\}) \subseteq \text{int}K$) and is denoted as $A \geqslant_K 0$ (respectively, $A >_K 0$). For $A,\ B \in \mathbb{R}^{n\times n}$, $A \geqslant_K B$ (respectively, $A >_K B$) means $A - B \geqslant_K 0$ (respectively,

$A - B >_K 0$). Let $\pi(K)$ denote the set of matrices $A \in \mathbb{R}^{n\times n}$ for which $AK \subseteq K$ [6], $A \in \mathbb{R}^{n\times n}$ is nonnegative over the proper cone K is equivalent to $A \in \pi(K)$, and $A \in \mathbb{R}^{n\times n}$ is a monotone matrix over the proper cone K if $A^{-1} \geqslant_K 0$, i.e., $A^{-1} \in \pi(K)$ [14]. The properties of nonnegative matrices over a proper cone are similar to that of nonnegative matrices, see for example [6, 14, 22]. Next, let K_1 be a proper cone in $\mathbb{R}^n$, and we give the definition of the nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{n\times n}$.

DEFINITION 1. Let K_1 be a proper cone in $\mathbb{R}^n$, A be a nonsingular matrix. Then,

(i). the single splitting $A = U - V$ is a single nonnegative splitting over the proper cone K_1 if $U^{-1}V \geqslant_{K_1} 0$ [15];

(ii). the double splitting $A = P - R - S$ is a double nonnegative splitting over the proper cone K_1 if $P^{-1}R \geqslant_{K_1} 0$ and $P^{-1}S \geqslant_{K_1} 0$ [33].

If $A = P - R - S$ is a double nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{n\times n}$, then

$$W = \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{pmatrix} \geqslant_{K_{1_{2n}}} 0.$$

Lastly, we will give some concepts related to rectangular matrices.

DEFINITION 2. Let K_1 and K_2 be proper cones, in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A matrix $A \in \mathbb{R}^{m\times n}$ is called

(1). nonnegative over proper cones if $AK_1 \subseteq K_2$;

(2). positive over proper cones if $A(K_1 - \{0\}) \subseteq \mathrm{int}K_2$.

Denote by $\pi(K_1, K_2)$ the set of matrices $A \in \mathbb{R}^{m\times n}$ for which $AK_1 \subseteq K_2$, we give the following definition.

DEFINITION 3. Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A real rectangular matrix $A \in \mathbb{R}^{m\times n}$ is called semimonotone over proper cones if $A^\dagger \in \pi(K_2, K_1)$.

In the following, we extend the concepts of the different types of proper nonnegative splittings that appear in [19] for the particular case $K = \mathbb{R}^n_+$ to general proper cones.

DEFINITION 4. Let K_1 be a proper cone in $\mathbb{R}^n$, the single splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper single nonnegative splitting over the proper cone K_1 if it is a proper splitting such that $U^\dagger V \geqslant_{K_1} 0$.

DEFINITION 5. Let K_1 be a proper cone in $\mathbb{R}^n$. For $A \in \mathbb{R}^{m\times n}$, the splitting $A = P - R - S$ is called a proper double nonnegative splitting over the proper cone K_1 if it is a proper splitting such that $P^\dagger R \geqslant_{K_1} 0$ and $P^\dagger S \geqslant_{K_1} 0$.

If $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$, then

$$W = \begin{pmatrix} P^\dagger R & P^\dagger S \\ I & 0 \end{pmatrix} \geqslant_{K_{1_{2n}}} 0.$$

3. Results for proper single nonnegative splittings over proper cones

Convergence results for the proper single nonnegative splitting over proper cones of a rectangular matrix are studied extensively in [5, 8]. In what follows of this section, we will propose the comparison results for proper single nonnegative splittings over proper cones of rectangular matrices.

Let us first consider different splittings of one rectangular matrix A, let K_1 be a proper cone in $\mathbb{R}^n$. Assume that $A = U_1 - V_1 = U_2 - V_2$ are proper single nonnegative splittings over the proper cone K_1 of A. Comparing $\rho(U_1^\dagger V_1)$ with $\rho(U_2^\dagger V_2)$, we have the following results.

THEOREM 1. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A = U_1 - V_1 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger V_2 \geqslant_{K_1} A^\dagger V_1 >_{K_1} 0$, then*

$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1.$$

Proof. As $A^\dagger V_2 \geqslant_{K_1} A^\dagger V_1 >_{K_1} 0$, it follows from [5, Theorem 2] that $\rho(U_i^\dagger V_i) < 1$ for $i = 1, 2$. Thus all we need to show is $\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2)$.

It also follows from [5, Theorem 2] that

$$\rho(U_i^\dagger V_i) = \frac{\rho(A^\dagger V_i)}{1 + \rho(A^\dagger V_i)}.$$

[6, Corollary 3.29] and [14, Corollary 2.6.] yield $\rho(A^\dagger V_1) < \rho(A^\dagger V_2)$. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, it is easy to see that $f(\lambda)$ is a strictly increasing function for $\lambda \geqslant 0$. Hence the inequality $\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2)$ holds. $\square$

THEOREM 2. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A = U_1 - V_1 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger U_2 \geqslant_{K_1} A^\dagger U_1 >_{K_1} 0$, then*

$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1.$$

Proof. As $A^\dagger U_2 \geqslant_{K_1} A^\dagger U_1 >_{K_1} 0$, it follows from [8, Theorem 2] that $\rho(U_i^\dagger V_i) < 1$ for $i = 1, 2$. Moreover, [8, Theorem 2] gives

$$\rho(U_i^\dagger V_i) = \frac{\rho(A^\dagger U_i) - 1}{\rho(A^\dagger U_i)}.$$

[6, Corollary 3.29] and [14, Corollary 2.6.] yield $\rho(A^{\dagger}U_1) < \rho(A^{\dagger}U_2)$. Let $f(\lambda) = \frac{\lambda-1}{\lambda}$, then $f(\lambda)$ is a strictly increasing function for $\lambda > 0$. Hence the inequality $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2)$ holds. $\square$

If we consider the proper single nonnegative splitting over the proper cone K_1 of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones, from Theorem 1 and Theorem 2, the following corollaries can be obtained.

COROLLARY 1. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones. If $(V_2 - V_1) \in \pi(K_1, K_2)$ and $A^{\dagger}V_1 >_{K_1} 0$, then*

$$\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1.$$

Proof. The semi-monotonicity of A over proper cones implies that $A^{\dagger} \in \pi(K_2, K_1)$, combining $(V_2 - V_1) \in \pi(K_1, K_2)$, we have

$$A^{\dagger}(V_2 - V_1)K_1 \subseteq A^{\dagger}K_2 \subseteq K_1,$$

i.e., $A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1$. As $A^{\dagger}V_1 >_{K_1} 0$, then we have

$$A^{\dagger}V_2 \geqslant_{K_1} A^{\dagger}V_1 >_{K_1} 0.$$

Theorem 1 yields $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$. $\square$

COROLLARY 2. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = U_1 - V_1 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones. If $(U_2 - U_1) \in \pi(K_1, K_2)$ and $A^{\dagger}U_1 >_{K_1} 0$, then*

$$\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1.$$

Proof. Similar to the proof of Corollary 1, under the assumptions, we get that $A^{\dagger}U_2 \geqslant_{K_1} A^{\dagger}U_1 >_{K_1} 0$. Theorem 2 gives $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$. $\square$

Next, we consider comparison results between the spectral radii of matrices arising from proper single nonnegative splittings over proper cones of different rectangular matrices. Let K_1 be a proper cone in $\mathbb{R}^n$, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. Comparing $\rho(U_1^{\dagger}V_1)$ with $\rho(U_2^{\dagger}V_2)$, the following results can be obtained.

