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ON A GENERALIZATION OF LEFT AND RIGHT INVERTIBLE OPERATORS ON BANACH SPACES

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Abstract. The purpose of this paper is to define and study left and right versions of the large class of Drazin invertible operators on Banach spaces, namely left and right Drazin invertible operators, as a generalization of left and right invertible operators. It is shown in particular that the operators introduced can be characterized by means of Kato decompositions.

1. Introduction

Given a Banach space X, we recall that a bounded operator T acting on X is said to be Drazin invertible if there exists a nonnegative integer j and the (unique) bounded operator S such that $ST = TS$, $STS = S$ and $ST^{j+1} = T^j$. Drazin invertible operators are a generalization of invertible operators and they have been extensively studied in the litterature, for instance see [1], [2] [5], [6]. It is well-known that Drazin invertible operators can be characterized in many ways, among which we shall recall the characterization by means of Kato decomposition. Precisely, a bounded operator T on a Banach space X is Drazin invertible if and only if it can be decomposed into a direct sum of an invertible operator and a nilpotent one, namely, if there exists some closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is invertible and T_N is nilpotent.

The purpose of this paper is to suggest left and right versions of Drazin invertible operators that subsume the classes of left and right invertible operators. It will be explored how far various known results for Drazin invertible operators have corresponding versions for left and right Drazin invertible operators, most importantly, the property of Kato decomposition.

The paper is composed of 4 sections. In details, section 2 is meant to recall some elementary facts on linear operators on linear and normed linear spaces. In section 3, we define left and right Drazin invertible operators and we show that these operators can be decomposed into a direct sum of an operator with the property of semi-regularity and another which is nilpotent. From such decompositions, several characterizations of left and right Drazin invertible operators will be given. In the final section, we investigate the relationships between left and right Drazin invertible operators and other classes of operators related to left and right invertible operators.

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2. Preliminary results

Let X be a real or complex linear space and T be a linear operator in X. Let $\mathscr{N}(T)$ denotes its kernel and $\mathscr{R}(T)$ denotes its range. We can define iterates T^2, T^3,..., T^n,... of T for $n > 0$ and we follow the convention that $T^0 = I$. Consider now $\mathscr{N}(T^n)$ and $\mathscr{R}(T^n)$. We recall the following result.

LEMMA 2.1. [4] *For $k = 0,1,2,\dots$ and $i = 0,1,2,\dots$, we have*

$$(a)\quad \frac{\mathscr{N}(T^{i+k})}{\mathscr{N}(T^i)} \simeq \mathscr{N}(T^k)\cap\mathscr{R}(T^i) \text{ and } \qquad (b)\quad \frac{\mathscr{R}(T^i)}{\mathscr{R}(T^{i+k})} \simeq \frac{X}{\mathscr{R}(T^k)+\mathscr{N}(T^i)}.$$

It is known that if $\mathscr{N}(T^k) = \mathscr{N}(T^{k+1})$, then $\mathscr{N}(T^n) = \mathscr{N}(T^k)$ for all $n \geqslant k$. In this case, the smallest nonnegative integer k such that $\mathscr{N}(T^k) = \mathscr{N}(T^{k+1})$ is called the ascent of T; it is denoted by $\alpha(T)$. If no such k exists, we define $\alpha(T) = \infty$. Similarly, if $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$, then $\mathscr{R}(T^n) = \mathscr{R}(T^k)$ when $n \geqslant k$. If there is such a k, the smallest such k is called the descent of T and denoted by $\delta(T)$. We write $\delta(T) = \infty$ if $\mathscr{R}(T^{n+1})$ is always a proper subset of $\mathscr{R}(T^n)$.

The notion of the degree $\delta(T)$ of an operator T is a modification of the notion of descent. It is defined on the following set

$$\Delta(T) := \{n \in \mathbb{N} : \forall m \geqslant n,\ \mathscr{N}(T)\cap\mathscr{R}(T^n) = \mathscr{N}(T)\cap\mathscr{R}(T^m)\}$$

to be the quantity

$$\delta(T) := \begin{cases} \min\Delta(T) & \text{if} \quad \Delta(T) \neq \emptyset \\ \infty & \text{if} \quad \Delta(T) = \emptyset \end{cases}$$

We recall that a subspace M of X is T-invariant if $TM \subseteq M$, in this case, the restriction of T to M is a linear operator in M. It is denoted by T_M. A subspace M reduces T if there is another subspace N such that M and N are T-invariant and $X = M \oplus N$. In this case, the restrictions T_M and T_N act on M and N respectively, and $T = T_M \oplus T_N$ in the sense that any $x \in X$ has a unique decomposition $x = x_M + x_N$ with $x_M \in M$, $x_N \in N$ and $Tx = T_M x_M + T_N x_N$.

LEMMA 2.2. *If there exist T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M satisfies $\mathscr{N}(T_M) \subseteq \mathscr{R}(T_M^n)$ (or equivalently, $\mathscr{N}(T_M^n) \subseteq \mathscr{R}(T_M)$) for all nonnegative integer n and T_N is nilpotent of some degree d, then*

$$\mathscr{N}(T)\cap\mathscr{R}(T^n) = \mathscr{N}(T_M) \text{ and } \mathscr{R}(T)+\mathscr{N}(T^n) = \mathscr{R}(T_M)\oplus N \text{ for all } n \geqslant d.$$

