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STANDARD OPERATOR JORDAN RINGS ON BANACH SPACES, AND THEIR DERIVATIONS

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Abstract. Let X be a real or complex Banach space, let $\mathscr{L}(X)$ denote the algebra of all bounded linear operators on X, and let $\mathscr{F}(X)$ stand for the ideal of $\mathscr{L}(X)$ consisting of those operators in $\mathscr{L}(X)$ having finite-dimensional range. We introduce *standard operator Jordan algebras* on X as those Jordan subalgebras of $\mathscr{L}(X)$ which contain $\mathscr{F}(X)$.

As main results, we prove the following:

— If X is a real or complex Banach space, if $\mathscr{A}$ is a standard operator Jordan algebra on X, and if D is an $\mathscr{L}(X)$-valued linear (Jordan) derivation of $\mathscr{A}$, then there exists $B \in \mathscr{L}(X)$ such that $D(A) = [B,A]$ for every $A \in \mathscr{A}$.

— Every standard operator Jordan algebra $\mathscr{A}$ has minimum norm topology, i.e. the topology of any algebra norm on $\mathscr{A}$ is greater than or equal to that of the operator norm.

— Surjective algebra homomorphisms from complete normed Jordan algebras to standard operator Jordan algebras are continuous.

Actually, the first of the results just quoted is discussed when $\mathscr{A}$ is merely a *standard operator Jordan ring* on X (i.e, a Jordan subring of $\mathscr{L}(X)$ containing $\mathscr{F}(X)$) and D is assumed to be additive (as linearity of D could have not a meaning in this setting). In turn, the last of the results quoted above remains true if the completeness of the starting normed algebras is substantially weakened.

1. Introduction, announcement of results, and a theorem of existence

For notions not introduced in this paper, the reader is referred to [8].

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$, let X be a Banach space over $\mathbb{K}$, let $\mathscr{L}(X)$ denote the algebra over $\mathbb{K}$ of all bounded linear operators on X, and let $\mathscr{F}(X)$ stand for the ideal of $\mathscr{L}(X)$ consisting of those operators in $\mathscr{L}(X)$ having finite-dimensional range. *Standard operator algebras* (respectively, *standard operator rings*) on X are defined as those subalgebras (respectively, subrings) of $\mathscr{L}(X)$ which contain $\mathscr{F}(X)$. In this paper we deal with a larger class of subsets of $\mathscr{L}(X)$, namely that of those Jordan subalgebras (respectively, Jordan subrings) of $\mathscr{L}(X)$ which contain $\mathscr{F}(X)$. These subsets of $\mathscr{L}(X)$ shall be called *standard operator Jordan algebras* (respectively, *standard operator Jordan rings*) on X. By the definition of a Jordan subring of an associative algebra over $\mathbb{K}$, every standard operator Jordan ring (so a fortiori every standard

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operator Jordan algebra) is closed under the so-called *Jordan product* $\bullet$ defined by $A_1 \bullet A_2 := \frac{1}{2}(A_1A_2 + A_2A_1)$, so in particular under the passing to squares of its elements. Since the inequality $\|A_1 \bullet A_2\| \leqslant \|A_1\|\|A_2\|$ holds for all $A_1, A_2 \in \mathscr{L}(X)$, every standard operator Jordan algebra on X shall be considered without notice as a normed algebra under the operator norm.

The following theorem was proved by Chernoff [10].

THEOREM 1.1. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard operator algebra on X, and let $D : \mathscr{A} \to \mathscr{L}(X)$ be a linear mapping such that*

$$D(A_1A_2) = D(A_1)A_2 + A_1D(A_2) \text{ for all } A_1, A_2 \in \mathscr{A}. \tag{1.1}$$

Then there exists $B \in \mathscr{L}(X)$ such that $D(A) = AB - BA$ for every $A \in \mathscr{A}$.

The main goal in this paper is to formulate and prove the appropriate variant of Theorem 1.1 when standard operator Jordan rings replace standard operator algebras. This is done by proving Theorem 1.5 below. To this end, the basic concept and fact which follow should be considered.

