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JORDAN $\{g,h\}$–DERIVATIONS OF UNITAL ALGEBRAS

DOMINIK BENKOVIČ AND MATEJA GRAŠIČ

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Abstract. In this paper we study Jordan $\{g,h\}$-derivations of unital algebras. For algebras having nontrivial idempotents we give the sufficient and necessary conditions that every Jordan $\{g,h\}$-derivation is a $\{g,h\}$-derivation. We are particularly interested in the class of algebras $\mathscr{A}$ having the property, that every Jordan $\{g,h\}$-derivation of $\mathscr{A}$ is a $\{g',h'\}$-derivation for some linear maps g' and h'.

1. Introduction

Throughout the paper R will be a commutative unital ring containing $\frac{1}{2}$ and $\mathscr{A}$ will be a unital algebra over R. As usual, with $x\circ y = xy + xy$ we denote the Jordan product and with $[x,y] = xy - yx$ the Lie product of elements $x,y \in \mathscr{A}$.

Let $g,h : \mathscr{A} \to \mathscr{A}$ be linear maps. A linear map $f : \mathscr{A} \to \mathscr{A}$ is called a $\{g,h\}$-*derivation* if

$$f(xy) = g(x)y + xh(y) = h(x)y + xg(y)$$

hold for all $x,y \in \mathscr{A}$ and is called a *Jordan* $\{g,h\}$-*derivation* if

$$f(x\circ y) = g(x)\circ y + x\circ h(y)$$

holds for all $x,y \in \mathscr{A}$. Clearly, each $\{g,h\}$-derivation is a Jordan $\{g,h\}$-derivation. Also, every derivation f is an $\{f,f\}$-derivation and every Jordan derivation f is a Jordan $\{f,f\}$-derivation. In 1957 Herstein [8] proved that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. This result has been extended to different rings and algebras in various directions (see e.g. [2, 3, 4, 6, 11, 12, 13] and references therein); one might very roughly summarize these results by saying that proper Jordan derivations (i.e. those that are not derivations) from rings (algebras) into themselves are rather rare and very special.

The notions of a $\{g,h\}$-derivation and a Jordan $\{g,h\}$-derivation were introduced by Brešar [5], where he used them to obtain [5, Corollary 4.4], which states that every Jordan derivation on the tensor product $\mathscr{A}\otimes\mathscr{B}$ of a semiprime algebra $\mathscr{A}$ and a commutative algebra $\mathscr{B}$ is a derivation. Let us point out, see [3, 6] for more details, that proper Jordan derivations on a semiprime algebra do not exist. Furthermore,

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even the tensor product of semiprime algebras is not necessarily a semiprime algebra. The above mentioned result from Brešar follows from [5, Theorem 4.3] which states that every Jordan $\{g,h\}$-derivation of a semiprime algebra $\mathscr{A}$ is a $\{g,h\}$-derivation. Therefore, naturally, the question arises, what algebras $\mathscr{A}$ have the property that every Jordan $\{g,h\}$-derivation of $\mathscr{A}$ is a $\{g,h\}$-derivation.

In this article we will consider two classes of algebras. We will study algebras $\mathscr{A}$ having property (P1) or (P2), where:

(P1) Every Jordan $\{g,h\}$-derivation of $\mathscr{A}$ is a $\{g',h'\}$-derivation for some linear maps g' and h'.

(P2) Every Jordan $\{g,h\}$-derivation of $\mathscr{A}$ is a $\{g,h\}$-derivation.

Using these notations we know, as already mentioned before, that every semiprime algebra $\mathscr{A}$ is an algebra with property (P2). Furthermore, Ghosh and Prakash [7, Theorem 3.1] proved that every matrix algebra $M_n(R)$ over a commutative ring R has property (P2). Let us point out that the upper triangular matrix algebra $T_n(R)$ does not have property (P2). In [10, Theorem 3.1] Kong and Zhang characterised nest algebras $\mathscr{T}(\mathscr{N})$ on a complex separable Hilbert space H having property (P2). One of the results obtained in [10] states, that every Jordan $\{g,h\}$-derivation of a triangular algebra $\mathscr{A}$ is a $\{g,h\}$-derivation if and only if $g(1)$ or $h(1)$ lies in the center of the algebra $\mathscr{A}$.

Clearly, every algebra having property (P2) also has property (P1). A converse of this claim does not hold. On an algebra $\mathscr{A}$ with the property (P2) every Jordan derivation is a derivation. It turns out, see Remark 2.4, that this also holds for every algebra $\mathscr{A}$ which has property (P1). In Remark 2.3 we point out that in an algebra $\mathscr{A}$ with property (P2) only the central elements $a\in\mathscr{A}$ satisfy the identity $[[x,y],a]=0$ for all $x,y\in\mathscr{A}$. As it turns out, this is the property that characteristically distinguishes between the classes of algebras having property (P2) and (P1). Theorem 3.1 namely states:

An algebra $\mathscr{A}$ has property (P2) if and only if $\mathscr{A}$ has property (P1) and $Z(\mathscr{A})=\{a\in\mathscr{A};[[\mathscr{A},\mathscr{A}],a]=0\}$.