THEOREM 3. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. If $A_2^{\dagger}V_2 \geqslant_{K_1} A_1^{\dagger}V_1 >_{K_1} 0$ and $A_1^{\dagger}V_1 \neq A_2^{\dagger}V_2$, then*

$$\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1.$$

THEOREM 4. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be proper single nonnegative splittings over the proper cone K_1 of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. If $A_2^{\dagger}U_2 \geqslant_{K_1} A_1^{\dagger}U_1 >_{K_1} 0$ and $A_1^{\dagger}U_1 \neq A_2^{\dagger}U_2$, then*

$$\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1.$$

The proofs of Theorem 3 and Theorem 4 are similar to the proofs of Theorem 1 and Theorem 2, respectively, which we haved omitted here.

The example given below demonstrates that the condition $A_2^{\dagger}V_2 \geqslant_{K_1} A_1^{\dagger}V_1 >_{K_1} 0$ cannot be dropped in Theorem 3. At the same time, the following examples shows that the condition $A_2^{\dagger}U_2 \geqslant_{K_1} A_1^{\dagger}U_1 >_{K_1} 0$ cannot be dropped in Theorem 4.

EXAMPLE 1. Consider proper cones $K_1 = \{x \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{x \in \mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Let

$$A_1 = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ \frac{1}{8} & \frac{5}{2} & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{8} & 0 & 0 \\ \frac{1}{10} & 3 & 1 \end{pmatrix}.$$

Assume that A_1 and A_2 are splitted as

$$A_1 = U_1 - V_1 \text{ and } A_2 = U_2 - V_2$$

with

$$U_1 = \begin{pmatrix} 4 & 0 & 0 \\ 1 & \frac{5}{2} & 2 \end{pmatrix}, \quad V_1 = \begin{pmatrix} \frac{23}{6} & 0 & 0 \\ \frac{7}{8} & 0 & 0 \end{pmatrix}$$

and

$$U_2 = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \frac{23}{8} & 0 & 0 \\ \frac{9}{10} & 0 & 0 \end{pmatrix},$$

respectively.

Following the operations, we have

$$U_1^{\dagger} = \begin{pmatrix} 0.25 & 0 \\ -0.0610 & 0.2439 \\ -0.0488 & 0.1951 \end{pmatrix}, \quad U_2^{\dagger} = \begin{pmatrix} 0.3333 & 0 \\ -0.1 & 0.3 \\ -0.0333 & 0.1 \end{pmatrix}$$

and

$$A_1^{\dagger}V_1 = \begin{pmatrix} 23 & 0 & 0 \\ -0.4878 & 0 & 0 \\ -0.3902 & 0 & 0 \end{pmatrix}, \quad A_2^{\dagger}V_2 = \begin{pmatrix} 23 & 0 & 0 \\ -0.42 & 0 & 0 \\ -0.14 & 0 & 0 \end{pmatrix}.$$

Moreover, we can get

$$A_1^{\dagger}U_1 = \begin{pmatrix} 24 & 0 & 0 \\ -0.4878 & 0.6098 & 0.4878 \\ -0.3902 & 0.4878 & 0.3902 \end{pmatrix} \text{ and } A_2^{\dagger}U_2 = \begin{pmatrix} 24 & 0 & 0 \\ -0.42 & 0.9 & 0.3 \\ -0.14 & 0.3 & 0.1 \end{pmatrix}.$$

It is easy to verify that $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are proper single nonnegative splittings over the proper cone K_1 of A_1 and A_2, respectively. But

$$A_2^{\dagger}V_2 - A_1^{\dagger}V_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.0678 & 0 & 0 \\ 0.2502 & 0 & 0 \end{pmatrix}$$

and

$$A_2^{\dagger}U_2 - A_1^{\dagger}U1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.0678 & 0.2902 & -0.1878 \\ 0.2502 & -0.1878 & -0.2902 \end{pmatrix},$$

i.e., $A_2^{\dagger}V_2 \ngeqslant_K A_1^{\dagger}V_1$ and $A_2^{\dagger}U_2 \ngeqslant_K A_1^{\dagger}U_1$.

In fact, we have $\rho(U_2^{\dagger}V_2) = 0.9583 = \rho(U_1^{\dagger}V_1) < 1$.

4. Results for proper double nonnegative splittings over proper cones

In this section, some convergence and comparison results for proper double nonnegative splittings over proper cones of rectangular matrices are considered.

4.1. The results of convergence

Some convergence results are presented in this subsection. First, we show the convergence equivalence of the proper single nonnegative splitting and the proper double nonnegative splitting over the same proper cone.

THEOREM 5. *Let K_1 be a proper cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$. If $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1, then it is convergent if and only if the proper single nonnegative splitting $A = P - (R + S)$ over the proper cone K_1 is convergent.*

Proof. The splitting $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1 implies that $W = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & 0 \end{pmatrix} \geqslant_{K_{1_{2n}}} 0$, it follows from [33, Lemma 2] that it is convergent if and only if $(I-W)^{-1} \geqslant_{K_{1_{2n}}} 0$.

On the one hand, notice that

$$\begin{aligned}(I-W)^{-1} &= \begin{pmatrix} [I-P^{\dagger}(R+S)]^{-1} & 0 \\ 0 & [I-P^{\dagger}(R+S)]^{-1} \end{pmatrix} \begin{pmatrix} I & P^{\dagger}S \\ I & I-P^{\dagger}R \end{pmatrix} \\ &= \begin{pmatrix} [I-P^{\dagger}(R+S)]^{-1} & [I-P^{\dagger}(R+S)]^{-1}P^{\dagger}S \\ [I-P^{\dagger}(R+S)]-1 & I+[I-P^{\dagger}(R+S)]^{-1}P^{\dagger}S \end{pmatrix}.\end{aligned}$$

On the other hand, we know that $[I-P^{\dagger}(R+S)]^{-1} \geqslant_{K_1} 0$ if and only if $\rho(P^{\dagger}(R+S)) < 1$ [33, Lemma 2]. The proper double nonnegative splitting $A = P - R - S$ over the proper cone K_1 and $[I-P^{\dagger}(R+S)]^{-1} \geqslant_{K_1} 0$ give $[I-P^{\dagger}(R+S)]^{-1}P^{\dagger}S \geqslant_{K_1} 0$ and $I+[I-P^{\dagger}(R+S)]^{-1}P^{\dagger}S \geqslant_{K_1} 0$. Hence, $(I-W)^{-1} \geqslant_{K_{1_{2n}}} 0$ if and only if $\rho(P^{\dagger}(R+$

$S)) < 1$, i.e., the proper single nonnegative splitting $A = P - (R+S)$ over the proper cone K_1 is convergent.

Therefore, the proper double nonnegative splitting $A = P - R - S$ over the proper cone K_1 is convergent if and only if the proper single nonnegative splitting $A = P - (R+S)$ over the proper cone K_1 is convergent. $\square$

REMARK 1. If $K_1 = \mathbb{R}^n_+$, then Theorem 5 becomes Theorem 4.3 of [19].

THEOREM 6. *Let K_1 be a proper cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$. If $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1 of A, and $A^\dagger P \geqslant_{K_1} 0$, then $\rho(W) < 1$.*

Proof. The fact of $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1 implies that $P^\dagger R \geqslant_{K_1} 0$ and $P^\dagger S \geqslant_{K_1} 0$. So we have $P^\dagger R + P^\dagger S = P^\dagger(R+S) \geqslant_{K_1} 0$. Setting $U = P$ and $V = R+S$, then we get that $A = U - V$ is a proper single nonnegative splitting over the proper cone K_1 and $A^\dagger P = A^\dagger U \geqslant_{K_1} 0$. It follows from [8, Theorem 2] that

$$\rho(P^\dagger(R+S)) = \rho(U^\dagger V) = \frac{\rho(A^\dagger U) - 1}{\rho(A^\dagger U)} < 1.$$

Theorem 5 then gives $\rho(W) < 1$. $\square$

REMARK 2. If K_1 is a nonnegative cone, which is a particular proper cone, then Theorem 6 becomes Theorem 4.5 of [19].

The following example shows that the condition which is not true for Theorem 4.5 in [19] is true for Theorem 6.