Proof. Let $T = T_M \oplus T_N$ where T_M satisfies $\mathscr{N}(T_M) \subseteq \mathscr{R}(T_M^n)$ for all nonnegative integer n and $T_N^d = 0$ for some nonnegative integer d. Then,

$$\begin{aligned}
\mathscr{N}(T)\cap\mathscr{R}(T^d) &= (\mathscr{N}(T_M)+\mathscr{N}(T_N))\cap(\mathscr{R}(T_M^d)+\mathscr{R}(T_N^d)) \\
&= (\mathscr{N}(T_M)+\mathscr{N}(T_N))\cap\mathscr{R}(T_M^d) \\
&= [\mathscr{N}(T_M)\cap\mathscr{R}(T_M^d)]+[\mathscr{N}(T_N)\cap\{0\}] \\
&= \mathscr{N}(T_M),
\end{aligned}$$

and

$$\begin{aligned}\mathscr{R}(T)+\mathscr{N}(T^d) &= (\mathscr{R}(T_M)\oplus\mathscr{R}(T_N))+(\mathscr{N}(T_M^d)\oplus\mathscr{N}(T_N^d))\\ &= \mathscr{R}(T_M)\oplus\mathscr{R}(T_N)+\mathscr{N}(T_M^d)\oplus N\\ &= \mathscr{R}(T_M)\oplus N. \quad \square\end{aligned}$$

Assume now that X is normed. A closed subspace M of X is said to be topologically complemented in X if it is algebraically complemented by a closed subspace N. The following lemma is purely algebraic and topological. It will be however a central result to the rest of the paper.

LEMMA 2.3. *Let M and N be closed subspaces of X such that $X = M\oplus N$. Let F be a subspace of M and G be a subspace of N. Then, $F\oplus G$ is topologically complemented in X if and only if F is topologically complemented in M and G is topologically complemented in N.*

Proof. The proof of this lemma will be divided into two steps.

• *Step I*: We shall prove that $F\oplus G$ is closed if and only if F and G are closed. Let P be the bounded projection onto M along N. Towards the first implication, let $\{x_n\}$ be a sequence in F which converges to $x\in X$. Obviously, $x\in F\oplus G$ so that $x = x_F + x_G$ where $x_F\in F$ and $x_G\in G$. On the other hand, we note that $Px = x$ so that $x\in M$. Thus, $x_G = x - x_F \in M$ and hence $x_G = 0$. Consequently, F is closed. To prove that G is closed, a similar argument yields to the result. Conversely, let $\{x_n\}$ be a sequence in $F\oplus G$ which converges to $x\in X$ and let $\{y_n\}$ be a sequence of F and $\{z_n\}$ be a sequence of G such that $x_n = y_n + z_n$. We have $x_n = P(x_n)+(I-P)(x_n)$ which converges to $P(x)+(I-P)(x) = x$. Also, we know by the uniqueness of the decomposition that $y_n = P(x_n)$ and that $z_n = (I-P)(x_n)$. Thus, since F and G are closed, it follows that $x\in F\oplus G$.

• *Step II*: Assume that $F\oplus G$ is topologically complemented in X. Then $F\oplus G$ is closed and there is Z closed in X such that $X = (F\oplus G)\oplus Z$. Let $H := M\cap(G\oplus Z)$ which we claim to be a topological complement of F in M. Towards proving this, we shall prove first that $G\oplus Z$ is closed in X (which entails that H is closed in M). Let $\{x_n\}$ be a sequence in $G\oplus Z$ which converges to $x\in X$ and let $\{y_n\}$ be a sequence of G and $\{z_n\}$ be a sequence of Z such that $x_n = y_n + z_n$. Take now the bounded projection P onto Z along $F\oplus G$. We have, $x_n = (I-P)(x_n)+P(x_n)$ which converges to $(I-P)(x)+P(x) = x$. But G is closed by Step I and Z is closed by construction. Thus, $x\in G\oplus Z$. On the other hand, we have

$$\begin{aligned}F+H &= F+(G\oplus Z)\cap M\\ &= (F+G\oplus Z)\cap M\\ &= X\cap M\\ &= M,\end{aligned}$$

and $F\cap H=F\cap M\cap(G\oplus Z)=\{0\}$. Since F is closed in M by Step I, it is topologically complemented in M by H. Finally, the proof that G is topologically complemeted in N can be done in the same way by taking $N\cap(F\oplus Z)$ as its topological complement in N.

Conversely, assume that F is topologically complemented in M and that G is topologically complemented in N. By Step I, $F\oplus G$ is closed in X. Let now Z_1 be a topological complement of F in M and Z_2 be a topological complement of G in N, then take $Z:=Z_1\oplus Z_2$ which we claim to be a topological complement of $F\oplus G$ in X. Indeed, firstly Z is closed in X by Step I. On the other hand, obviously, $(F\oplus G)+Z=X$. Hence, in order to finish the proof, we shall demonstrate that $(F\oplus G)\cap Z=\{0\}$. Towards this, let $x\in(F\oplus G)\cap Z=(F\oplus G)\cap(Z_1\oplus Z_2)$. Then $x=x_F+x_G=x_{Z_1}+x_{Z_2}$ where $x_F\in F$, $x_G\in G$, $x_{Z_1}\in Z_1$ and $x_{Z_2}\in Z_2$. It follows that $x_F=x_{Z_1}+x_{Z_2}-x_G$ so that $x_F\in(Z_1+N)\cap F\subseteq(Z_1+N)\cap M$. As $(Z_1+N)\cap M=Z_1+(N\cap M)=Z_1$, it follows that $x_F=0$ and hence $x=x_G$. Consequently, $x=x_G\in(Z_2+Z_1)\cap G\subseteq(Z_2+Z_1)\cap N$. But $(Z_2+Z_1)\cap N=Z_2+(Z_1\cap N)=Z_2$. Hence, $x\in G\cap Z_2=\{0\}$. $\square$