Let us look from another perspective. We mentioned before that on algebras with property (P1) there do not exist proper Jordan derivations. Could this property be characteristic for algebras having property (P1)? In general this is not the case (see Example following Theorem 3.1). But, if an algebra is unital and contains a nontrivial idempotent, then under some conditions this property is indeed characteristic. Let us assume that $\mathscr{A}$ has an idempotent $p\neq 0,1$ and let q denote the idempotent $1-p$. In this case $\mathscr{A}$ can be represented in the Peirce decomposition form $\mathscr{A}=p\mathscr{A}p+p\mathscr{A}q+q\mathscr{A}p+q\mathscr{A}q$, where $p\mathscr{A}p$ and $q\mathscr{A}q$ are subalgebras with unitary elements p and q, respectively, $p\mathscr{A}q$ is an $(p\mathscr{A}p,q\mathscr{A}q)$-bimodule and $q\mathscr{A}p$ is an $(q\mathscr{A}q,p\mathscr{A}p)$-bimodule. We will assume that $\mathscr{A}$ satisfies

$$\begin{aligned} pxp\cdot p\mathscr{A}q=\{0\}=q\mathscr{A}p\cdot pxp \quad &\Rightarrow \quad pxp=0,\\ p\mathscr{A}q\cdot qxq=\{0\}=qxq\cdot q\mathscr{A}p \quad &\Rightarrow \quad qxq=0 \end{aligned} \tag{1}$$

for all $x \in \mathscr{A}$. Examples of such unital algebras are triangular algebras, matrix algebras, and prime (hence in particular simple) algebras with nontrivial idempotents. Our main result Theorem 3.3 states:

Let $\mathscr{A}$ be a unital algebra containing a nontrivial idempotent p satisfying (1). Then $\mathscr{A}$ has property (P1) if and only if every Jordan derivation of $\mathscr{A}$ is a derivation.

Therefore, any triangular algebra $\mathscr{A}$ has property (P1) but does not necessarily have property (P2) (see Corollary 4.3). From the main theorem it also follows that every matrix algebra $M_n(A)$, $n \geqslant 2$, where A is a unital algebra, has property (P2) (see Corollary 4.2).

In the second section we give basic remarks that will be used in the proofs of the main theorems (Theorem 3.1, Theorem 3.3) presented in the third section. In the last section we apply these results to unital algebras containing nontrivial idempotents.

2. Remarks on Jordan $\{g,h\}$-derivations

Let $\mathscr{A}$ be an algebra and let $Z(\mathscr{A})$ denote the center of $\mathscr{A}$. It is known (see [5]) that every $\{g,h\}$-derivation f can be written as $f(x) = \lambda x + d(x)$ for all $x \in \mathscr{A}$, where $d : \mathscr{A} \to \mathscr{A}$ is a derivation and $\lambda \in Z(\mathscr{A})$.

REMARK 2.1. A linear map $f : \mathscr{A} \to \mathscr{A}$ is a $\{g,h\}$-derivation and a $\{g',h'\}$-derivation if and only if $g'(x) - g(x) = h(x) - h'(x) = ax$ for all $x \in \mathscr{A}$, where $a \in Z(\mathscr{A})$.

Proof. Let us assume that

$$\begin{aligned} f(xy) &= g(x)y + xh(y) = g'(x)y + xh'(y) \\ &= h(x)y + xg(y) = h'(x)y + xg'(y) \end{aligned}$$

for all $x,y \in \mathscr{A}$. Define $G = g' - g$ and $H = h' - h$. Then the above identities can be written as

$$G(x)y + xH(y) = 0 = H(x)y + xG(y) \tag{2}$$

for all $x,y \in \mathscr{A}$. Let $a = G(1)$. Substituting $x = y = 1$ in (2) we get $H(1) = -G(1) = -a$. Now let us consecutively make the substitutions $y = 1$ and $x = 1$ in (2). We get

$$\begin{aligned} &G(x) = xa \quad \text{and} \quad H(x) = -xa \quad \text{for all } x \in \mathscr{A}, \\ &H(y) = -ay \quad \text{and} \quad G(y) = ay \quad \text{for all } y \in \mathscr{A}. \end{aligned}$$

Therefore $a \in Z(\mathscr{A})$, $G(x) = g'(x) - g(x) = xa$ and $H(x) = h'(x) - h(x) = -ax$ for all $x \in \mathscr{A}$. The second implication is an easy calculation, so we omit it. □

REMARK 2.2. A linear map $f : \mathscr{A} \to \mathscr{A}$ is a Jordan $\{g,h\}$-derivation and a Jordan $\{g',h'\}$-derivation if and only if $g'(x) - g(x) = h(x) - h'(x) = a \circ x$ for all $x \in \mathscr{A}$, where $a \in \mathscr{A}$ is such that $[[\mathscr{A},\mathscr{A}],a] = 0$.

Proof. Let f be a Jordan $\{g,h\}$-derivation and a Jordan $\{g',h'\}$-derivation. Then

$$f(x\circ y) = g(x)\circ y + x\circ h(y) = g'(x)\circ y + x\circ h'(y)$$

for all $x,y\in\mathscr{A}$. Now let $G = g'-g$ and $H = h'-h$. Then the above identity can be written as

$$G(x)\circ y + x\circ H(y) = 0 \tag{3}$$

for all $x,y\in\mathscr{A}$. Let $a = \frac{1}{2}G(1)$. As in the proof of the previous proposition we make the substitutions $x = y = 1$, getting $2G(1)+2H(1) = 0$ and $H(1) = -G(1) = -2a$. By letting $y = 1$ and then $x = 1$ in (3) we further have

$$2G(x) = -H(1)\circ x \quad \text{for all } x\in\mathscr{A},$$
$$2H(y) = -G(1)\circ y \quad \text{for all } y\in\mathscr{A}.$$

Therefore, $G(x) = g'(x) - g(x) = a\circ x$ and $H(x) = h'(x) - h(x) = -a\circ x$ for all $x\in\mathscr{A}$. Using this in (3) we get

$$y\circ(x\circ a) - x\circ(y\circ a) = 0$$

for all $x,y\in\mathscr{A}$. Knowing that $[[x,y],a] = x\circ(y\circ a) - y\circ(x\circ a)$ we have proven that a satisfies the identity $[[x,y],a] = 0$ for all $x,y\in\mathscr{A}$. The second implication can be verified by a direct computation. $\square$