EXAMPLE 2. Consider proper cones $K_1 = \{x \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{x \in \mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Assume that

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{5}{2} & 1 \end{pmatrix}.$$

Let A be splitted as $A = P - R - S$ with

$$P = \begin{pmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{pmatrix}.$$

It is easy to see that

$$A^\dagger P = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0.8621 & 0.3448 \\ 0 & 0.3448 & 0.1379 \end{pmatrix}.$$

Following the operations, we have

$$P^{\dagger}R = \begin{pmatrix} 0.5 & 0 & 0 \\ -0.0862 & 0.0862 & 0 \\ -0.0345 & 0.0345 & 0 \end{pmatrix} \text{ and } P^{\dagger}S = \begin{pmatrix} 0.25 & 0 & 0 \\ 0.0862 & -0.0862 & 0 \\ 0.0345 & -0.0345 & 0 \end{pmatrix}.$$

It is easy to verify that $A = P - R - S$ is not a proper double nonnegative splitting [19, Definition 4.1].

However, the assumptions of Theorem 6 are satisfied. We then have

$$\rho(W) = 0.8090 < 1.$$

If we consider $A^{\dagger} \in \pi(K_2, K_1)$, then the following corollary is obtained.

COROLLARY 3. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume that $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$. If $A^{\dagger} \in \pi(K_2, K_1)$ and $P \in \pi(K_1, K_2)$, then $\rho(W) < 1$.*

Proof. The assumptions $A^{\dagger} \in \pi(K_2, K_1)$ and $P \in \pi(K_1, K_2)$ imply $A^{\dagger}P \geqslant_{K_1} 0$. Theorem 6 then gives $\rho(W) < 1$. □

In the next theorem, we establish other convergence conditions for the proper double nonnegative splitting $A = P - R - S$ over the proper cone K_1, they are generalizations of convergence conditions of [19, Theorem 4.7].

THEOREM 7. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A = P - R - S$ be a proper double nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$. Then the following are equivalent:*

(1) $\rho(W) < 1$;

(2) $[I - P^{\dagger}(R+S)]^{-1} \geqslant_{K_1} 0$;

(3) $A^{\dagger}(R+S) \geqslant_{K_1} 0$;

(4) $A^{\dagger}(R+S) \geqslant_{K_1} P^{\dagger}(R+S)$.

Proof. Since $A = P - R - S$ is a proper double nonnegative splitting over the proper cone K_1, we have $P^{\dagger}R \geqslant_{K_1} 0$ and $P^{\dagger}S \geqslant_{K_1} 0$, so $P^{\dagger}R + P^{\dagger}S = P^{\dagger}(R+S) \geqslant_{K_1} 0$. Setting $U = P$ and $V = R+S$, then $A = U - V$ is a proper single nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$.

(1) ⇒ (2) As $\rho(W) < 1$, it follows from Theorem 5 that $\rho(P^{\dagger}(R+S)) < 1$. Since $\rho(P^{\dagger}(R+S)) = \rho(U^{\dagger}V) < 1$ and $U^{\dagger}V \geqslant_{K_1} 0$, so from [33, Lemma 2] we get that $[I - P^{\dagger}(R+S)]^{-1} = (I - U^{\dagger}V)^{-1} \geqslant_{K_1} 0$.

(2) ⇒ (3) Since $A = U - V$ is a proper single nonnegative splitting over the proper cone K_1, so from [5, Theorem 1] we have $A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}$. Therefore $A^{\dagger}VK_1 = [I - P^{\dagger}(R+S)]^{-1}P^{\dagger}(R+S)K_1 \subseteq K_1$, i.e., $A^{\dagger}(R+S) = A^{\dagger}V \geqslant_{K_1} 0$.

(3) $\Rightarrow$ (4) As $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$, we have $A^\dagger - U^\dagger = U^\dagger V A^\dagger$. And then $A^\dagger V - U^\dagger V = U^\dagger V A^\dagger V$. We have $(A^\dagger V - U^\dagger V)K_1 = U^\dagger V A^\dagger V K_1 \subseteq U^\dagger V K_1 \subseteq K_1$, i.e., $A^\dagger V \geqslant_{K_1} U^\dagger V$. Then we have $A^\dagger(R+S) \geqslant_{K_1} P^\dagger(R+S)$.

(4) $\Rightarrow$ (1) Since $A^\dagger V = A^\dagger(R+S) \geqslant_{K_1} P^\dagger(R+S) = U^\dagger V$ and $U^\dagger V \geqslant_{K_1} 0$, so we have $A^\dagger V \geqslant_{K_1} 0$. [5, Theorem 2] gives $\rho(P^\dagger(R+S)) = \rho(U^\dagger V) < 1$. As $\rho(P^\dagger(R+S)) = \rho(U^\dagger V) < 1$, it follows from Theorem 5 that $\rho(W) < 1$. $\square$

4.2. Comparison results

Let $A \in \mathbb{R}^{m\times n}$, $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of A. Then, we define

$$W_1 = \begin{pmatrix} P_1^\dagger R_1 & P_1^\dagger S_1 \\ I & 0 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} P_2^\dagger R_2 & P_2^\dagger S_2 \\ I & 0 \end{pmatrix}.$$

In the following, we give comparison results for proper double nonnegative splittings over the proper cone K_1 of A. The first result of comparing $\rho(W_1)$ with $\rho(W_2)$ is shown in the following theorem.

THEOREM 8. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$. Suppose $P_2^\dagger A \geqslant_{K_1} P_1^\dagger A$, $A^\dagger P_i \geqslant_{K_1} 0$ for $i = 1,2$ and any one of the following conditions*

(1). $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$;

(2). $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. By Theorem 6, the conditions $A^\dagger P_i \geqslant_{K_1} 0$ for $i = 1,2$ imply $\rho(W_i) < 1$. Assume that $\rho(W_2) = 0$, then the conclusion holds clearly. Assume that $\rho(W_2) \neq 0$, i.e., $0 < \rho(W_2) < 1$. By [6, Theorem 3.2], there exists a nonzero vector

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_{1_{2n}},$$

in conformity with W_2 such that $W_2 X = \rho(W_2)X$, which can be rewritten into

$$\begin{aligned} P_2^\dagger R_2 x_1 + P_2^\dagger S_2 x_2 &= \rho(W_2)x_1, \\ x_1 &= \rho(W_2)x_2, \end{aligned}$$

where $x_1, x_2 \in K_1$.

Then we have

$$\begin{aligned} W_1 X - \rho(W_2)X &= \begin{pmatrix} P_1^\dagger R_1 x_1 + P_1^\dagger S_1 x_2 - \rho(W_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + \frac{1}{\rho(W_2)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}. \end{aligned}$$

(i). Since $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} &\Delta - \frac{1}{\rho(W_2)}((P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1) \\ &= \left(\frac{1}{\rho(W_2)} - 1\right)(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 \\ &\geqslant_{K_1} 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \Delta &\geqslant_{K_1} \frac{1}{\rho(W_2)}(P_1^\dagger R_1 - P_2^\dagger R_2 + P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ &= \frac{1}{\rho(W_2)}(P_2^\dagger A - P_1^\dagger A)x_1 \\ &\geqslant_{K_1} 0. \end{aligned}$$

Thus,

$$W_1X - \rho(W_2)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\rho(W_2) \leqslant \rho(W_1) < 1$.

(ii). Since $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} &\Delta - ((P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1) \\ &= \left(\frac{1}{\rho(W_2)} - 1\right)(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ &\geqslant_{K_1} 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \Delta &\geqslant_{K_1} (P_1^\dagger R_1 - P_2^\dagger R_2 + P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ &= (P_2^\dagger A - P_1^\dagger A)x_1 \\ &\geqslant_{K_1} 0. \end{aligned}$$

Consequently,

$$W_1X - \rho(W_2)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\rho(W_2) \leqslant \rho(W_1) < 1$. □

An example of Theorem 8 is shown below.