In the remaining of the paper, we let X denote an infinite dimensional Banach space and we let $\mathscr{B}(X)$ denote the set of all linear bounded operators on X. We recall that an operator $T\in\mathscr{B}(X)$ is said to be left invertible if there exists an operator $S\in\mathscr{B}(X)$ such that $ST=I$ (where I is the identity operator) while T is said to be right invertible if there exists $A\in\mathscr{B}(X)$ such that $TA=I$. We also recall that an operator $T\in\mathscr{B}(X)$ is said to be semi-regular if $\mathscr{R}(T)$ is closed and $\mathscr{N}(T)\subseteq\mathscr{R}(T^n)$ (or equivalently, $\mathscr{N}(T^n)\subseteq\mathscr{R}(T)$) for all nonnegative integer n, upper Drazin invertible if $p:=\alpha(T)<\infty$ and $\mathscr{R}(T^{p+1})$ is closed, lower Drazin invertible if $q:=\delta(T)<\infty$ and $\mathscr{R}(T^q)$ is closed ([1]), and quasi-Fredholm of degree d if $d=\delta(T)$ and the subspaces $\mathscr{N}(T)\cap\mathscr{R}(T^d)$ and $\mathscr{R}(T)+\mathscr{N}(T^d)$ are closed. We note that an operator $T\in\mathscr{B}(X)$ is quasi-Fredholm of degree 0 if and only if it is semi-regular.

We end up this section by recalling that an operator $T\in\mathscr{B}(X)$ is said to have a Kato decomposition if there exist closed T-invariant subspaces M and N of X such that $X=M\oplus N$ and $T=T_M\oplus T_N$ where T_M is semi-regular and T_N is nilpotent.

3. Left and right Drazin invertible operators

It is well-known that an operator $T\in\mathscr{B}(X)$ is left invertible if and only if it is one-to-one and its range is topologically complemented. In such a case, $p:=\alpha(T)=0$ so that $\mathscr{N}(T)=\{0\}$. It is also known that an operator $T\in\mathscr{B}(X)$ is right invertible if and only if it is surjective with a topologically complemented kernel. In such a case, $q:=\delta(T)=0$ and $\mathscr{R}(T)=X$. Motivated by these two characterizations, we propose the following two generalizations of left and right invertible operators.

DEFINITION 3.1. An operator $T\in\mathscr{B}(X)$ will be called left Drazin invertible if $p:=\alpha(T)<\infty$ and the subspace $\mathscr{R}(T)+\mathscr{N}(T^p)$ is topologically complemented in X, while $T\in\mathscr{B}(X)$ will be called right Drazin invertible if $q:=\delta(T)<\infty$ and the subspace $\mathscr{N}(T)\cap\mathscr{R}(T^q)$ is topologically complemented in X.

Obviously, an operator $T \in \mathscr{B}(X)$ is Drazin invertible if and only if it is both left and right Drazin invertible. In fact, the converse implication is clear. For the direct one, if $d := \alpha(T) = \delta(T) < \infty$, it follows from Lemma 2.1 that $\mathscr{N}(T) \cap \mathscr{R}(T^d) = \{0\}$ and $X = \mathscr{R}(T) + \mathscr{N}(T^d)$.

Observe now that any left or right Drazin invertible operator is quasi-Fredholm of some degree d. To see this, let $T \in \mathscr{B}(X)$ be left Drazin invertible. If $\alpha(T) = 0$, then T is obviously quasi-Fredholm if degree 0. Otherwise, if $1 \leqslant p := \alpha(T) < \infty$, then it follows from Lemma 2.1 (a) that $\mathscr{N}(T) \cap \mathscr{R}(T^{p-1}) \neq \{0\}$ and $\mathscr{N}(T) \cap \mathscr{R}(T^k) = \{0\}$ for all $k \geqslant p$. Hence, since $\mathscr{N}(T) \cap \mathscr{R}(T^p) = \{0\}$ and $\mathscr{R}(T) + \mathscr{N}(T^p)$ are closed, T is quasi-Fredholm of degree p. Similarly, if T is right Drazin invertible, it is quasi-Fredholm if degree 0 whenever $\delta(T) = 0$. Otherwise, if $1 \leqslant q := \delta(T) < \infty$, it follows from Lemma 2.1 (b) that $\mathscr{R}(T) + \mathscr{N}(T^{q-1}) \neq X$ and $\mathscr{R}(T) + \mathscr{N}(T^k) = X$ all $k \geqslant q$ so that T is quasi-Fredholm of degree q since $\mathscr{R}(T) + \mathscr{N}(T^q) = X$ and $\mathscr{N}(T) \cap \mathscr{R}(T^q)$ are closed. Thus, [7, Proposition 3] implies that each left Drazin invertible operator is upper Drazin invertible and that each right Drazin invertible operator is lower Drazin invertible. We shall also note that if X is a Hilbert space, then the converse implications hold.

3.1. Kato decomposition of left Drazin invertible operators and consequences

We begin this section by giving the Kato decomposition of left Drazin invertible operators.

THEOREM 3.2. *Let $T \in \mathscr{B}(X)$. Then, the following statements are equivalent:*

(i) *T is left Drazin invertible.*

(ii) *There exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is nilpotent.*

Proof. $(i) \Rightarrow (ii)$ Since $p := \alpha(T) < \infty$, it follows from Lemma 2.1(a) that $\mathscr{N}(T) \cap \mathscr{R}(T^p) = \{0\}$ so that $\mathscr{N}(T) \cap \mathscr{R}(T^p)$ is topologically complemented. Now, since T is quasi-Fredholm of degree p and the subspace $\mathscr{R}(T) + \mathscr{N}(T^p)$ is also topologically complemented, it follows from [7, Theorem 5] that there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is semi-regular and T_N satisfies $T_N^p = 0$, that is nilpotent. We shall prove now that T_M is left invertible. Towards this, we know from Lemma 2.1 (a) that $\mathscr{N}(T) \cap \mathscr{R}(T^p) = \{0\}$ since $p := \alpha(T) < \infty$. Thus, Lemma 2.2 implies that T_M is one-to-one. On the other hand, we also know from Lemma 2.2 that $\mathscr{R}(T) + \mathscr{N}(T^p) = \mathscr{R}(T_M) \oplus N$. Thus, since $\mathscr{R}(T) + \mathscr{N}(T^p)$ is topologically complemented in X, it follows from Lemma 2.3 that $\mathscr{R}(T_M)$ is topologically complemented in M.