REMARK 2.3. Let $\mathscr{A}$ be an algebra with property (P2). Then

$$Z(\mathscr{A}) = \{a\in\mathscr{A}; [[\mathscr{A},\mathscr{A}],a] = 0\}.$$

Proof. It suffices to prove that every $a\in\mathscr{A}$ satisfying $[[x,y],a] = 0$ for all $x,y\in\mathscr{A}$ lies in $Z(\mathscr{A})$. Let f be a $\{g,h\}$-derivation. Then f is a Jordan $\{g,h\}$-derivation and by Remark 2.2 f is also a Jordan $\{g',h'\}$-derivation, where $g'(x)-g(x) = h(x) - h'(x) = a\circ x$ for all $x\in\mathscr{A}$. From this it follows, since $\mathscr{A}$ has property (P2), that f is a $\{g',h'\}$-derivation. Remark 2.1 further tells us that there exists $\lambda\in Z(\mathscr{A})$ such that $g'(x)-g(x) = h(x)-h'(x) = \lambda x$ for all $x\in\mathscr{A}$. We have come to $a\circ x = \lambda x$ for all $x\in\mathscr{A}$. By putting $x = 1$ in this identity we have $2a = \lambda\in Z(\mathscr{A})$ and therefore $a\in Z(\mathscr{A})$. $\square$

REMARK 2.4. Let $\mathscr{A}$ be an algebra with property (P1). Then every Jordan derivation of $\mathscr{A}$ is a derivation.

Proof. Let f be a Jordan derivation of $\mathscr{A}$. Then f is also a Jordan $\{f,f\}$-derivation. Since $\mathscr{A}$ satisfies (P1), there exist linear maps $g,h:\mathscr{A}\to\mathscr{A}$ such that f is a $\{g,h\}$-derivation. Therefore f is of the form $f(x) = \lambda x + d(x)$, where $d:\mathscr{A}\to\mathscr{A}$ is a derivation and $\lambda\in Z(\mathscr{A})$. The following holds

$$\begin{aligned} f(x\circ y) &= f(x)\circ y + x\circ f(y)\\ \lambda(x\circ y) + d(x\circ y) &= (\lambda x + d(x))\circ y + x\circ(\lambda y + d(y))\\ &= 2\lambda(x\circ y) + d(x)\circ y + x\circ d(y)\end{aligned}$$

for all $x,y \in \mathscr{A}$. Since d is (also) a Jordan derivation, it follows that $\lambda(x \circ y) = 0$ for all $x,y \in \mathscr{A}$. Letting $x = y = 1$, we get $2\lambda = 0$. Therefore $\lambda = 0$ and $f = d$ is a derivation. $\square$

3. Main Theorem

Let $\mathscr{A}$ be an algebra with property (P2). Then $\mathscr{A}$ also has property (P1) and by Remark 2.3 it follows that only the central elements of $\mathscr{A}$ satisfy identity $[[\mathscr{A},\mathscr{A}],a] = 0$. The question arises whether these two properties are characteristic properties for algebras satisfying (P2)? Our first result states that this is so:

THEOREM 3.1. *An algebra $\mathscr{A}$ has property (P2) if and only if $\mathscr{A}$ is an algebra with property (P1) and $Z(\mathscr{A}) = \{a \in \mathscr{A}; [[\mathscr{A},\mathscr{A}],a] = 0\}$.*

Proof. It suffices to prove the 'right to left' implication. Let f be a Jordan $\{g,h\}$-derivation. By assumption there exist linear maps $g',h' : \mathscr{A} \to \mathscr{A}$ such that f is a $\{g',h'\}$-derivation. Using Remark 2.2 it follows that $g'(x) - g(x) = h(x) - h'(x) = a \circ x$ for all $x \in \mathscr{A}$, where $a \in \mathscr{A}$ is such that $[[x,y],a] = 0$ for all $x,y \in \mathscr{A}$. Using the assumption this gives that $a \in Z(\mathscr{A})$ and therefore $g'(x) - g(x) = h(x) - h'(x) = 2ax$ for all $x \in \mathscr{A}$. Since $2a \in Z(\mathscr{A})$, by Remark 2.1 it follows that f is a $\{g,h\}$-derivation. We have thereby proven that $\mathscr{A}$ has property (P2). $\square$

So now we know that algebras $\mathscr{A}$ containing a noncentral element a satisfying $[[\mathscr{A},\mathscr{A}],a] = 0$ do not have property (P2). Do these algebras have property (P1)? This can not be expected in general. See the following Example. Let us assume that $\mathscr{A}$ is an algebra with property (P1). By Remark 2.4 we know that every Jordan derivation of $\mathscr{A}$ is also a derivation. Is the contrary true? Again, this is not the case in general. There exist algebras having the property that every Jordan derivation is a derivation and not having property (P1). The following example was constructed using Brešar's example [5, Example 2.1]:

EXAMPLE 3.2. Let

$$\mathscr{A} = \left\{ \begin{pmatrix} \mu & t & s \\ & \mu & r \\ & & \mu \end{pmatrix} ; \mu,t,r,s \in R \right\}.$$

Obviously, $\mathscr{A}$ is a subalgebra of the upper triangular matrix algebra $T_3(R)$. Each element $x \in \mathscr{A}$ can be represented in the form $x = \mu 1 + te_{12} + se_{13} + re_{23}$ where e_{ij} are matrix units. Note that $\mathscr{A}$ is a noncommutative algebra satisfying the polynomial identity $[[X,Y],Z]$ and that the center of $\mathscr{A}$ consists of all elements: $\lambda = \mu 1 + se_{13}$, where $\mu,s \in R$. Let us define linear maps $f,g : \mathscr{A} \to \mathscr{A}$ by $f(x) = 2\mu e_{12}$ and $h(x) = re_{13}$ for all $x \in \mathscr{A}$. A direct calculation gives

$$f(x \circ y) = f(x) \circ y + x \circ h(y)$$

for all $x,y \in \mathscr{A}$. The map f is therefore a Jordan $\{f,h\}$-derivation. Let us assume that f is a $\{g',h'\}$-derivation for some linear maps g',h'. Then f is of the form $f(x) = \lambda x + d(x)$, where $d : \mathscr{A} \to \mathscr{A}$ is a derivation and $\lambda \in Z(\mathscr{A})$. This is in contradiction with $f(1) = 2e_{12} \notin Z(\mathscr{A})$ and $f(1) = \lambda \in Z(\mathscr{A})$. The algebra $\mathscr{A}$ therefore does not have the property (P1). But, see [5, Example 2.1], every Jordan derivation of $\mathscr{A}$ is a derivation.