EXAMPLE 3. Consider proper cones $K_1 = \{x \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{x \in \mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{5}{2} & 1 \end{pmatrix}$$

with

$$P_1 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & \frac{5}{2} & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{3}{8} & \frac{1}{8} & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{3}{8} & -\frac{1}{8} & 0 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{pmatrix}.$$

It is easy to see that

$$A^{\dagger}P_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0.1724 & 0.8621 & 0.3448 \\ 0.0690 & 0.3448 & 0.1379 \end{pmatrix},$$

$$A^{\dagger}P_2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0.8621 & 0.3448 \\ 0 & 0.3448 & 0.1379 \end{pmatrix} \text{ and } P_2^{\dagger}A - P_1^{\dagger}A = \begin{pmatrix} 0.0833 & 0 & 0 \\ 0.0287 & 0 & 0 \\ 0.0115 & 0 & 0 \end{pmatrix}.$$

Following the operations, we have

$$P_1^{\dagger}R_1 = \begin{pmatrix} 0.1667 & 0 & 0 \\ -0.0718 & 0.0431 & 0 \\ -0.0287 & 0.0172 & 0 \end{pmatrix}, \quad P_1^{\dagger}S_1 = \begin{pmatrix} 0.6667 & 0 & 0 \\ 0.1006 & -0.0431 & 0 \\ 0.0402 & -0.0172 & 0 \end{pmatrix}$$

and

$$P_2^{\dagger}R_2 = \begin{pmatrix} 0.5 & 0 & 0 \\ -0.0862 & 0.0862 & 0 \\ -0.0345 & 0.0345 & 0 \end{pmatrix}, \quad P_2^{\dagger}S_2 = \begin{pmatrix} 0.25 & 0 & 0 \\ 0.0862 & -0.0862 & 0 \\ 0.0345 & -0.0345 & 0 \end{pmatrix}.$$

It is easy to prove that $A = P_1 - R_1 - S_1$ and $A = P_2 - R_2 - S_2$ are not proper double nonnegative splittings of A, but proper double nonnegative splittings over the proper cone K_1 of A.

Clearly, the assumptions of Theorem 8 are satisfied. We then have

$$\rho(W_2) = 0.8090 < 0.9041 = \rho(W_1) < 1.$$

When we consider proper double nonnegative splittings over the proper cone K_1 of a nonnegative matrix $A \in \mathbb{R}^{m\times n}$ over proper cones, the following corollary is a direct result of Theorem 8.

COROLLARY 4. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of a nonnegative matrix $A \in \mathbb{R}^{m\times n}$ over proper cones. Suppose $(P_2^{\dagger} - P_1^{\dagger}) \in \pi(K_2, K_1)$, $A^{\dagger}P_i \geqslant_{K_1} 0$ for $i = 1, 2$ and any one of the following conditions*

(1). $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$;

(2). $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. The nonnegativity of A over proper cones implies that $A \in \pi(K_1,K_2)$, combining $(P_2^\dagger - P_1^\dagger) \in \pi(K_2,K_1)$, we can obtain $P_2^\dagger A \geqslant_{K_1} P_1^\dagger A$. Theorem 8 then yields $\rho(W_2) \leqslant \rho(W_1) < 1$. □

If we consider proper double nonnegative splittings over the proper cone K_1 of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones, we have the following corollary.

COROLLARY 5. *Let* K_1 *and* K_2 *be proper cones in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively. Let* $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *be proper double nonnegative splittings over the proper cone* K_1 *of a semimonotone matrix* $A \in \mathbb{R}^{m\times n}$ *over proper cones. Suppose* $P_2^\dagger A \geqslant_{K_1} P_1^\dagger A$*,* $P_i \in \pi(K_1,K_2)$ *for* $i = 1,2$ *and any one of the following conditions*

(1). $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$;

(2). $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. The fact of A is a semimonotone matrix over proper cones implies that $A^\dagger \in \pi(K_2,K_1)$. As $A^\dagger \in \pi(K_2,K_1)$ and $P_i \in \pi(K_1,K_2)$, we then have $A^\dagger P_i \geqslant_{K_1} 0$ for $i = 1,2$. Theorem 8 gives $\rho(W_2) \leqslant \rho(W_1) < 1$. □

If we consider a nonnegative and semimonotone matrix $A \in \mathbb{R}^{m\times n}$ over proper cones, from Theorem 8, we can get the following result directly.

COROLLARY 6. *Let* K_1 *and* K_2 *be proper cones in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively. Let* $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *be proper double nonnegative splittings over the proper cone* K_1 *of a nonnegative and semimonotone matrix* $A \in \mathbb{R}^{m\times n}$ *over proper cones. Suppose* $(P_2^\dagger - P_1^\dagger) \in \pi(K_2,K_1)$*,* $P_i \in \pi(K_1,K_2)$ *for* $i = 1,2$ *and any one of the following conditions*

(1). $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$;

(2). $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. By Definition 2 and Definition 3, we have $A \in \pi(K_1,K_2)$ and $A^\dagger \in \pi(K_2,K_1)$, respectively. Similar to the proofs of Corollary 4 and Corollary 5, under the assumptions, we get that $P_2^\dagger A \geqslant_{K_1} P_1^\dagger A$ and $A^\dagger P_i \geqslant_{K_1} 0$ for $i = 1,2$. Theorem 8 then yields $\rho(W_2) \leqslant \rho(W_1) < 1$. □

Another comparison theorem of proper double nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$ is as follows.

THEOREM 9. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$. Suppose $P_1^\dagger R_1 - P_2^\dagger R_2 + P_1^\dagger S_1 - P_2^\dagger S_2 \geqslant_{K_1} 0$, $A^\dagger P_i \geqslant_{K_1} 0$ for $i = 1,2$ and any one of the following conditions*

(1). $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$*;*

(2). $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$*,*

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. From Theorem 6, we know that both proper double nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$ are convergent, that is, $\rho(W_1) < 1$ and $\rho(W_2) < 1$. Assume that $\rho(W_2) = 0$, then the conclusion holds clearly. Assume that $\rho(W_2) \neq 0$, i.e., $0 < \rho(W_2) < 1$. By [6, Theorem 3.2], there exists a nonzero vector

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_{1_{2n}},$$

in conformity with W_2 such that $W_2X = \rho(W_2)X$, which can be rewritten into

$$\begin{aligned} P_2^\dagger R_2 x_1 + P_2^\dagger S_2 x_2 &= \rho(W_2)x_1, \\ x_1 &= \rho(W_2)x_2, \end{aligned}$$

where $x_1, x_2 \in K_1$.

Then we have

$$\begin{aligned} W_1X - \rho(W_2)X &= \begin{pmatrix} P_1^\dagger R_1 x_1 + P_1^\dagger S_1 x_2 - \rho(W_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + \frac{1}{\rho(W_2)}(P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}. \end{aligned}$$

(i). Since $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} &\Delta - \frac{1}{\rho(W_2)}((P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + (P_1^\dagger S_1 - P_2^\dagger S_2)x_1) \\ = &\left(\frac{1}{\rho(W_2)} - 1\right)(P_2^\dagger R_2 - P_1^\dagger R_1)x_1 \\ \geqslant_{K_1} &\ 0. \end{aligned}$$

Since $P_1^\dagger R_1 - P_2^\dagger R_2 + P_1^\dagger S_1 - P_2^\dagger S_2 \geqslant_{K_1} 0$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} \Delta &\geqslant_{K_1} \frac{1}{\rho(W_2)}(P_1^\dagger R_1 - P_2^\dagger R_2 + P_1^\dagger S_1 - P_2^\dagger S_2)x_1 \\ &\geqslant_{K_1} 0. \end{aligned}$$

Thus,

$$W_1X - \rho(W_2)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\rho(W_2) \leqslant \rho(W_1) < 1$.