Let X be a vector space over $\mathbb{K}$. Following [19, Definition IV.14.1], we say that a mapping $T : X \to X$ is a *differential operator* on X if it is additive and there exists a ring derivation d of $\mathbb{K}$ [1] such that

$$T(\lambda x) = \lambda T(x) + d(\lambda)x \text{ for all } \lambda \in \mathbb{K} \text{ and } x \in X. \tag{1.2}$$

It is easily realized that the set of all differential operators on X is a Lie subring of the associative ring of all additive operators on X, and contains the set (say $L(X)$) of all linear operators on X as an ideal. Therefore we are provided with the following.

FACT 1.2. *Let X be a vector space over $\mathbb{K}$, and let T be a differential operator on X. Then $[A,T] := AT - TA$ lies in $L(X)$ whenever A does, and the mapping $A \to [A,T]$ is a ring derivation of $L(X)$.*

The key tool to prove Theorem 1.5 is the following theorem, proved in [28].

THEOREM 1.3. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard operator algebra on X, and let $D : \mathscr{A} \to \mathscr{L}(X)$ be an additive mapping satisfying*

$$D(A_1A_2) = D(A_1)A_2 + A_1D(A_2) \text{ for all } A_1, A_2 \in \mathscr{A}.$$

We have:

(i) *There exists a differential operator T on X such that $D(A) = [A,T]$ for every $A \in \mathscr{A}$.*

(ii) *If X is infinite-dimensional, then the differential operator T above actually lies in $\mathscr{L}(X)$.*

[1] The reader is referred to [25] to realize that (automatically discontinuous) nonzero ring derivations of $\mathbb{K}$ do exist.

Comments on the proof. Assertion (i) corresponds to the first part of the proof of [28, Theorem 3] (from the beginning of that proof to line 16 in page 321). It is worth noticing that, essentially, the differential operator T whose existence is assured is uniquely determined by D, in the sense that two such operators differ only in a scalar multiple of the identity operator on X (see Proposition 2.2(ii) in the current paper). This remark is important to the well-understanding of the words 'the differential operator T above' in assertion (ii). We note that assertion (i) also follows from Jacobson's Theorem IV.14.3 in [19], by 'pairing' the Banach space X with its dual [2].

The crucial assertion (ii) corresponds to the remaining part of the proof of [28, Theorem 3]. □

To complement Theorem 1.3, let us consider the case that the Banach space X over $\mathbb{K}$ is finite-dimensional. Then $X = \mathbb{K}^m$ for some positive integer m, and $\mathscr{L}(X) = M_m(\mathbb{K})$ is the unique standard operator Jordan ring on X. In this case, each ring derivation d of $\mathbb{K}$ gives rise to a differential operator $\widehat{d}$ on X defined by $\widehat{d}(\lambda_1,\ldots,\lambda_m) := \big(d(\lambda_1),\ldots,d(\lambda_m)\big)$. With this notation, [27, Theorem 2.2] can be read as follows.

THEOREM 1.4. *Set $X = \mathbb{K}^m$, and let T be a differential operator on X. Then there exist $A \in \mathscr{L}(X)$ and a ring derivation d of $\mathbb{K}$ such that $T = A + \widehat{d}$.*

Now our main result reads as follows.

THEOREM 1.5. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard operator Jordan ring on X, and let $D : \mathscr{A} \to \mathscr{L}(X)$ be an additive mapping such that*

$$D(A^2) = 2D(A) \bullet A \quad \text{for every} \quad A \in \mathscr{A}. \tag{1.3}$$

Then there exists a differential operator T on X such that $D(A) = [A,T]$ for every $A \in \mathscr{A}$. Moreover the differential operator T above lies in $\mathscr{L}(X)$ whenever X is infinite-dimensional, or D is linear on $\mathscr{F}(X)$, or D is continuous.