Notice, that the algebra in the example does not contain nontrivial idempotents. Our main theorem gives a characterization of algebras having property (P1) in the special case when an algebra is unital and contains a nontrivial idempotent p satisfying (1).

THEOREM 3.3. *Let $\mathscr{A}$ be a unital algebra containing a nontrivial idempotent p satisfying (1). Then $\mathscr{A}$ has property (P1) if and only if every Jordan derivation of $\mathscr{A}$ is a derivation.*

Before proving Theorem 3.3 some auxiliary results are needed. Let $\mathscr{A} = p\mathscr{A}p + p\mathscr{A}q + q\mathscr{A}p + q\mathscr{A}q$ be a unital algebra containing a nontrivial idempotent p, which satisfies (1). Every element $x \in \mathscr{A}$ can be represented in the form

$$x = pxp + pxq + qxp + qxq = a + m + n + b, \tag{4}$$

where $a \in p\mathscr{A}p$, $m \in p\mathscr{A}q$, $n \in q\mathscr{A}p$ and $b \in q\mathscr{A}q$. Let us point out, see [1, Proposition 2.1], that the center of $\mathscr{A}$ consists of all elements of the form $\lambda \in p\mathscr{A}p + q\mathscr{A}q$ satisfying $[\lambda, m] = [\lambda, n] = 0$ for all $m \in p\mathscr{A}q, n \in q\mathscr{A}p$. Also the following holds:

REMARK 3.4. Let $x \in \mathscr{A}$. If $[x, p\mathscr{A}q] = 0$ and $[x, q\mathscr{A}p] = 0$, then $pxp + qxq \in Z(\mathscr{A})$.

Let f be a Jordan $\{g,h\}$-derivation. Set $\mathfrak{a} = \frac{1}{2}g(1)$ and $\mathfrak{b} = \frac{1}{2}h(1)$. Since $f(x \circ y) = g(x) \circ y + x \circ h(y)$ for all $x,y \in \mathscr{A}$ it follows that

$$g(x) = f(x) - \mathfrak{b} \circ x \quad \text{and} \quad h(x) = f(x) - \mathfrak{a} \circ x \tag{5}$$

for all $x \in \mathscr{A}$. This gives us an important identity:

$$f(x \circ y) = f(x) \circ y + x \circ f(y) - x \circ (y \circ \mathfrak{a}) - y \circ (x \circ \mathfrak{b}) \tag{6}$$

for all $x,y \in \mathscr{A}$. Our goal is to give the form of elements $\mathfrak{a}$ and $\mathfrak{b}$. Our first observation, that was used already by Brešar in the proof of [5, Lemma 4.1], is the following:

LEMMA 3.5. *The element $\mathfrak{a} - \mathfrak{b}$ satisfies the identity $[[x,y], \mathfrak{a} - \mathfrak{b}] = 0$ for all $x,y \in \mathscr{A}$.*

Next, the following holds for the element $\mathfrak{a} + \mathfrak{b}$:

LEMMA 3.6. *Let e be an idempotent of $\mathscr{A}$. Then*

$$\lambda = \mathfrak{a} + \mathfrak{b} \in e\mathscr{A}e + (1-e)\mathscr{A}(1-e).$$

Moreover $[\lambda, e] = 0$.

Proof. Substituting $x = y = e$ in (6) we have

$$\begin{aligned} 2f(e) &= f(e) \circ e + e \circ f(e) - e \circ (e \circ (\mathfrak{a} + \mathfrak{b})) \\ &= 2f(e)e + 2ef(e) - e\lambda - 2e\lambda e - \lambda e. \end{aligned}$$

Multiplying this identity by e from the left side and by $1-e$ from the right side, we get $e\lambda(1-e) = 0$. Similarly, multiplying the above identity from the left side by $1-e$ and from the right side by e, we get $(1-e)\lambda e = 0$. Therefore $\lambda = e\lambda e + (1-e)\lambda(1-e)$ is of the desired form and obviously also $[\lambda, e] = 0$ holds. $\square$

LEMMA 3.7. *The elements $\mathfrak{a}$ and $\mathfrak{b}$ are of the form $\mathfrak{a} = \alpha + \gamma$ and $\mathfrak{b} = \beta - \gamma$, where $\alpha, \beta \in Z(\mathscr{A})$ and $\gamma \in p\mathscr{A}q + q\mathscr{A}p$ is such that $[[x,y], \gamma] = 0$ holds for all $x, y \in \mathscr{A}$.*

Proof. Let p be a nontrivial idempotent of $\mathscr{A}$. Decomposing $\mathscr{A}$ using this idempotent we can write the elements $\mathfrak{a}, \mathfrak{b}$ as in (4): $\mathfrak{a} = a_1 + m_1 + n_1 + b_1$ and $\mathfrak{b} = a_2 + m_2 + n_2 + b_2$. Since p is an idempotent, by Lemma 3.6 we get

$$\begin{aligned} 0 &= p(\mathfrak{a} + \mathfrak{b})q = p\mathfrak{a}q + p\mathfrak{b}q = m_1 + m_2 \quad \text{and} \\ 0 &= q(\mathfrak{a} + \mathfrak{b})p = q\mathfrak{a}p + q\mathfrak{b}p = n_1 + n_2. \end{aligned}$$