(ii). Since $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} & \Delta - ((P_1^{\dagger}R_1 - P_2^{\dagger}R_2)x_1 + (P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1) \\ = & \left(\frac{1}{\rho(W_2)} - 1\right)(P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1 \\ \geqslant_{K_1} & \ 0. \end{aligned}$$

Since $P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \geqslant_{K_1} 0$, then

$$\begin{aligned} \Delta & \geqslant_{K_1} (P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1 \\ & \geqslant_{K_1} 0. \end{aligned}$$

Thus,

$$W_1X - \rho(W_2)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\rho(W_2) \leqslant \rho(W_1) < 1$. □

If we consider $A^{\dagger} \in \pi(K_2, K_1)$, from Theorem 9, we have the following corollary.

COROLLARY 7. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$, and $A^{\dagger} \in \pi(K_2, K_1)$. Suppose $P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \geqslant_{K_1} 0$, $P_i \in \pi(K_1, K_2)$ for $i = 1, 2$ and any one of the following conditions*

(1). $P_2^{\dagger}R_2 \geqslant_{K_1} P_1^{\dagger}R_1$;

(2). $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. The assumptions $A^{\dagger} \in \pi(K_2, K_1)$ and $P_i \in \pi(K_1, K_2)$ imply $A^{\dagger}P_i \geqslant_{K_1} 0$, for $i = 1, 2$. Theorem 9 then gives $\rho(W_2) \leqslant \rho(W_1) < 1$. □

In the following, we will provide comparison results for proper double nonnegative splittings over the proper cone K_1 of different rectangular matrices.

Let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. Then, we define

$$W_1 = \begin{pmatrix} P_1^{\dagger}R_1 & P_1^{\dagger}S_1 \\ I & 0 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} P_2^{\dagger}R_2 & P_2^{\dagger}S_2 \\ I & 0 \end{pmatrix}.$$

For general rectangular matrices $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, comparing $\rho(W_1)$ with $\rho(W_2)$, we have the following theorem.

THEOREM 10. *Let K_1 be a proper cone in $\mathbb{R}^n$, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$, respectively. Suppose $P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \geqslant_{K_1} 0$, $A_i^{\dagger}P_i \geqslant_{K_1} 0$ for $i = 1,2$ and any one of the following conditions*

(1). $P_2^{\dagger}R_2 \geqslant_{K_1} P_1^{\dagger}R_1$*;*

(2). $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$*,*

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

The proof of Theorems 10 is very similar to the proof of Theorems 9, so we omitted it here.

The following example demonstrates that the condition $P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \geqslant_{K_1} 0$ cannot be dropped in Theorem 10.

EXAMPLE 4. Consider proper cones $K_1 = \{x \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{x \in \mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Assume that

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{5}{2} & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{6} & 3 & 1 \end{pmatrix}.$$

Let A_1 and A_2 be splitted as

$$A_1 = P_1 - R_1 - S_1 \text{ and } A_2 = P_2 - R_2 - S_2,$$

respectively. Setting

$$P_1 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & \frac{5}{2} & 2 \end{pmatrix}, \quad R_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{3}{8} & \frac{1}{6} & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{6} & 0 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 3 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 3 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{6} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{6} & 0 \end{pmatrix}.$$

Then we can see that

$$A_1^{\dagger}P_1 = \begin{pmatrix} 6 & 0 & 0 \\ -0.0610 & 0.6098 & 0.4878 \\ -0.0488 & 0.4878 & 0.3902 \end{pmatrix} \text{ and } A_2^{\dagger}P_2 = \begin{pmatrix} 16 & 0 & 0 \\ 0.5 & 0.9 & 0.3 \\ 0.1667 & 0.3 & 0.1 \end{pmatrix}.$$

Following the operations, we have

$$P_1^{\dagger}R_1 = \begin{pmatrix} 0.1667 & 0 & 0 \\ -0.0508 & 0.0407 & 0 \\ -0.0407 & 0.0325 & 0 \end{pmatrix}, \quad P_1^{\dagger}S_1 = \begin{pmatrix} 0.6667 & 0 & 0 \\ 0.0407 & -0.0407 & 0 \\ 0.0325 & -0.0325 & 0 \end{pmatrix}$$

and

$$P_2^{\dagger}P_2 = \begin{pmatrix} 0.75 & 0 & 0 \\ 0.075 & 0.05 & 0 \\ 0.025 & 0.0167 & 0 \end{pmatrix}, \quad P_2^{\dagger}S_2 = \begin{pmatrix} 0.1875 & 0 & 0 \\ -0.0437 & -0.05 & 0 \\ -0.0146 & -0.0167 & 0 \end{pmatrix}.$$

It should be noted that $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are not proper double nonnegative splittings, but proper double nonnegative splittings over the proper cone K_1.

Clearly, we have $P_2^{\dagger}R_2 \geqslant_{K_1} P_1^{\dagger}R_1$, $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$ and $A_i^{\dagger}P_i \geqslant_{K_1} 0$ for $i = 1,2$. But

$$P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 = \begin{pmatrix} -0.1041 & 0 & 0 \\ -0.0414 & 0 & 0 \\ -0.0186 & 0 & 0 \end{pmatrix},$$

i.e. $P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \ngeqslant_{K_1} 0$.

In fact, we have $\rho(W_1) = 0.9041 < 0.9478 = \rho(W_2) < 1$.

Similar examples can be constructed for proper double nonnegative splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$.

For semimonotone matrices $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$ over proper cones, comparing $\rho(W_1)$ with $\rho(W_2)$, we have the following corollary, which is a direct result of Theorem 10.

COROLLARY 8. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$ be semimonotone matrices over proper cones, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of A_1 and A_2, respectively. Suppose $P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \geqslant_{K_1} 0$, $P_i \in \pi(K_1,K_2)$ for $i = 1,2$ and any one of the following conditions*

(1). $P_2^{\dagger}R_2 \geqslant_{K_1} P_1^{\dagger}R_1$;

(2). $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. Since A_1 and A_2 are semimonotone matrices over proper cones, we have $A_1^{\dagger} \in \pi(K_2,K_1)$ and $A_2^{\dagger} \in \pi(K_2,K_1)$. As $P_i \in \pi(K_1,K_2)$, then we have $A_i^{\dagger}P_i \geqslant_{K_1} 0$ for $i = 1,2$. Moreover, Theorem 10 gives $\rho(W_2) \leqslant \rho(W_1) < 1$. □

When A_1 and A_2 have the same null space, we have the following comparison theorem.

THEOREM 11. *Let K_1 be a proper cone in $\mathbb{R}^n$. Let $A_1 \in \mathbb{R}^{m\times\times n}$ and $A_2 \in \mathbb{R}^{m\times\times n}$ be two matrices having the same null space, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of A_1 and A_2, respectively. Suppose $P_2^{\dagger}A_2 \geqslant_{K_1} P_1^{\dagger}A_1$, $A_i^{\dagger}P_i \geqslant_{K_1} 0$ for $i = 1,2$ and any one of the following conditions*

(1). $P_2^{\dagger}R_2 \geqslant_{K_1} P_1^{\dagger}R_1$;

(2). $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. As $A_i^{\dagger}P_i \geqslant_{K_1} 0$, Theorem 6 then yields $\rho(W_i) < 1$ for $i = 1,2$. Assume that $\rho(W_2) = 0$, then the conclusion holds clearly. Assume that $\rho(W_2) \neq 0$, i.e., $0 < \rho(W_2) < 1$. By [6, Theorem 3.2], there exists a nonzero vector

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_{1_{2n}},$$

in conformity with W_2 such that $W_2X = \rho(W_2)X$, which can be rewritten into

$$\begin{aligned} P_2^{\dagger}R_2x_1 + P_2^{\dagger}S_2x_2 &= \rho(W_2)x_1, \\ x_1 &= \rho(W_2)x_2, \end{aligned}$$

where $x_1, x_2 \in K_1$.

Then we have

$$\begin{aligned} W_1X - \rho(W_2)X &= \begin{pmatrix} P_1^{\dagger}R_1x_1 + P_1^{\dagger}S_1x_2 - \rho(W_2)x_1 \\ x_1 - \rho(W_2)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_1^{\dagger}R_1 - P_2^{\dagger}R_2)x_1 + \frac{1}{\rho(W_2)}(P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}. \end{aligned}$$

Now $N(A_1) = N(A_2)$ implies $R(A_1^T) = R(A_2^T) = R(P_1^T) = R(P_2^T)$. Then $P_1^{\dagger}P_1 = P_2^{\dagger}P_2$.