$(ii) \Rightarrow (i)$ Suppose that there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is nilpotent of degree p. We know from Lemma 2.2 that $\mathscr{N}(T) \cap \mathscr{R}(T^p) = \mathscr{N}(T_M)$ and that $\mathscr{R}(T) + \mathscr{N}(T^p) = \mathscr{R}(T_M) \oplus N$. Thus, since $\mathscr{N}(T) \cap \mathscr{R}(T^{p-1}) \neq \{0\}$ and $\mathscr{N}(T) \cap \mathscr{R}(T^p) = \mathscr{N}(T_M) = \{0\}$, it follows from Lemma 2.1 (a) that $p = \alpha(T) < \infty$. Finally, the fact that $\mathscr{R}(T_M)$ is topologically complemented in M entails from Lemma 2.3 that $\mathscr{R}(T)+\mathscr{N}(T^p)$ is topologically complemented in X. $\square$

Based on Theorem 3.2, left Drazin invertible operators can be also characterized in the following way.

THEOREM 3.3. *Let $T \in \mathscr{B}(X)$. Then, T is left Drazin invertible if and only if $p := \alpha(T) < \infty$ and $\mathscr{N}(T^p)$ and $\mathscr{R}(T^{p+1})$ are topologically complemented.*

Proof. If T is left Drazin invertible, then by Theorem 3.2, there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is nilpotent of degree $p := \alpha(T)$. It follows that $T^p = T_M^p \oplus 0$ and that $T^{p+1} = T_M^{p+1} \oplus 0$, so that $\mathscr{N}(T^p) = N$ which is topologically complemented and $\mathscr{R}(T^{p+1}) = \mathscr{R}(T_M^{p+1})$. Since $\mathscr{R}(T_M^{p+1})$ is topologically complemented in M, it follows from Lemma 2.3 that $\mathscr{R}(T^{p+1})$ is topologically complemented in X. Conversely, let $p := \alpha(T) < \infty$ and assume that $\mathscr{N}(T^p)$ and $\mathscr{R}(T^{p+1})$ are topologically complemented. Let F and G be closed subspaces of X such that $X = \mathscr{R}(T^{p+1}) \oplus F = \mathscr{N}(T^p) \oplus G$. First, note that $T^{-p}(\mathscr{R}(T^{p+1})) = \mathscr{R}(T) + \mathscr{N}(T^p)$ and that $T^{-p}(\mathscr{R}(T^{p+1}) + F) = T^{-p}(\mathscr{R}(T^{p+1})) + T^{-p}(F)$ since $\mathscr{R}(T^{p+1}) \subset \mathscr{R}(T^p)$. Let now $H := G \cap T^{-p}(F)$ which we claim to be a topological complement of $\mathscr{R}(T) + \mathscr{N}(T^p)$. Indeed, we have

$$
\begin{aligned}
(\mathscr{R}(T)+\mathscr{N}(T^p))+H &= \mathscr{R}(T)+\mathscr{N}(T^p)+G\cap T^{-p}(F)\\
&= \mathscr{R}(T)+(\mathscr{N}(T^p)+G)\cap T^{-p}(F)\\
&= \mathscr{R}(T)+T^{-p}(F).
\end{aligned}
$$

Since $\mathscr{N}(T^p) = T^{-p}(\{0\}) \subset T^{-p}(F)$, it follows that

$$
\begin{aligned}
(\mathscr{R}(T)+\mathscr{N}(T^p))+H &= \mathscr{R}(T)+T^{-p}(F)\\
&= \mathscr{R}(T)+\mathscr{N}(T^p)+T^{-p}(F)\\
&= X.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(\mathscr{R}(T)+\mathscr{N}(T^p))\cap H &= (\mathscr{R}(T)+\mathscr{N}(T^p))\cap T^{-p}(F)\cap G\\
&= \mathscr{N}(T^p)\cap G\\
&= \{0\}.
\end{aligned}
$$

Finally, the closedness of $\mathscr{R}(T)+\mathscr{N}(T^p)$ and H are obvious and thus the proof is completed. $\square$

We are now interested in giving other characterizations of left Drazin invertible operators. For this purpose, we let T_n denote the restriction of an operator $T \in \mathscr{B}(X)$ to $\mathscr{R}(T^n)$ viewed as a map from $\mathscr{R}(T^n)$ into itself.

THEOREM 3.4. *Let $T \in \mathscr{B}(X)$. Then, the following statements are equivalent:*

(i) *T is left Drazin invertible.*

(ii) *There exists a nonnegative integer n such that $\mathscr{N}(T^n)$ is topologically complemented, $\mathscr{R}(T^n)$ is closed and $T_n : \mathscr{R}(T^n) \longrightarrow \mathscr{R}(T^n)$ is left invertible.*

(iii) *There exists a projection $P \in \mathscr{B}(X)$ such that $PT = TP$, $T + P$ is left invertible and TP is nilpotent.*