Let us set $\gamma = m_1 + n_1 = -m_2 - n_2 \in p\mathscr{A}q + q\mathscr{A}p$. Further, let $\alpha = a_1 + b_1$ and $\beta = a_2 + b_2$. Then $\mathfrak{a} = \alpha + \gamma$ and $\mathfrak{b} = \beta - \gamma$. Note that $p + m$ and $p + n$ are idempotents for all $m \in p\mathscr{A}q$, $n \in q\mathscr{A}p$. Next, let $\lambda = \mathfrak{a} + \mathfrak{b} = \alpha + \beta$. By Lemma 3.6 we know that $[\lambda, p+m] = 0$ and $[\lambda, p+n] = 0$. Since also $[\lambda, p] = 0$, we have

$$\begin{aligned} [\lambda, m] &= [\lambda, p+m] - [\lambda, p] = 0 \quad \text{and} \\ [\lambda, n] &= [\lambda, p+n] - [\lambda, p] = 0 \end{aligned}$$

for all $m \in p\mathscr{A}q$, $n \in q\mathscr{A}p$. Since $\lambda = \alpha + \beta \in p\mathscr{A}p + q\mathscr{A}q$ and because $[\lambda, p\mathscr{A}q] = 0$ and $[\lambda, q\mathscr{A}p] = 0$, Remark 3.4 tells us that $\alpha + \beta \in Z(\mathscr{A})$.

Next we will prove that also the element $\alpha - \beta$ is central. Using Lemma 3.5 we first observe that the element $\mathfrak{a} - \mathfrak{b} = \alpha - \beta + 2\gamma$ satisfies $[[x,y], \alpha - \beta + 2\gamma] = 0$ for all $x, y \in \mathscr{A}$. Substitutions $x = p$, $y = m$ and $x = n$, $y = p$ in this identity give us

$$[m, \alpha - \beta + 2\gamma] = 0 \quad \text{and} \quad [n, \alpha - \beta + 2\gamma] = 0$$

for all $m \in p\mathscr{A}q$, $n \in q\mathscr{A}p$. Since $\alpha - \beta \in p\mathscr{A}p + q\mathscr{A}q$ and $2\gamma \in p\mathscr{A}q + q\mathscr{A}p$, by Remark 3.4 we know that $\alpha - \beta \in Z(\mathscr{A})$. Therefore $[[x,y], 2\gamma] = 0$ for all $x, y \in \mathscr{A}$ and consequently the element γ satisfies $[[\mathscr{A}, \mathscr{A}], \gamma] = 0$.

We have thereby proven that both elements $\alpha+\beta$ and $\alpha-\beta$ lie in the center of $\mathscr{A}$, therefore also α and β are central elements and the proof is completed. $\square$

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. By Remark 2.4, we only have to prove that if every Jordan derivation of $\mathscr{A}$ is a derivation, then $\mathscr{A}$ has property (P1). Let f be a Jordan $\{g,h\}$-derivation. By Lemma 3.7 we know that $\mathfrak{a}=\alpha+\gamma$ and $\mathfrak{b}=\beta-\gamma$, where $\alpha,\beta\in Z(\mathscr{A})$. Therefore, setting $\lambda=\alpha+\beta$ and using the identity $[[x,y],\gamma]=x\circ(y\circ\gamma)-y\circ(x\circ\gamma)$, (6) can be rewritten as

$$\begin{aligned} f(x\circ y) &= f(x)\circ y+x\circ f(y)-2(\alpha+\beta)(x\circ y)-x\circ(y\circ\gamma)+y\circ(x\circ\gamma)\\ &= f(x)\circ y+x\circ f(y)-2\lambda(x\circ y)-[[x,y],\gamma]. \end{aligned}$$

Since $[[\mathscr{A},\mathscr{A}],\gamma]=0$, we further have

$$f(x\circ y)=f(x)\circ y+x\circ f(y)-2\lambda(x\circ y).$$

Subtracting $2\lambda(x\circ y)$ on both sides of this identity and using the fact that $\lambda\in Z(\mathscr{A})$, we further have

$$f(x\circ y)-2\lambda(x\circ y)=(f(x)-2\lambda x)\circ y+x\circ(f(y)-2\lambda y)$$

for all $x,y\in\mathscr{A}$. Now set $d(x)=f(x)-2\lambda x$ for all $x\in\mathscr{A}$. Since $d(x\circ y)=d(x)\circ y+x\circ d(x)$ holds for all $x,y\in\mathscr{A}$, the map d is a Jordan derivation. By assumption it follows that d is also a derivation. So we have $f(x)=2\lambda x+d(x)=(\alpha+\beta)\circ x+d(x)$ for all $x\in\mathscr{A}$. Using (5) we further see that the maps g and h are of the form

$$\begin{aligned} g(x) &= f(x)-\mathfrak{b}\circ x=\alpha\circ x+d(x)+\gamma\circ x \quad\text{and}\\ h(x) &= f(x)-\mathfrak{a}\circ x=\beta\circ x+d(x)-\gamma\circ x \end{aligned}$$

for all $x\in\mathscr{A}$.

Finally, let us prove that f is a $\{g',h'\}$-derivation, where $g'(x)=\alpha\circ x+d(x)$ and $h'(x)=\beta\circ x+d(x)$ for all $x\in\mathscr{A}$. Knowing that $\alpha,\beta\in Z(\mathscr{A})$ and d is a derivation, on the one hand we have

$$\begin{aligned} f(xy) &= (\alpha+\beta)\circ(xy)+d(xy)\\ &= (\alpha\circ(xy)+d(x)y)+(\beta\circ(xy)+xd(y))\\ &= (\alpha\circ x+d(x))y+x(\beta\circ y+d(y))\\ &= g'(x)y+xh'(y) \end{aligned}$$

for all $x,y\in\mathscr{A}$ and on the other hand

$$\begin{aligned} f(xy) &= (\beta\circ(xy)+d(x)y)+(\alpha\circ(xy)+xd(y))\\ &= (\beta\circ x+d(x))y+x(\alpha\circ y+d(y))\\ &= h'(x)y+xg'(x) \end{aligned}$$

for all $x,y\in\mathscr{A}$. The map f is therefore a $\{g',h'\}$-derivation. $\square$

4. Applications

In this section we will present some applications of Theorem 3.1 and Theorem 3.3 to some special classes of unital algebras. Notice, that every simple unital algebra containing a nontrivial idempotent and every unital prime algebra containing a nontrivial idempotent satisfy (1). These are examples of unital semiprime algebras which have property (P2) as Brešar [5] proved. We will focus on applications in matrix and triangular algebras.