Actually, Theorem 1.5 above is derived from a more general result in the spirit of [21] (see Theorem 2.4).

As a straightforward consequence of Theorem 1.5, we obtain the following.

COROLLARY 1.6. *Let X, $\mathscr{A}$, and D be as in Theorem 1.5. Then D is continuous if and only if there exists $B \in \mathscr{L}(X)$ such that $D(A) = [A,B]$ for every $A \in \mathscr{A}$.*

REMARK 1.7. Let $\mathscr{E}$ be a (possibly non-commutative) associative algebra over $\mathbb{K}$, and let $\mathscr{W}$ be a left module over $\mathscr{E}$ in the usual sense (see for example [3, Definition 9.11]). Adapting to this setting the notion that we have given of a differential operator on a vector space over $\mathbb{K}$, we say that a mapping $T : \mathscr{W} \to \mathscr{W}$ is a *differential operator* on $\mathscr{W}$ if it is linear and there exists a linear derivation d of $\mathscr{E}$ such that $T(ex) = eT(x) + d(e)x$ for all $e \in \mathscr{E}$ and $x \in \mathscr{W}$. It is clear that both all $\mathscr{E}$-linear

[2] Howhever, it is worth noticing that Jacobson omits the proof of his theorem, giving only some vague indications for that proof.

operators on $\mathscr{W}$ and all operators of the form $S_e : x \to ex$ (when e runs over $\mathscr{E}$) are differential operators on $\mathscr{W}$. To obtain the description of derivations of the so-called 'structurable H^*-algebras' [7], differential operators on the so-called 'Hilbert modules over associative H^*-algebras with zero annihiltor' [26] were studied in [6]. As the main result in [6], it is proved that, given a Hilbert module $\mathscr{W}$ over an associative H^*-algebra $\mathscr{E}$ with zero annihilator, the set of all bounded differential operators on X is a weak-operator closed Lie subalgebra of $\mathscr{L}(W)$, and that every bounded differential operator on $\mathscr{W}$ is the sum of a continuous $\mathscr{E}$-linear operator and of an operator in the closure of the set $\{S_e : e \in \mathscr{E}\}$ in $\mathscr{L}(X)$ for the weak-operator topology.

In the last section of this paper we prove that every standard operator Jordan algebra $\mathscr{A}$ over $\mathbb{K}$ has *minimum norm topology*, i.e. the topology of any algebra norm on $\mathscr{A}$ is greater than or equal to that of its natural norm (see Theorem 3.2). This result (whose associative forerunner is well-known [13]) becomes one of the ingredients to show that surjective algebra homomorphisms from complete normed Jordan algebras over $\mathbb{K}$ to standard operator Jordan algebras on any Banach space over $\mathbb{K}$ are continuous. Actually, this last result remains true if the completeness of the starting normed algebra is substantially relaxed (see Theorem 3.7 for details).

Let X be a Hilbert space over $\mathbb{K}$. Jordan subalgebras of $\mathscr{L}(X)$ have been considered in the literature, specially in the case that they actually are *Jordan ideals*, i.e. subspaces $\mathscr{A}$ of $\mathscr{L}(X)$ such that $\mathscr{L}(X) \bullet \mathscr{A} \subseteq \mathscr{A}$. It was proved in [15, Theorem 3] that Jordan ideals of $\mathscr{L}(X)$ are (associative) ideals whenever X is separable. Later it is shown in [5, Item (iii) in p. 3 or Corollary 2.6] that the requirement of separability of X in the result just reviewed can be altogether removed. On the other hand, it is well-known that, if $\dim(X) \geqslant 2$, then Jordan subalgebras of $\mathscr{L}(X)$ which are not (associative) subalgebras do exist, the simplest example being the so-called three-dimensional spin factor (see [8, p. 553]). Nevertheless, it seems to us that the existence of standard operator Jordan algebras on X which are not subalgebras of $\mathscr{L}(X)$ has not been settled in the literature. (Note that such standard operator Jordan algebras on X may exist only if X is infinite-dimensional.) Therefore we conclude this section by proving the following.