(i). Since $P_2^{\dagger}R_2 \geqslant_{K_1} P_1^{\dagger}R_1$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned} &\Delta - \frac{1}{\rho(W_2)}((P_1^{\dagger}R_1 - P_2^{\dagger}R_2)x_1 + (P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1) \\ &= \left(\frac{1}{\rho(W_2)} - 1\right)(P_2^{\dagger}R_2 - P_1^{\dagger}R_1)x_1 \\ &\geqslant_{K_1} 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \Delta &\geqslant_{K_1} \frac{1}{\rho(W_2)}(P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1 \\ &= \frac{1}{\rho(W_2)}(P_2^{\dagger}A_2 - P_1^{\dagger}A_1)x_1 \\ &\geqslant_{K_1} 0. \end{aligned}$$

Thus,

$$W_1X - \rho(W_2)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\rho(W_2) \leqslant \rho(W_1) < 1$.

(ii). Since $P_1^{\dagger}S_1 \geqslant_{K_1} P_2^{\dagger}S_2$ and $0 < \rho(W_2) < 1$, then

$$\begin{aligned}
&\Delta - ((P_1^{\dagger}R_1 - P_2^{\dagger}R_2)x_1 + (P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1) \\
= &\left(\frac{1}{\rho(W_2)} - 1\right)(P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1 \\
\geqslant_{K_1} &\ 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
\Delta &\geqslant_{K_1} (P_1^{\dagger}R_1 - P_2^{\dagger}R_2 + P_1^{\dagger}S_1 - P_2^{\dagger}S_2)x_1 \\
&= (P_2^{\dagger}A_2 - P_1^{\dagger}A_1)x_1 \\
&\geqslant_{K_1} 0.
\end{aligned}$$

Consequently,

$$W_1X - \rho(W_2)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\rho(W_2) \leqslant \rho(W_1) < 1$. □

The following example demonstrates that the condition $P_2^{\dagger}A_2 \geqslant_{K_1} P_1^{\dagger}A_1$ cannot be dropped in Theorem 11.

EXAMPLE 5. Consider proper cones $K_1 = \{x \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{x \in \mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

Let

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{6} & 3 & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{6} & 3 & 2 \end{pmatrix}.$$

If A_1 and A_2 are splitted as

$$A_1 = P_1 - R_1 - S_1 \text{ and } A_2 = P_2 - R_2 - S_2,$$

respectively, here

$$P_1 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 2 \end{pmatrix}, \quad R_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{8} & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{8} & 0 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 3 & 2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 3 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{8} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{8} & 0 \end{pmatrix}.$$

Following the operations, we have

$$P_1^\dagger R_1 = \begin{pmatrix} 0.1667 & 0 & 0 \\ -0.0385 & 0.0288 & 0 \\ -0.0256 & 0.0192 & 0 \end{pmatrix}, \quad P_1^\dagger S_1 = \begin{pmatrix} 0.6667 & 0 & 0 \\ 0.0385 & -0.0288 & 0 \\ 0.0256 & -0.0192 & 0 \end{pmatrix}$$

and

$$P_2^\dagger R_2 = \begin{pmatrix} 0.75 & 0 & 0 \\ 0.0577 & 0.0288 & 0 \\ 0.0385 & 0.0192 & 0 \end{pmatrix}, \quad P_2^\dagger S_2 = \begin{pmatrix} 0.1875 & 0 & 0 \\ -0.0337 & -0.0288 & 0 \\ -0.0224 & -0.0192 & 0 \end{pmatrix}.$$

It is easy to verify that $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are proper double nonnegative splittings over the proper cone K_1 of A_1 and A_2, respectively.

It is easy to see that

$$A_1^\dagger P_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0.6923 & 0.4615 \\ 0 & 0.4615 & 0.3077 \end{pmatrix}, \quad A_2^\dagger P_2 = \begin{pmatrix} 16 & 0 & 0 \\ 0.3846 & 0.6923 & 0.4615 \\ 0.2564 & 0.4615 & 0.3077 \end{pmatrix}$$

and

$$P_1^\dagger A_1 = \begin{pmatrix} 0.1667 & 0 & 0 \\ 0 & 0.6923 & 0.4615 \\ 0 & 0.4615 & 0.3077 \end{pmatrix}, \quad P_2^\dagger A_2 = \begin{pmatrix} 0.0625 & 0 & 0 \\ -0.0240 & 0.6923 & 0.4615 \\ -0.0160 & 0.4615 & 0.3077 \end{pmatrix}.$$

Then we can get $A_i^\dagger P_i \geqslant_{K_1} 0$ for $i = 1,2$. However

$$P_2^\dagger A_2 - P_1^\dagger A_1 = \begin{pmatrix} -0.1042 & 0 & 0 \\ -0.0240 & 0 & 0 \\ -0.0160 & 0 & 0 \end{pmatrix} \not\geqslant_{K_1} 0.$$

In fact, we have $\rho(W_1) = 0.9041 < 0.9478 = \rho(W_2) < 1$.

Similar examples can be constructed for proper double nonnegative splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$.

For semimonotone matrices $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$ over proper cones with the same null space, comparing $\rho(W_1)$ with $\rho(W_2)$, we have the following comparison result.

COROLLARY 9. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A_1 \in \mathbb{R}^{m\times n}$ and $A_2 \in \mathbb{R}^{m\times n}$ be semimonotone matrices over proper cones having the same null space, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be proper double nonnegative splittings over the proper cone K_1 of A_1 and A_2, respectively. Suppose $P_2^\dagger A_2 \geqslant_{K_1} P_1^\dagger A_1$, $P_i \in \pi(K_1,K_2)$ for $i = 1,2$ and any one of the following conditions*

(1). $P_2^\dagger R_2 \geqslant_{K_1} P_1^\dagger R_1$;

(2). $P_1^\dagger S_1 \geqslant_{K_1} P_2^\dagger S_2$,

holds, then $\rho(W_2) \leqslant \rho(W_1) < 1$.

Proof. The semi-monotonicity of A_i over proper cones imply that $A_i^\dagger \in \pi(K_2, K_1)$, combining $P_i \in \pi(K_1, K_2)$, we can get $A_i^\dagger P_i \geqslant_{K_1} 0$ for $i = 1, 2$. Theorem 11 then yields $\rho(W_2) \leqslant \rho(W_1) < 1$. □

5. The application

Particularly, when the rectangular linear system (1) is an ill-posed linear system that formed by discretization of Fredholm integral equations of the first kind [12], to find the least squares solution $A^\dagger b$, we give a corresponding modified the well-posed system

$$(A^T A + \lambda I)x = A^T b \tag{7}$$

which is based on Tikhonov's regularization [30], where λ is a regularization parameter and $\lambda > 0$. Let $B_\lambda = A^T A + \lambda I$, where B_λ is nonsingular, then the system (7) can be rewritten in the following equivalent form

$$B_\lambda x = A^T b. \tag{8}$$

Moreover, the authors of the literature [3] have shown that $B_\lambda^{-1} A^T b \to A^\dagger b$ as $\lambda \to 0$. There are two forms of splitting iteration methods for solving the linear system (8):

a). If $B_\lambda \in \mathbb{R}^{n\times n}$ has the single splitting [32]

$$B_\lambda = U_\lambda - V_\lambda, \tag{9}$$

where U_λ is invertible, then the associated iterative is given by

$$x_{i+1} = U_\lambda^{-1} V_\lambda x_i + U_\lambda^{-1} A^T b. \tag{10}$$

It is well known that this iterative method converges to $B_\lambda^{-1} A^T b$ $(= A^\dagger b$ as $\lambda \to 0)$ if and only if $\rho(U_\lambda^{-1} V_\lambda) < 1$.