Let $\mathrm{Id}([\mathscr{A},\mathscr{A}])$ denote the ideal of an algebra $\mathscr{A}$ which is generated by all commutators of $\mathscr{A}$.

PROPOSITION 4.1. *Let $\mathscr{A}$ be a unital algebra containing a nontrivial idempotent p satisfying (1). Suppose that $\mathrm{Id}([p\mathscr{A}p,p\mathscr{A}p]) = p\mathscr{A}p$ or $\mathrm{Id}([q\mathscr{A}q,q\mathscr{A}q]) = q\mathscr{A}q$. Then $\mathscr{A}$ has property (P2).*

Proof. From [2, Proposition 3.3 and Theorem 4.1] it follows that every Jordan derivation of $\mathscr{A}$ is a derivation. Therefore $\mathscr{A}$ is an algebra with property (P1). By Theorem 3.1 we have to prove that $Z(\mathscr{A}) = \{a \in \mathscr{A}; [[\mathscr{A},\mathscr{A}],a] = \{0\}\}$.

Let us assume that $\mathrm{Id}([p\mathscr{A}p,p\mathscr{A}p]) = p\mathscr{A}p$. The case, when $\mathrm{Id}([q\mathscr{A}q,q\mathscr{A}q]) = q\mathscr{A}q$ is treated analogously. Let $x_0 \in \mathscr{A}$ satisfy $[[x,y],x_0] = 0$ for all $x,y \in \mathscr{A}$. Decomposing $\mathscr{A}$ using idempotent p we can write element x_0 in the form (4) as: $x_0 = a_0 + m_0 + n_0 + b_0$. Now let $x = e$ and $y = m \in p\mathscr{A}q$. Then the above identity yields $[m,x_0] = 0$. So $[p\mathscr{A}q,x_0] = 0$. Similarly, letting $x = n \in q\mathscr{A}p$ and $y = e$ in the above identity, we have $[n,x_0] = 0$ and consequently $[q\mathscr{A}p,x_0] = 0$. Using Remark 3.4 we deduce that $a_0 + b_0 \in Z(\mathscr{A})$. Therefore $[[x,y],m_0 + n_0] = 0$ for all $x,y \in \mathscr{A}$ and, in particular, for arbitrary elements $a_1,a_2 \in p\mathscr{A}p$ we have

$$0 = [[a_1,a_2],m_0+n_0] = [a_1,a_2]m_0 - n_0[a_1,a_2].$$

So $[p\mathscr{A}p,p\mathscr{A}p]m_0 = \{0\} = n_0[p\mathscr{A}p,p\mathscr{A}p]$. Using this, since

$$a_1[a_2,a_3a_4]m_0 = a_1a_3[a_2,a_4]m_0 + a_1[a_2,a_3]a_4m_0$$

for all $a_1,a_2,a_3,a_4 \in p\mathscr{A}p$, we get

$$0 = a_1[a_2,a_3]a_4m_0.$$

Therefore we can deduce that $\mathrm{Id}([p\mathscr{A}p,p\mathscr{A}p])m_0 = \{0\}$. By assumption we know that $\mathrm{Id}([p\mathscr{A}p,p\mathscr{A}p]) = p\mathscr{A}p$, so $pm_0 = m_0 = 0$. Similarly it can be shown that $n_0 = 0$. Therefore $x_0 = a_0 + b_0 \in Z(\mathscr{A})$. $\square$

Matrix algebras

Let $\mathscr{A} = M_n(A)$, $n \geqslant 2$, be a matrix algebra, where A is a unital algebra. Let $\{e_{ij} | i,j = 1,2,\ldots,n\}$ be the system of matrix units of $\mathscr{A}$ and let 1 be the identity of $\mathscr{A}$. Let us denote the idempotent $p = e_{11}$ and $q = 1 - e_{11}$. In this case $\mathscr{A}$ and p

satisfy (1). Note that the subalgebra $p\mathscr{A}p$ is isomorphic to A and $q\mathscr{A}q$ is isomorphic to the matrix algebra $M_{n-1}(A)$. Clearly, $(p\mathscr{A}p, q\mathscr{A}q)$-bimodule $p\mathscr{A}q \cong M_{1\times(n-1)}(A)$ is faithful as a left $p\mathscr{A}p$-module and as a right $q\mathscr{A}q$-module. Recall, that the ideal generated by all commutators of $M_n(A)$ coincides with $M_n(A)$. Assume that a is an arbitrary matrix in $M_n(A)$. Since $ae_{ii} = [ae_{ii}, e_{ij}]e_{ji}$ for all $i \neq j$ and $a = ae_{11} + \ldots + ae_{nn}$ we see that even the right ideal generated by all commutators of $M_n(A)$ coincides with $M_n(A)$.

COROLLARY 4.2. *The matrix algebra $M_n(A)$, $n \geqslant 2$, has property (P2).*

Proof. Let $n \geqslant 3$. Let us decompose the algebra $M_n(A)$ using the idempotent $p = e_{11}$. Then the algebra $q\mathscr{A}q$ is isomorphic to the matrix algebra $M_{n-1}(A)$. Since $\mathrm{Id}([M_{n-1}(A), M_{n-1}(A)]) = M_{n-1}(A)$ the result follows from Proposition 4.1.