Proof. $(i) \Leftrightarrow (ii)$ If T is left Drazin invertible, then $p := \alpha(T) < \infty$ and T is quasi-Fredholm of degree p so that $\mathscr{R}(T^p)$ is closed by [7, Proposition 3]. Also, by Theorem 3.3, $\mathscr{N}(T^p)$ and $\mathscr{R}(T^{p+1})$ are topologically complemented. Let now $T_p : \mathscr{R}(T^p) \longrightarrow \mathscr{R}(T^p)$. From Lemma 2.1 (a) we have $\mathscr{N}(T_p) = \mathscr{N}(T) \cap \mathscr{R}(T^p) = \{0\}$ so that T_p is one-to-one. On the other hand, take F a topological complement of $\mathscr{R}(T^{p+1})$ in X. Then, $F \cap \mathscr{R}(T^p)$ is a topological complement of $\mathscr{R}(T^{p+1})$ in $\mathscr{R}(T^p)$ and hence, T_p is left invertible. Conversely, assume that there exists a nonnegative integer n such that $\mathscr{N}(T^n)$ is topologically complemented, $\mathscr{R}(T^n)$ is closed and $T_n : \mathscr{R}(T^n) \longrightarrow \mathscr{R}(T^n)$ is left invertible. Since $\mathscr{N}(T_n) = \mathscr{N}(T) \cap \mathscr{R}(T^n) = \{0\}$, it follows from Lemma 2.1 (a) that $p := \alpha(T) \leqslant n < \infty$. Finally, since $\mathscr{R}(T^{n+1})$ is topologically complemented in $\mathscr{R}(T^n)$ and $\mathscr{N}(T^n)$ is topologically complemented in X, we can prove that $\mathscr{R}(T) + \mathscr{N}(T^n) = \mathscr{R}(T) + \mathscr{N}(T^p)$ is topologically complemented in X by repeating the same construction which was done in the proof of Theorem 3.3.

$(i) \Leftrightarrow (iii)$ If T is left Drazin invertible, then there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is nilpotent of degree $p := \alpha(T)$. Consider the bounded projection P onto N along M. We have, $N = \mathscr{R}(P)$ and $M = \mathscr{N}(P)$. Take now $x \in X$. Then, x can be decomposed in a unique way as $x_M + x_N$ where $x_M \in M$ and $x_N \in N$. Thus we have, $TP(x) = Tx_N$ and $PT(x) = P(Tx_M + Tx_N) = Tx_N$, so that $TP = PT$. Also, we have $(TP)^p(x) = T^pP^p(x_M + x_N) = T^p(x_N) = (T_M^p \oplus \{0\})(x_N) = 0$ which entails that TP is nilpotent. On the other hand, obviously $(T+P)_{\mathscr{N}(P)} = T_{\mathscr{N}(P)} = T_M$ which is left invertible and $(T+P)_{\mathscr{R}(P)} = T_{\mathscr{R}(P)} + I_{\mathscr{N}(P)} = T_N + I_N$ which is invertible since T_N is nilpotent. Therefore, $T + P = (T+P)_M \oplus (T+P)_N = T_M \oplus (T_N + I_N)$ so that $T + P$ is one-to-one and $\mathscr{R}(T+P) = \mathscr{R}(T_M) \oplus N$. Since $\mathscr{R}(T_M)$ is topologically complemented in M, it follows from Lemma 2.3 that $\mathscr{R}(T+P)$ is topologically complemented in X so that $T + P$ is left invertible. Towards the converse implication, assume that there exists a projection $P \in \mathscr{B}(X)$ such that $PT = TP$, $T + P$ is left invertible and TP is nilpotent of degree d. Since P is bounded, we know that $\mathscr{N}(P)$ and $\mathscr{R}(P)$ are closed and $X = \mathscr{N}(P) \oplus \mathscr{R}(P)$. Since $PT = TP$, $\mathscr{N}(P)$ and $\mathscr{R}(P)$ are T-invariant. Consider now $T_{\mathscr{N}(P)} : \mathscr{N}(P) \longrightarrow \mathscr{N}(P)$ and $T_{\mathscr{R}(P)} : \mathscr{R}(P) \longrightarrow \mathscr{R}(P)$. Then $T = T_{\mathscr{N}(P)} \oplus T_{\mathscr{R}(P)}$ and $T + P = T_{\mathscr{N}(P)} \oplus (T_{\mathscr{R}(P)} + I_{\mathscr{R}(P)})$. Also, $T_{\mathscr{R}(P)}$ is nilpotent since for $x \in \mathscr{R}(P)$, we have $T^d_{\mathscr{R}(P)}(x) = T^dP^d(x) = (TP)^d(x) = 0$. To finish the proof, we shall prove that $T_{\mathscr{N}(P)}$ is left invertible. Towards this, we know that $T + P = T_{\mathscr{N}(P)} \oplus (T_{\mathscr{R}(P)} + I_{\mathscr{R}(P)})$ and that $T + P$ is left invertible. Thus, $T_{\mathscr{N}(P)}$ is one-to-one and $\mathscr{R}(T_{\mathscr{N}(P)}) \oplus \mathscr{R}(P)$ is topologically complemented in X. Finally, the fact that $\mathscr{R}(T_{\mathscr{N}(P)})$ is topologically complemented in $\mathscr{N}(P)$ follows from Lemma 2.3. $\square$

REMARKS 3.5. Let $T \in \mathscr{B}(X)$ be an operator that satisfies the following conditions: there exists a nonnegative integer j and a bounded operator S such that $\mathscr{R}(T^j)$ is closed, $\mathscr{N}(T^j)$ is topologically complemented, $ST^{j+1} = T^j$ and $TS(\mathscr{R}(T^j)) \subset \mathscr{R}(T^{j+1})$. Then, we claim that T is left Drazin invertible. Indeed, to prove this, it suffices to apply Theorem 3.4 (ii) and demonstrate that $T_j : \mathscr{R}(T^j) \longrightarrow \mathscr{R}(T^j)$ is left invertible. Towards this, consider the bounded operator STS. Then, $STS(\mathscr{R}(T^j)) \subset \mathscr{R}(T^j)$ and for all $y = T^j x \in \mathscr{R}(T^j)$, we have

$$STST_j(y) = STSTT^j(x) = ST^{j+1}(x) = T^j(x) = y,$$

so that T_j is left invertible as desired.