b). Given the double splitting of $B_\lambda \in \mathbb{R}^{n\times n}$ as [32]

$$B_\lambda = P_\lambda - R_\lambda - S_\lambda, \tag{11}$$

where P is invertible, then the regularized iterative scheme is given by

$$x_{i+1} = P_\lambda^{-1} R_\lambda x_i + P_\lambda^{-1} S_\lambda x_{i-1} + P_\lambda^{-1} A^T b,\ i = 1, 2, \cdots. \tag{12}$$

In order to study the convergence, the iterative scheme (12) can be written as the following equivalent form

$$\begin{pmatrix} x_{i+1} \\ x_i \end{pmatrix} = \begin{pmatrix} P_\lambda^{-1} R_\lambda & P_\lambda^{-1} S_\lambda \\ I & 0 \end{pmatrix} \begin{pmatrix} x_i \\ x_{i-1} \end{pmatrix} + \begin{pmatrix} P_\lambda^{-1} A^T b \\ 0 \end{pmatrix},\ i = 1, 2, \cdots, \tag{13}$$

where I denotes the identity matrix with compatible size and

$$W_\lambda = \begin{pmatrix} P_\lambda^{-1}R_\lambda & P_\lambda^{-1}S \\ I & 0 \end{pmatrix}$$

is the iteration matrix. The iterative scheme (13) converges to the unique solution $B_\lambda^{-1}A^Tb\ (=A^\dagger b\ as\ \lambda \to 0)$ of (8) if and only if $\rho(W_\lambda) < 1$ [13, 28].

The numerical solutions over field of both systems (1) and (8) have been studied and compared in [3, 26].

In the following, we consider the application of the research results of proper nonnegative splittings over proper cones in the regularized iterative method for the ill-posed linear system. When A_2 in Theorem 3 and Theorem 4 is a nonsingular matrix, we have the following results.

THEOREM 12. *Let K_1 be a proper cone in $\mathbb{R}^n$. Let $A = U - V$ be a proper single nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$, $B_\lambda = U_\lambda - V_\lambda$ be a single nonnegative splitting over the proper cone K_1 of $B_\lambda \in \mathbb{R}^{n\times n}$. Suppose $A^\dagger V \geqslant_{K_1} \lim\limits_{\lambda\to 0} B_\lambda^{-1}V_\lambda >_{K_1} 0$ and $A^\dagger V \neq \lim\limits_{\lambda\to 0} B_\lambda^{-1}V_\lambda$, then*

$$\lim_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) < \rho(U^\dagger V) < 1.$$

Proof. Since $\lim\limits_{\lambda\to 0} B_\lambda^{-1}V_\lambda >_{K_1} 0$, so we have

$$\lim_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) = \lim_{\lambda\to 0}\frac{\rho(B_\lambda^{-1}V_\lambda)}{1+\rho(B_\lambda^{-1}V_\lambda)} < 1$$

by applying [15, Lemma 2.5]. Similarly, under the assumption $A^\dagger V >_{K_1} 0$, [5, Theorem 2] implies

$$\rho(U^\dagger V) = \frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)} < 1.$$

In the following, in order to prove that $\lim\limits_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) < \rho(U^\dagger V)$, we first need to show that $\rho(A^\dagger V) > \lim\limits_{\lambda\to 0}\rho(B_\lambda^{-1}V_\lambda)$.

As $A^\dagger V \geqslant_{K_1} \lim\limits_{\lambda\to 0} B_\lambda^{-1}V_\lambda >_{K_1} 0$, so [6, Corollary 3.29] and [14, Corollary 2.6.] imply $\rho(A^\dagger V) > \lim\limits_{\lambda\to 0}\rho(B_\lambda^{-1}V_\lambda)$. Since $f(\lambda) = \frac{\lambda}{1+\lambda}$ is a strictly increasing function for $\lambda \geqslant 0$, so the inequality $\lim\limits_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) < \rho(U^\dagger V)$ is true. □

THEOREM 13. *Let K_1 be a proper cone in $\mathbb{R}^n$. Let $A = U - V$ be a proper single nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$, $B_\lambda = U_\lambda - V_\lambda$ be a single nonnegative splitting over the proper cone K_1 of $B_\lambda \in \mathbb{R}^{n\times n}$. Suppose $A^\dagger U \geqslant_{K_1} \lim\limits_{\lambda\to 0} B_\lambda^{-1}U_\lambda >_{K_1} 0$ and $A^\dagger U \neq \lim\limits_{\lambda\to 0} B_\lambda^{-1}U_\lambda$, then*

$$\lim_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) < \rho(U^\dagger V) < 1.$$

Proof. By applying [15, Lemma 2.5], the assumption $\lim_{\lambda\to 0} B_\lambda^{-1}U_\lambda >_{K_1} 0$ implies

$$\lim_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) = \lim_{\lambda\to 0}\frac{\rho(B_\lambda^{-1}U_\lambda)-1}{\rho(B_\lambda^{-1}U_\lambda)} < 1.$$

As $A^\dagger U >_{K_1} 0$, it follows from [8, Theorem 2] that

$$\rho(U^\dagger V) = \frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)} < 1.$$

So what we need to show now is that $\lim_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) < \rho(U^\dagger V)$. To do this, we first need to demonstrate that $\rho(A^\dagger U) > \lim_{\lambda\to 0}\rho(B_\lambda^{-1}U_\lambda)$.

As $A^\dagger U \geqslant_{K_1} \lim_{\lambda\to 0} B_\lambda^{-1}U_\lambda >_{K_1} 0$, so [6, Corollary 3.29] and [14, Corollary 2.6.] imply $\rho(A^\dagger U) > \lim_{\lambda\to 0}\rho(B_\lambda^{-1}U_\lambda)$. Since $f(\lambda) = \frac{\lambda-1}{\lambda}$ is a strictly increasing function for $\lambda > 0$, so $\lim_{\lambda\to 0}\rho(U_\lambda^{-1}V_\lambda) < \rho(U^\dagger V)$. □

REMARK 3. Theorem 12 and Theorem 13 are special cases of Theorem 3 and Theorem 4, respectively. Particularly, Theorem 12 is Theorem 3.2 of [26] if $K_1 = \mathbb{R}^n_+$.

The example given below shows that $A^\dagger V \geqslant_{K_1} \lim_{\lambda\to 0} B_\lambda^{-1}V_\lambda >_{K_1} 0$ and $A^\dagger U \geqslant_{K_1} \lim_{\lambda\to 0} B_\lambda^{-1}U_\lambda >_{K_1} 0$ cannot be dropped in Theorem 12 and Theorem 13, respectively.

EXAMPLE 6. Consider proper cones $K_1 = \{x\in\mathbb{R}^3 | (x_2^2+x_3^2)^{\frac{1}{2}} \leqslant x_1\}$ and $K_2 = \{x\in\mathbb{R}^2 | (x_2^2)^{\frac{1}{2}} \leqslant x_1\}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

and $\lambda = 10^{-4}$, then

$$B_\lambda = \begin{pmatrix} 20.0001 & 2 & 0 \\ 2 & 1.0001 & 0 \\ 0 & 0 & 0.0001 \end{pmatrix}.$$

Assume that A and B_λ are splitted as

$$A = U - V \text{ and } B_\lambda = U_\lambda - V_\lambda$$

with

$$U = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

and

$$U_\lambda = \begin{pmatrix} 30.0001 & 0 & 0 \\ 3 & 1.0001 & 0 \\ 0 & 0 & 0.0001 \end{pmatrix}, \quad V_\lambda = \begin{pmatrix} 10 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively.

Following the operations, we have

$$U^{\dagger}V = \begin{pmatrix} 0.3333 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } U_{\lambda}^{-1}V_{\lambda} = \begin{pmatrix} 0.3333 & -0.0667 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $B_{\lambda} = U_{\lambda} - V_{\lambda}$ is not a single nonnegative splittings, but a single nonnegative splitting over the proper cone K_1.