THEOREM 1.8. *Let X be an infinite-dimensional Hilbert space over $\mathbb{K}$. Then there exists a standard operator Jordan algebra on X which is self-adjoint and closed in $\mathscr{L}(X)$, but is not a standard operator algebra.*

Proof. Case $\mathbb{K} = \mathbb{R}$.

In this case, by [24, Theorem 3], X becomes an absolute-valued algebra with a left unit under its own norm and a suitable product. Let e denote the left unit of this algebra, for $x \in X$ let $\Phi(x)$ denote the operator of left multiplication by x on this algebra and let us set $x^* := 2(x|e)e - x$, and for $T \in \mathscr{L}(X)$ let $T^* \in \mathscr{L}(X)$ denote the adjoint of T. Then, according to [8, Proposition 2.7.33], for every $x \in X$ we have $\Phi(x^*) = \Phi(x)^*$ and $\Phi(x)^*\Phi(x) = \|x\|^2 I_X$, where I_X denotes the identity operator on X. Therefore $\Phi(X)$ is a self-adjoint subspace of $\mathscr{L}(X)$, the mapping $\Phi : x \to \Phi(x)$ from X to $\mathscr{L}(X)$ is a linear isometry (hence $\Phi(X)$ is closed in $\mathscr{L}(X)$), and, for each nonzero $x \in X$, $\Phi(x)$

is invertible in $\mathscr{L}(X)$ with $\|\Phi(x)^{-1}\| = \|\Phi(x)\|^{-1}$. On the other hand, for $x \in X$ we have

$$\begin{aligned}\Phi(x)^2 &= \Phi(2(x|e)e - x^*)\Phi(x) = 2(x|e)\Phi(x) - \Phi(x)^*\Phi(x)\\ &= 2(x|e)\Phi(x) - \|x\|^2 I_X = \Phi(2(x|e)x - \|x\|^2 e) \in \Phi(X).\end{aligned}$$

Therefore $\Phi(X)$ is a Jordan subalgebra of $\mathscr{L}(X)$. Now denote by $\mathscr{K}(X)$ the ideal of $\mathscr{L}(X)$ consisting of all compact operators of X. Let $0 \neq x \in X$ and $B \in \mathscr{K}(X)$. Then, since $\Phi(x)$ is invertible in $\mathscr{L}(X)$ and B is not, it follows from [8, Corollary 1.1.21(ii)] that $\|\Phi(x) - B\| \geqslant \|\Phi(x)^{-1}\|^{-1} = \|\Phi(x)\|$. This shows that $\Phi(X) \cap \mathscr{K}(X) = 0$, that the direct sum $\Phi(X) \oplus \mathscr{K}(X)$ is topological, and that consequently $\mathscr{A} := \Phi(X) \oplus \mathscr{K}(X)$ is closed in $\mathscr{L}(X)$ (as both $\Phi(X)$ and $\mathscr{K}(X)$ are Banach spaces). Moreover, since $\Phi(X)$ is a self-adjoint Jordan subalgebra of $\mathscr{L}(X)$, we realize that $\mathscr{A}$ is a self-adjoint standard Jordan operator algebra on X.

Now only remains to prove that $\mathscr{A}$ is not a subalgebra of $\mathscr{L}(X)$. To this end, recall that $\|\Phi(x) - B\| \geqslant \|\Phi(x)\|$ whenever $x \in X$ and $B \in \mathscr{K}(X)$. Then, since Φ is a linear isometry, we derive that $\|\Phi(x) + \mathscr{K}(X)\| = \|\Phi(x)\| = \|x\|$ for every $x \in X$. In this way the mapping $x \to \Phi(x) + \mathscr{K}(X)$ becomes a surjective linear isometry from X to the norm-unital complete normed Jordan algebra $\mathscr{A}/\mathscr{K}(X)$, and takes e to the unit of $\mathscr{A}/\mathscr{K}(X)$. Therefore, since X is a Hilbert space, and every Hilbert space is smooth at any point of its unit sphere, we conclude that $\mathscr{A}/\mathscr{K}(X)$ is smooth at its unit, i.e. $\mathscr{A}/\mathscr{K}(X)$ is a 'smooth normed algebra' in the sense of [8, p. 204]. Now, if $\mathscr{A}$ were a subalgebra of $\mathscr{L}(X)$, then $\mathscr{A}/\mathscr{K}(X)$ would be an associative smooth normed algebra, and therefore, by the implication (ii)$\Rightarrow$(iii) in [8, Theorem 2.6.21], $\mathscr{A}/\mathscr{K}(X)$ would be finite-dimensional, which is obviously impossible.