We define now the left Drazin spectrum of an operator $T \in \mathscr{B}(X)$ to be the following set of the complex plane:

$$\sigma_{ld}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}.$$

Obviously, we have the following inclusions:

$$\sigma_{ld}(T) \subset \sigma_l(T) \subset \sigma(T),$$

where $\sigma_l(T)$ and $\sigma(T)$ are the left spectrum and the spectrum of T, respectively.

As a consequence of Theorem 3.2, the following result about the compactness of the left Drazin spectrum arises.

THEOREM 3.6. *The left Drazin spectrum $\sigma_{ld}(T)$ is a compact subset of the complex plane.*

Proof. Towards proving the compactness of $\sigma_{ld}(T)$, since $\sigma_{ld}(T) \subset \sigma(T)$ and $\sigma(T)$ is bounded, we only need to prove that $\sigma_{ld}(T)$ is closed in the complex plane and thus in $\sigma(T)$. Towards this, it suffices to prove that if T is left Drazin invertible, then there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{C}$ satisfying $0 < |\lambda| < \varepsilon$, $T - \lambda I$ is left Drazin invertible. Let then T be a left Drazin invertible operator. Then, it follows from Theorem 3.2 that there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is nilpotent. Since $\sigma_l(T_M)$ is closed, there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{C}$ verifying $0 < |\lambda| < \varepsilon$, $T_M - \lambda I$ is left invertible. But $T - \lambda I = T_M - \lambda I \oplus T_N - \lambda I$ and $T_N - \lambda I$ is invertible. Consequently, $T - \lambda I$ is one-to-one and $\mathscr{R}(T - \lambda I) = \mathscr{R}(T_M - \lambda I) \oplus N$ which is topologically complemented in X by Lemma 2.3. $\square$

3.2. Kato decomposition of right Drazin invertible operators and consequences

Right Drazin invertible operators allow the following Kato decomposition.

THEOREM 3.7. *Let $T \in \mathscr{B}(X)$. Then the following statements are equivalent:*

(i) *T is right Drazin invertible.*

(ii) *There exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right invertible and T_N is nilpotent.*

Proof. $(i) \Rightarrow (ii)$ Since $q := \delta(T) < \infty$, it follows from Lemma 2.1(b) that $\mathscr{R}(T) + \mathscr{N}(T^q) = X$ so that $\mathscr{R}(T) + \mathscr{N}(T^q)$ is topologically complemented. Now, since T is quasi-Fredholm of degree q and the subspace $\mathscr{N}(T) \cap \mathscr{R}(T^q)$ is also topologically complemented, it follows from [7, Theorem 5] that there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is semi-regular and T_N satisfies $T_N^q = 0$. We shall prove now that $\mathscr{R}(T_M) = M$ and that $\mathscr{N}(T_M)$ is topologically complemented in M. In fact, we know from Lemma 2.2 that $\mathscr{R}(T) + \mathscr{N}(T^q) = \mathscr{R}(T_M) \oplus N$ and that $\mathscr{N}(T) \cap \mathscr{R}(T^q) = \mathscr{N}(T_M)$. Hence, $\mathscr{R}(T_M) \oplus N = M \oplus N$, so that $\mathscr{R}(T_M) = M$ since $\mathscr{R}(T_M) \subset M$. Finally, the fact that $\mathscr{N}(T_M)$ is topologically complemented in M follows from Lemma 2.3.

$(ii) \Rightarrow (i)$ Suppose that there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right invertible and T_N is nilpotent of degree q. We know that $\mathscr{R}(T) + \mathscr{N}(T^{q-1}) \neq X$ and from Lemma 2.2 that $\mathscr{R}(T) + \mathscr{N}(T^q) = \mathscr{R}(T_M) \oplus N = M \oplus N = X$. Therefore, Lemma 2.1 (b) implies that $q = \delta(T)$. Finally, since by Lemma 2.2 $\mathscr{N}(T) \cap \mathscr{R}(T^q) = \mathscr{N}(T_M)$ and since $\mathscr{N}(T_M)$ is topologically complemented in M, it follows from Lemma 2.3 that $\mathscr{N}(T) \cap \mathscr{R}(T^q)$ is topologically complemented in X. $\square$

THEOREM 3.8. *Let $T \in \mathscr{B}(X)$. Then, the following statements are equivalent:*

(i) *T is right Drazin invertible.*

(ii) *There exists a nonnegative integer n such that $\mathscr{R}(T^n)$ is topologically complemented and $T_n : \mathscr{R}(T^n) \longrightarrow \mathscr{R}(T^n)$ is right invertible.*

(iii) *There exists a projection $P \in \mathscr{B}(X)$ such that $PT = TP$, $T + P$ is right invertible and TP is nilpotent.*

Proof. $(i) \Leftrightarrow (ii)$ If T is right Drazin invertible, then by Theorem 3.7, there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right invertible and T_N is nilpotent of degree $q := \delta(T)$. We have $T^q = T_M^q \oplus 0$ so that $\mathscr{R}(T^q) = M$ which is topologically complemented in X. Take now $T_q := T_M$ which is right invertible. Conversely, assume that for some nonnegative integer n, $\mathscr{R}(T^n)$ is topologically complemented and $T_n : \mathscr{R}(T^n) \longrightarrow \mathscr{R}(T^n)$ is right invertible. We have $\mathscr{R}(T^{n+1}) = \mathscr{R}(T^n)$ so that $q := \delta(T) \leqslant n < \infty$. On the other hand, $\mathscr{N}(T) \cap \mathscr{R}(T^q) = \mathscr{N}(T) \cap \mathscr{R}(T^n)$ which is topologically complemented in $\mathscr{R}(T^n)$. Since this latter subspace is topologically complemented in X, it follows from Lemma 2.3 that $\mathscr{N}(T) \cap \mathscr{R}(T^q)$ is also topologically complemented in X.