Now let $n = 2$. From classical results of Jacobson and Rickart [9, Theorems 7 and 22] it follows that every Jordan derivation of $M_n(A)$ is a derivation. Regarding Theorem 3.1, Theorem 3.3 and the examples mentioned following (1) the proof will be completed if we show that only central elements a from $M_2(A)$ satisfy the identity $[[x,y],a] = 0$. Therefore, let $a = \sum_{i,j=1}^{2} a_{ij}e_{ij}$, $a_{ij} \in A$, satisfy this identity for all $x, y \in M_2(A)$. Since

$$\begin{aligned} 0 &= [[e_{11}, e_{12}], a] = [e_{12}, a] = a_{21}e_{11} + (a_{22} - a_{11})e_{12} - a_{21}e_{22}, \\ 0 &= [[e_{21}, e_{11}], a] = [e_{21}, a] = -a_{12}e_{11} + (a_{11} - a_{22})e_{21} + a_{12}e_{22}, \end{aligned}$$

we see, that $a_{12} = a_{21} = 0$ and $a_{11} = a_{22}$. Therefore $a = a_0(e_{11} + e_{22}) = a_0 1$ for some $a_0 \in A$. Let $a_1 \in A$ be an arbitrary element. Since

$$0 = [[a_1e_{11}, e_{12}], a] = [a_1e_{12}, a_0 1] = [a_1, a_0]e_{12}$$

it follows that $a_0 \in Z(A)$. We have proven that $a \in Z(A) \cdot 1 = Z(M_2(A))$. $\square$

Triangular algebras

In case $\mathscr{A}$ is a unital algebra containing a nontrivial idempotent p such that $q\mathscr{A}p = \{0\}$ and that the bimodule $p\mathscr{A}q$ is faithful as a left $p\mathscr{A}p$-module and also as a right $q\mathscr{A}q$-module, then $\mathscr{A}$ is a triangular algebra. The most important examples of triangular algebras are upper triangular matrix algebras $T_n(A)$, block upper triangular matrix algebras $B_n(A)$ over a unital algebra A and also nest algebras $T(\mathscr{N})$, where $\mathscr{N}$ is a nest in a Hilbert space H. Zhang and Yu [13] gave a proof that every Jordan derivation of a triangular algebra is a derivation. Using this and Theorems 3.1 and 3.3 we obtain:

COROLLARY 4.3. *For any triangular algebra $\mathscr{A}$ the following hold:*

(i) *$\mathscr{A}$ has property (P1),*

(ii) *$\mathscr{A}$ has property (P2) if and only if $Z(\mathscr{A}) = \{a \in \mathscr{A}; [[\mathscr{A}, \mathscr{A}], a] = 0\}$.*

Upper triangular matrix algebra. Let A be a unital algebra. By $T_n(A)$ we denote the algebra of all $n\times n$ upper triangular matrices with entries in A. For $n\geqslant 2$ the algebra $T_n(A)$ can be represented as a triangular algebra of the form

$$T_n(A)=\begin{pmatrix} A & M_{1\times(n-1)}(A) \\ & T_{n-1}(A)\end{pmatrix}.$$

Hence $T_n(A)$ is an algebra with property (P1). Note, that $T_n(A)$ is generated as an ideal by its commutators if and only if A is an algebra generated as an ideal by its commutators. Hence from Proposition 4.1 it follows:

COROLLARY 4.4. *Let A be a unital algebra satisfying $\mathrm{Id}([A,A])=A$. Then $T_n(A)$, $n\geqslant 2$, has property (P2).*

COROLLARY 4.5. *Let A be a unital commutative algebra. Then the upper triangular matrix algebra $T_n(A)$, $n\geqslant 2$, does not have property (P2).*

Proof. Note that the center of $T_n(A)$ does not contain the matrix unit e_{1n}, but the identity $[[x,y],e_{1n}]=0$ holds for all $x,y\in T_n(A)$. Hence the result follows from Corollary 4.3(ii). $\square$

Block upper triangular matrix algebras. Let A be a unital algebra. Let $n\geqslant 2$, $m\geqslant 1$ be positive integers and let $(k_1,k_2,\ldots,k_m)$ be a vector of positive integers such that $k_1+k_2+\ldots+k_m=n$. The block upper triangular matrix algebra $B_n(A)$ is a subalgebra of the matrix algebra $M_n(A)$ of the form

$$B_n(A)=\begin{pmatrix} M_{k_1}(A) & M_{k_1\times k_2}(A) & \cdots & M_{k_1\times k_m}(A) \\ 0 & M_{k_2}(A) & \cdots & M_{k_2\times k_m}(A) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(A)\end{pmatrix}.$$

If $m=1$, then $B_n(A)=M_n(A)$ is an algebra with property (P2). If $m\geqslant 2$ and $k_i=1$ for $i=1,2,\ldots,m$, then the algebra $B_n(A)$ is simply the upper triangular matrix algebra $T_n(A)$, and so Corollary 4.4 and Corollary 4.5 apply.

When $B_n(A)\neq M_n(A)$ then the algebra $B_n(A)$ is a triangular algebra and therefore has property (P1). Notice that the decomposition of algebra $B_n(A)$ using idempotent p need not satisfy (1). Let us give an example.

EXAMPLE 4.6. Let A be a unital algebra and consider the block upper triangular matrix algebra

$$\mathscr{A}=B_4(A)=\begin{pmatrix} A & A & A & A \\ 0 & A & A & A \\ 0 & A & A & A \\ 0 & 0 & 0 & A\end{pmatrix}.$$

Does the algebra $\mathscr{A}$ have property (P2)? Let us decompose $\mathscr{A}$ using the idempotent $p = e_{22} + e_{33}$: $\mathscr{A} = p\mathscr{A}p + p\mathscr{A}q + q\mathscr{A}p + q\mathscr{A}q$, where $q = e_{11} + e_{44}$. Since

$$p\mathscr{A}p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & A & 0 \\ 0 & A & A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong M_2(A),$$

we have $\mathrm{Id}([p\mathscr{A}p, p\mathscr{A}p]) = p\mathscr{A}p$. One might think to use Proposition 4.1. But, as we will see, this decomposition of the algebra $\mathscr{A}$ does not satisfy (1). Note that

$$p\mathscr{A}q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q\mathscr{A}p = \begin{pmatrix} 0 & A & A & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $e_{14} \in q\mathscr{A}q$ we have $p\mathscr{A}q \cdot e_{14} = \{0\} = e_{14} \cdot q\mathscr{A}p$. Therefore Proposition 4.1 can not be applied.