Case $\mathbb{K} = \mathbb{C}$.

For any complex vector space Z and any conjugate-linear involutive operator $\natural$ on Z, let us denote by $H(Z, \natural)$ the real subspace of Z consisting of all $\natural$-invariant elements of Z. Now let Y be a complex normed space, and let $\natural$ be a conjugation (i.e. a conjugate-linear involutive isometry) on Y. According to the notation and facts in [8, §3.4.71], by defining $T^\natural := \natural \circ T \circ \natural$ for $T \in \mathscr{L}(Y)$, we are provided with an isometric involutive conjugate-linear algebra automorphism (denoted also by $\natural$) on $\mathscr{L}(Y)$ such that the normed real algebras $\mathscr{L}(H(Y, \natural))$ and $H(\mathscr{L}(Y), \natural)$ are bicontinuously isomorphic in a natural way.

Now let X be our infinite-dimensional complex Hilbert space. Choose an orthonormal basis of X, let $\natural$ denote the unique conjugation on X which fixes all elements of the basis, and let $\phi : \mathscr{L}(H(X, \natural)) \to H(\mathscr{L}(X), \natural)$ be the natural algebra isomorphism. Then, according to [9, Lemma 8.1.105], ϕ is an isometry. On the other hand, since $H(X, \natural)$ is an infinite-dimensional real Hilbert space, the case $\mathbb{K} = \mathbb{R}$ of the theorem (already proved) provides us with a standard operator Jordan algebra $\mathscr{A}$ on $H(X, \natural)$ which is self-adjoint and closed in $\mathscr{L}(H(X, \natural))$, but is not a standard operator algebra. Therefore, since $\mathscr{L}(X) = H(\mathscr{L}(X), \natural) \oplus iH(\mathscr{L}(X), \natural)$, and this direct sum is topological, and $\mathscr{F}(X)$ is a $\natural$-invariant subalgebra of $\mathscr{L}(X)$, and $\phi(\mathscr{A}) \subseteq H(\mathscr{L}(X), \natural)$, we realize that $\phi(\mathscr{A}) \oplus i\phi(\mathscr{A})$ is a standard operator Jordan algebra on X which is self-adjoint and closed in $\mathscr{L}(X)$, but is not a standard operator algebra. $\square$

The result in [24] applied at the beginning of the above proof is very deep, as it depends heavily on the mathematical formulation of the so-called 'Canonical Anticommutation Relations' in Quantum Mechanics (see for example [4, Proposition 5.2.2]). In the case that the real Hilbert space X is separable, a clever and much simpler proof of the result in [24] just quoted has been provided in [11]. With a slightly different formalization, this proof can be found as the proof of [8, Theorem 2.7.38].

2. Derivations of standard operator Jordan rings are inner

Although not relevant in our development, the following fact has its own interest.