$(i) \Leftrightarrow (iii)$ Assume that T is right Drazin invertible. Then, there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right invertible and T_N is nilpotent of degree $q := \delta(T)$. Let P be the bounded projection onto N along M. Then, as is the proof of $(i) \Leftrightarrow (iii)$ in Theorem 3.4, we obtain that $PT = TP$ and TP is nilpotent. Finally, it is easy to see that $(T+P)_{\mathscr{N}(P)} =$

$T_{\mathscr{N}(P)} = T_M$ which is right invertible and that $(T+P)_{\mathscr{R}(P)} = T_{\mathscr{R}(P)} + I_{\mathscr{N}(P)} = T_N + I_N$ which is invertible. Therefore, $T+P = T_M \oplus (T_N + I_N)$ so that $T+P$ is surjective and $\mathscr{N}(T+P) = \mathscr{N}(T_M)$. Since $\mathscr{N}(T_M)$ is topologically complemented in M, it follows from Lemma 2.3 that $\mathscr{N}(T+P)$ is topologically complemeted in X. Towards the converse implication, assume that there exists a projection $P \in \mathscr{B}(X)$ such that $PT = TP$, TP is nilpotent and $T+P$ is right invertible. Then $X = \mathscr{N}(P) \oplus \mathscr{R}(P)$ and $\mathscr{N}(P)$ and $\mathscr{R}(P)$ are closed and T-invariant. Consider now $T_{\mathscr{N}(P)} : \mathscr{N}(P) \longrightarrow \mathscr{N}(P)$ and $T_{\mathscr{R}(P)} : \mathscr{R}(P) \longrightarrow \mathscr{R}(P)$. Then $T = T_{\mathscr{N}(P)} \oplus T_{\mathscr{R}(P)}$, $T+P = T_{\mathscr{N}(P)} \oplus (T_{\mathscr{R}(P)} + I_{\mathscr{R}(P)})$ and $T_{\mathscr{R}(P)}$ is nilpotent. Since $T+P$ is right invertible, it follows that $X = \mathscr{N}(P) \oplus \mathscr{R}(P) = \mathscr{R}(T_{\mathscr{N}(P)}) \oplus \mathscr{R}(P)$ so that $\mathscr{R}(T_{\mathscr{N}(P)}) = \mathscr{N}(P)$ since $\mathscr{R}(T_{\mathscr{N}(P)}) \subset \mathscr{N}(P)$. Finally, we have $\mathscr{N}(T+P) = \mathscr{N}(T_{\mathscr{N}(P)})$ which is topologically complemented in X. Thus, $\mathscr{N}(T_{\mathscr{N}(P)})$ is topologically complemented in $\mathscr{N}(P)$ by Lemma 2.3. □

REMARKS 3.9. Let $T \in \mathscr{B}(X)$ be an operator that satisfies the following conditions: there exist a nonnegative integer j and an operator $S \in \mathscr{B}(X)$ such that $\mathscr{R}(T^j)$ is topologically complemented, $TST^j = T^j$ and $S(\mathscr{R}(T^j)) \subset \mathscr{R}(T^j)$. Then, we note that it follows from Theorem 3.8 (ii) that T is right Drazin invertible.

We define now the right Drazin spectrum of an operator $T \in \mathscr{B}(X)$ by:

$$\sigma_{rd}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right Drazin invertible}\}.$$

Obviously, the following inclusions hold:

$$\sigma_{rd}(T) \subset \sigma_r(T) \subset \sigma(T),$$

where $\sigma_r(T)$ is the right spectrum of T.

Based on the compactness of the right spectrum, Theorem 3.7 yields to the following result about the compactness of the right Drazin spectrum.

THEOREM 3.10. *The right Drazin spectrum $\sigma_{rd}(T)$ is a compact subset of the complex plane.*

4. Some classes related to left and right invertible linear operators

The purpose of this section is to demonstrate that under some additional conditions, left and right Drazin invertible operators are equivalent to operators already introduced and studied in the litterature, namely left and right generalized Drazin invertible operators (see [3]) and left and right Browder operators (see [10]).

4.1. Left and right generalized Drazin invertible operators

For an operator $T \in \mathscr{B}(X)$, we recall that its quasi-nilpotent part is the subspace defined by:

$$\mathscr{H}_0(T) := \{x \in X : ||T^n x||^{\frac{1}{n}} \text{ converges to } 0\}$$

and that its analytical core part is the subspace defined by:

$$\mathcal{K}(T) := \{x \in X : \text{there exists a sequence } (x_n) \text{ in } X \text{ and } \delta > 0 : Tx_1 = x, \\ Tx_{n+1} = x_n \text{ and } ||x_n|| \leqslant \delta^n ||x|| \text{ for all nonnegative integer } n\}.$$

In [3], new classes of operators were introduced and defined by means of quasi-nilpotent parts and analytical core parts as follows.

DEFINITION 4.1. An operator $T \in \mathcal{B}(X)$ is said to be left generalized Drazin invertible if $\mathcal{H}_0(T)$ is topologically complemented in X by a closed subspace M such that TM is topologically complemented in M, and T is said to be right generalized Drazin invertible if $\mathcal{K}(T)$ is topologically complemented in X by a closed subspace M such that $TM \subseteq M \subseteq \mathcal{H}_0(T)$ and $\mathcal{N}(T) \cap \mathcal{K}(T)$ is topologically complemented in $\mathcal{K}(T)$.