The algebra $\mathscr{A}$ does not have property (P2) if the algebra A in commutative. Note that the algebra $\mathscr{A}$ contains a noncentral element e_{14} satisfying $[[\mathscr{A},\mathscr{A}], e_{14}] = 0$. On the other hand, if the algebra A satisfyies $\mathrm{Id}([A,A]) = A$, then $\mathscr{A}$ has property (P2). This follows from the following proposition.

COROLLARY 4.7. *Let $B_n(A)$, $n \geqslant 2$, be a block upper triangular matrix algebra.*

(i) If $k_1 \geqslant 2$ or $k_m \geqslant 2$, then $B_n(A)$ has property (P2).

(ii) If $k_1 = k_m = 1$ and

(a) $\mathrm{Id}([A,A]) = A$, then $B_n(A)$ has property (P2).

(b) A is commutative, then $B_n(A)$ does not have property (P2).

Proof. (i) If $m = 1$ and $k_1 \geqslant 2$, then $B_n(A) = M_{k_1}(A)$ is an algebra with property (P2).

Let $m \geqslant 2$ and let us assume that $k_1 \geqslant 2$. For the case $k_m \geqslant 2$ the proof is similar. If we choose the idempotent $p = e_{11} + \ldots + e_{k_1k_1}$ and $q = 1 - p$, then $\mathscr{A} = B_n(A)$ can be represented as a triangular algebra of the form $\mathscr{A} = p\mathscr{A}p + p\mathscr{A}q + q\mathscr{A}q$. In this case $\mathscr{A}$ and p satisfy (1). Let us recall that every matrix algebra $M_k(A)$, $k \geqslant 2$, coincides with the ideal generated by all commutators of $M_k(A)$. Since $p\mathscr{A}p \cong M_{k_1}(A)$, we have $\mathrm{Id}([p\mathscr{A}p, p\mathscr{A}p]) = p\mathscr{A}p$ and so, by Proposition 4.1, $\mathscr{A}$ has property (P2).

(ii) Let $k_1 = k_m = 1$ and $\mathrm{Id}([A,A]) = A$. Again, if we choose the idempotent $p = e_{11}$, then $\mathscr{A} = B_n(A)$ is a triangular algebra where $p\mathscr{A}p \cong A$. Consequently, by Proposition 4.1, $\mathscr{A}$ has property (P2).

In the last case, when A is a unital commutative algebra and $B_n(A)$ is a block upper triangular algebra with $k_1 = k_m = 1$, the identity $[[x,y], e_{1n}] = 0$ holds for all

$x,y \in B_n(A)$. Since the center of $B_n(A)$ does not contain the matrix unit e_{1n}, the result follows from Corollary 4.3(ii). $\square$

Let us point out that every block upper triangular matrix algebra $B_n(\mathbb{C})$ over the field of complex numbers is isomorphic to some nest algebra over a finite dimensional Hilbert space H.

Nest algebras. Let H be a complex Hilbert space and $\mathscr{B}(H)$ the algebra of all bounded linear operators on H. A nest is a chain $\mathscr{N}$ of closed subspaces of H containing $\{0\}$ and H, which is closed under arbitrary intersections and closed linear spans. The nest algebra associated to $\mathscr{N}$ is the algebra

$$\mathscr{T}(\mathscr{N}) = \{T \in \mathscr{B}(H) \mid T(N) \subseteq N \text{ for all } N \in \mathscr{N}\}.$$

A nest $\mathscr{N}$ is called trivial if $\mathscr{N} = \{0,H\}$. In this case $\mathscr{T}(\mathscr{N}) = \mathscr{B}(H)$ has property (P2). Namely, $\mathscr{B}(H)$ is a prime algebra. If $\mathscr{N}$ is a nontrivial nest, then $T(\mathscr{N})$ is a triangular algebra. Namely, if $N \in \mathscr{N} \setminus \{0,H\}$ and e is the orthonormal projection onto N, then $T(\mathscr{N}) = T(\mathscr{N})e + eT(\mathscr{N})f + fT(\mathscr{N})$, where $f = 1-e$ denotes the orthonormal projection onto $N^{\perp}$. Both subalgebras $T(\mathscr{N})e$ and $fT(\mathscr{N})$ are also nest algebras and the center of each of them coincides with $\mathbb{C}\cdot 1$. Clearly, every nest algebra $T(\mathscr{N})$ has property (P1). An algebra $\mathscr{T}(\mathscr{N})$ has property (P2) if and only if only central elements a satisfy the identity $[[x,y],a] = 0$ for all $x,y \in \mathscr{T}(\mathscr{N})$. So when is this the case? The answer to this question was given recently by Kong and Zhang [10, Theorem 3.1]. Let us present the result as a corollary:

COROLLARY 4.8. *Let $\mathscr{N}$ be a nest of a complex separable Hilbert space H. $\mathscr{T}(\mathscr{N})$ has property (P2) if and only if* $\dim 0_+ \neq 1$ *or* $\dim H_-^{\perp} \neq 1$.

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Dominik Benkovič
University of Maribor, Faculty of Natural Sciences and Mathematics
Department of Mathematics and Computer Science
2000 Maribor, Slovenia
e-mail: dominik.benkovic@um.si

Mateja Grašič
University of Maribor, Faculty of Natural Sciences and Mathematics
Department of Mathematics and Computer Science
2000 Maribor, Slovenia
e-mail: mateja.grasic@um.si