Moreover, we have

$$A^{\dagger}V = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{\lambda}^{-1}V_{\lambda} = \begin{pmatrix} 0.5 & -0.125 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A^{\dagger}U = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{\lambda}^{-1}U_{\lambda} = \begin{pmatrix} 1.5 & -0.125 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But

$$A^{\dagger}V - B_{\lambda}^{-1}V_{\lambda} = \begin{pmatrix} 0 & 0.125 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A^{\dagger}U - B_{\lambda}^{-1}U_{\lambda} = \begin{pmatrix} 0 & 0.125 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

i.e., $A^{\dagger}V \ngeqslant_{K_1} B_{\lambda}^{-1}V_{\lambda}$ and $A^{\dagger}U \ngeqslant_{K_1} B_{\lambda}^{-1}U_{\lambda}$.

In fact, we have $\rho(U_{\lambda}^{-1}V_{\lambda}) = 0.3333 = \rho(U^{\dagger}V) < 1$.

In what follows of this section, comparison theorems between the spectral radii of matrices arising from double splittings over proper cones of the rectangular matrix and the square matrix are presented.

THEOREM 14. *Let K_1 be a proper cone in $\mathbb{R}^n$. Let $A = P - R - S$ be a proper double nonnegative splitting over the proper cone K_1 of $A \in \mathbb{R}^{m\times n}$ with $A^{\dagger}P \geqslant_{K_1} 0$, $B_{\lambda} = P_{\lambda} - R_{\lambda} - S_{\lambda}$ be a double nonnegative splitting over the proper cone K_1 of $B_{\lambda} \in \mathbb{R}^{n\times n}$ with $\lim_{\lambda\to 0} B_{\lambda}^{-1}P_{\lambda} \geqslant_{K_1} 0$. If $P^{\dagger}R \geqslant_{K_1} \lim_{\lambda\to 0} P_{\lambda}^{-1}R_{\lambda}$ and $P^{\dagger}S \geqslant_{K_1} \lim_{\lambda\to 0} P_{\lambda}^{-1}S_{\lambda}$, then*

$$\lim_{\lambda\to 0} \rho(W_{\lambda}) \leqslant \rho(W) < 1.$$

Proof. As $A^{\dagger}P \geqslant_{K_1} 0$, Theorem 6 then yields $\rho(W) < 1$. Setting $U_{\lambda} = P_{\lambda}$ and $V_{\lambda} = R_{\lambda} + S_{\lambda}$, then we get that $B_{\lambda} = U_{\lambda} - V_{\lambda}$ is a single nonnegative splitting over the proper cone K_1 and $\lim_{\lambda\to 0} B_\lambda^{-1}P_\lambda = \lim_{\lambda\to 0} B_\lambda^{-1}U_\lambda \geqslant_{K_1} 0$. It follows from [15, Lemma 2.5] and [33, Theorem 1] that $\lim_{\lambda\to 0}\rho(W_\lambda) < 1$.

Assume that $\lim_{\lambda\to 0}\rho(W_\lambda) = 0$, then the conclusion holds clearly. Assume that $\lim_{\lambda\to 0}\rho(W_\lambda) \neq 0$, i.e., $0 < \lim_{\lambda\to 0}\rho(W_\lambda) < 1$. By [6, Theorem 3.2], there exists a nonzero vector

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_{1_{2n}},$$

in conformity with $\lim_{\lambda\to 0} W_\lambda$ such that $\lim_{\lambda\to 0} W_\lambda X = \lim_{\lambda\to 0}\rho(W_\lambda)X$, which can be rewritten into

$$\begin{aligned} \lim_{\lambda\to 0} P_\lambda^{-1}R_\lambda x_1 + \lim_{\lambda\to 0} P_\lambda^{-1}S_\lambda x_2 &= \lim_{\lambda\to 0}\rho(W_\lambda)x_1, \\ x_1 &= \lim_{\lambda\to 0}\rho(W_\lambda)x_2, \end{aligned}$$

where $x_1, x_2 \in K_1$.

Then we have

$$\begin{aligned} WX - \lim_{\lambda\to 0}\rho(W_\lambda)X &= \begin{pmatrix} P^\dagger R x_1 + P^\dagger S_1 x_2 - \lim_{\lambda\to 0}\rho(W_\lambda)x_1 \\ x_1 - \lim_{\lambda\to 0}\rho(W_\lambda)x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P^\dagger R - \lim_{\lambda\to 0} P_\lambda^{-1}R_\lambda)x_1 + \frac{1}{\lim_{\lambda\to 0}\rho(W_\lambda)}(P^\dagger S - \lim_{\lambda\to 0} P_\lambda^{-1}S_\lambda)x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}. \end{aligned}$$

As $P^\dagger R \geqslant_{K_1} \lim_{\lambda\to 0} P_\lambda^{-1}R_\lambda$ and $P^\dagger S \geqslant_{K_1} \lim_{\lambda\to 0} P_\lambda^{-1}S_\lambda$, so

$$\Delta = (P^\dagger R - \lim_{\lambda\to 0} P_\lambda^{-1}R_\lambda)x_1 + \frac{1}{\lim_{\lambda\to 0}\rho(W_\lambda)}(P^\dagger S - \lim_{\lambda\to 0} P_\lambda^{-1}S_\lambda)x_1 \geqslant_{K_1} 0.$$

Hence,

$$WX - \lim_{\lambda\to 0}\rho(W_\lambda)X \geqslant_{K_1} 0.$$

It follows from [22, Corollary 3.2] that $\lim_{\lambda\to 0}\rho(W_\lambda) \leqslant \rho(W) < 1$.

The numerical example is given below to demonstrate Theorem 14.

EXAMPLE 7. Consider the proper cone $K_1 = \mathbb{R}^3_+$. If

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

is splitted as

$$A = P - R - S,$$

here

$$P = \begin{pmatrix} 12 & 6 & 0 \\ 4 & 2 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 7 & 3 & 0 \\ 1 & 0.5 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0.5 & 0 \end{pmatrix}.$$

For $\lambda = 10^{-4}$, let

$$B_\lambda = \begin{pmatrix} 20.0001 & 10 & 0 \\ 10 & 5.0001 & 0 \\ 0 & 0 & 0.0001 \end{pmatrix}$$

be splitted as

$$B_\lambda = P_\lambda - R_\lambda - S_\lambda,$$

here

$$P_\lambda = \begin{pmatrix} 24.0001 & 12 & 0 \\ 12 & 6.0001 & 0 \\ 0 & 0 & 0.0001 \end{pmatrix}, \quad R_\lambda = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } S_\lambda = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$A^\dagger P = \begin{pmatrix} 2.24 & 1.12 & 0 \\ 1.12 & 0.56 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B_\lambda^{-1} P_\lambda = \begin{pmatrix} 1.16 & 0.08 & 0 \\ 0.08 & 1.04 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Following the operations, we have

$$P^\dagger R - P_\lambda^{-1} R_\lambda = \begin{pmatrix} 0.3733 & 0.19 & 0 \\ 0.1867 & 0.095 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P^\dagger S - P_\lambda^{-1} S_\lambda = \begin{pmatrix} 0.0133 & 0.0033 & 0 \\ 0.0067 & 0.0017 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, the assumptions of Theorem 14 are satisfied.

In fact, we have $\rho(W_\lambda) = 0.3513 < 0.6994 = \rho(W) < 1$.

What should be noted that Theorem 14 is a generalization of (i) of Theorem 3.20 in [26]. In a word, several results of the reference [26] are included in the theoretical results of this paper. Meanwhile, the numerical examples given above show that the regularized iterative method based on splittings over proper cones has stronger applicability.

6. Conclusion

In this paper, convergence results for the proper double nonnegative splitting over the proper cone K_1 of a rectangular matrix are established. Comparison theorems for the spectral radii of matrices arising from proper nonnegative splittings over the proper cone K_1 of the same rectangular matrix or different rectangular matrices are presented. The application of research results of proper nonnegative splittings over proper cones in ill-posed linear systems is given.

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