The sentences left generalized Drazin invertible and right generalized Drazin invertible suggest that these classes of operators subsume the classes of left and right Drazin invertible operators, respectively. In fact, this is true except that this cannot be directly seen from the definitions as the approach and the techniques used in [3] are different from those used in this paper. However, the characterizations of left and right Drazin invertible operators by means of Kato decompositions given in Theorems 3.2 and 3.7 imply that each left Drazin invertible operator is left generalized Drazin invertible and that each right Drazin invertible operator is right generalized Drazin invertible. Indeed, it was proven in [3, Theorems 3.3, 3.4] that an operator $T \in \mathcal{B}(X)$ is left (resp. right) generalized Drazin invertible if and only if there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left (resp. right) invertible and T_N is quasi-nilpotent. In the following result, we answer the question of when we have converse implications, in other words, under which condition left and right generalized Drazin invertible operators are left and right Drazin invertible, respectively.

THEOREM 4.2. *Let $T \in \mathcal{B}(X)$. Then, the following equivalences hold:*

(i) *T is left Drazin invertible if and only if T is left generalized Drazin invertible, $p := \alpha(T) < \infty$ and $\mathcal{R}(T^p)$ and $\mathcal{R}(T^{p+1})$ are closed.*

(ii) *T is right Drazin invertible if and only if T is right generalized Drazin invertible and $\delta(T) < \infty$.*

Proof. (i) If T is left Drazin invertible, then the Kato decomposition of T given in Theorem 3.2 implies that T is left generalized Drazin invertible by [3, Theorem 3.3] since each nilpotent operator is quasi-nilpotent. Also, $p := \alpha(T) < \infty$ and $\mathcal{R}(T^p)$ and $\mathcal{R}(T^{p+1})$ are closed by [7, Proposition 3]. Conversely, if T is left generalized Drazin invertible, there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is quasi-nilpotent. Since $p := \alpha(T)$ is finite, then $\alpha(T_N) \leqslant p$. On the other hand, we know that $\mathcal{R}(T^p) = \mathcal{R}(T_M^p) \oplus \mathcal{R}(T_N^p)$ and that $\mathcal{R}(T^{p+1}) = \mathcal{R}(T_M^{p+1}) \oplus \mathcal{R}(T_N^{p+1})$. Thus, the facts that $\mathcal{R}(T^p)$ and $\mathcal{R}(T^{p+1})$ are closed entail from Lemma 2.3 that $\mathcal{R}(T_N^p)$ and $\mathcal{R}(T_N^{p+1})$ are closed too, and hence so are $\mathscr{R}((T_N^p)^*)$ and $\mathscr{R}((T_N^{p+1})^*)$. Therefore, it follows from the equality $\mathscr{N}(T_N^p) = \mathscr{N}(T_N^{p+1})$ and from the classical closed range theorem that

$$\mathscr{R}((T_N^p)^*) = \mathscr{N}(T_N^p)^\perp = \mathscr{N}(T_N^{p+1})^\perp = \mathscr{R}((T_N^{p+1})^*),$$

so that $q := \delta(T_N^*) \leqslant p$. Now, since T_N^* is quasi-nilpotent, it follows from [9, Corollary 10.6] that $(T_N^*)^p = 0$. Finally, from the known relation $\mathscr{N}(T^*) = \mathscr{R}(T)^\perp$ applied to T_N^p and from the fact that $\mathscr{R}(T_N^p)$ is closed, we can easily prove that T_N is a nilpotent operator.

(ii) If T is right Drazin invertible, then the Kato decomposition of T given in Theorem 3.7 implies from [3, Theorem 3.4] that T is right generalized Drazin invertible. Conversely, if T is right generalized Drazin invertible, then there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right invertible and T_N is quasi-nilpotent. Since $\delta(T) < \infty$, we note that $\delta(T_N) < \infty$. Therefore, it follows from [9, Corollary 10.6] that T_N is nilpotent. $\square$

4.2. Left and right Browder operators

DEFINITION 4.3. An operator $T \in \mathscr{B}(X)$ is said to be left Browder if $\dim \mathscr{N}(T) < \infty$, $\alpha(T) < \infty$ and $\mathscr{R}(T)$ is topologically complemented, right Browder if $\operatorname{codim} \mathscr{R}(T) < \infty$, $\delta(T) < \infty$ and $\mathscr{N}(T)$ is topologically complemented and Browder if it is both left and right Browder.

It was proven in [10, Theorems 5, 6] that an operator $T \in \mathscr{B}(X)$ is left (right) Browder if and only if there exist closed T-invariant subspaces M and N of X such that $\dim N < \infty$, $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left (right) invertible and T_N is nilpotent. Hence always, a left Browder operator is left Drazin invertible and a right Browder operator is right Drazin invertible. In the following theorem, an additional condition is added to a left Drazin invertible operator to be left Browder and another to a right Drazin invertible operator to be right Browder.

THEOREM 4.4. *Let $T \in \mathscr{B}(X)$. Then, the following equivalences hold:*

(i) *T is left Browder if and only if T is left Drazin invertible and $\mathscr{N}(T^{\alpha(T)})$ is of finite dimension.*

(ii) *T is right Browder if and only if T is right Drazin invertible and $\mathscr{R}(T^{\delta(T)})$ is of finite codimension.*

(iii) *T is Browder if and only if T is Drazin invertible and either $\mathscr{N}(T^{\alpha}(T))$ is of finite dimension or $\mathscr{R}(T^{\delta(T)})$ is of finite codimension.*

Proof. The proof of (i) and (ii) are based on the Kato decompositions. They are straightforward and hence ommited. For the proof of (iii), the direct implication is obvious. For the converse one, if $\mathscr{N}(T^{\alpha}(T))$ is of finite dimension, then by (i), T is left Browder. Since $\alpha(T) = \delta(T) < \infty$, it follows from [8, Corollary 4.4] that $\dim \mathscr{N}(T) = \operatorname{codim} \mathscr{R}(T)$ so that T is Browder. If $\mathscr{R}(T^{\delta(T)})$ is of finite codimension, then again by [8, Corollary 4.4], T is Browder. $\square$

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