FACT 2.1. *Let X be a Hausdorff topological vector space over $\mathbb{K}$. Then every continuous differential operator on X is linear.*

Proof. We may suppose that $X \neq 0$. Let T be a continuous differential operator on X, and let d be the associated ring derivation of $\mathbb{K}$. It is well-known and easy to prove that there is no continuous ring derivation of $\mathbb{K}$ other than 0. Therefore, to prove the lemma it is enough to show that d is continuous. Then, since d is additive, it suffices to show that d is continuous at 0. Let λ_n be a sequence in $\mathbb{K}$ converging to 0. Take $0 \neq x \in X$. Then it follows from (1.2) that $\lim_n d(\lambda_n)x = 0$. But, by Tihonov's theorem (see for example [18, p. 144]), the mapping $\lambda \to \lambda x$ from $\mathbb{K}$ to X is a topological embedding. Therefore the sequence $d(\lambda_n)$ converges to 0. $\square$

Let X be a Banach space over $\mathbb{K}$, and let $\mathscr{A}$ be a standard operator Jordan ring on X. We note that, if A belongs to $\mathscr{A}$, then A^n lies in $\mathscr{A}$ for every positive integer n. Indeed, we have clearly $A^{n+1} = A^n \bullet A$, and the induction principle applies. Now, given a positive integer n, we are going to studying additive mappings $D : \mathscr{A} \to \mathscr{L}(X)$ satisfying

$$D(A^{2n}) = 2D(A^n) \bullet A^n \quad \text{for every} \quad A \in \mathscr{A}. \tag{2.1}$$

Given a Banach space X over $\mathbb{K}$, we denote by X' the (topological) dual of X and, for $(x, f) \in X \times X'$, we denote by $x \otimes f \in \mathscr{F}(X)$ the operator on X defined by $(x \otimes f)(y) := f(y)x$ for every $y \in X$.

PROPOSITION 2.2. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard operator Jordan ring on X, and let $D : \mathscr{A} \to \mathscr{L}(X)$ be an additive mapping vanishing on $\mathscr{F}(X)$. We have:*

(i) *If D satisfies (2.1) for a certain positive integer number n, then $D(A^n) = 0$ for every $A \in \mathscr{A}$.*

(ii) *If D is of the form $A \to [A, T]$ for some differential operator T on X, then $T \in \mathbb{K}I_X$ (where I_X denotes the identity on X) and hence $D = 0$.*

Proof. Assertion (i) is proved by arguing verbatim as in the proof of the implication (ii)$\Rightarrow$(i) in [21, Proposition 2.3] (a rather long argument).

Suppose that D is of the form $A \to [A,T]$ for some differential operator T on X, and that D vanishes on $\mathscr{F}(X)$. Let d be the ring derivation of $\mathbb{K}$ associated to T, let x,y be in X with $y \neq 0$, take f be in X' such that $f(y)=1$, and set $A := x \otimes f \in \mathscr{F}(X)$. Then, since D vanishes on $\mathscr{F}(X)$, we have

$$\begin{aligned} 0 &= ([A,T])(y) = f(T(y))x - T(f(y)x) \\ &= f(T(y))x - f(y)T(x) - d(f(y))x = f(T(y))x - T(x). \end{aligned}$$

Therefore, since x is arbitrary in X, we see that $T = \lambda I_X$, where $\lambda := f(T(y)) \in \mathbb{K}$. Thus assertion (ii) has been proved. □

PROPOSITION 2.3. *Let X be a Banach space over $\mathbb{K}$, set $\mathscr{A} := \mathscr{F}(X)$, let n be a positive integer, and let $D : \mathscr{A} \to \mathscr{L}(X)$ be an additive mapping satisfying (2.1). Then there exists a differential operator T on X such that $D(A) = [A,T]$ for every $A \in \mathscr{A}$. Moreover the differential operator T above lies in $\mathscr{L}(X)$ whenever X is infinite-dimensional, or D is linear, or D is continuous.*

Proof. Let A be in $\mathscr{A}$. Then, according to Litoff's theorem [2, Theorem 4.3.11], there is a projection $P \in \mathscr{A}$ such that $AP = PA = A$. Now we follow the argument in the proof of [21, Proposition 2.1], with the appropriate changes. Indeed, as in that proof, condition (2.1) leads to

$$PD(P)P = 0. \tag{2.2}$$

Now, for any integer number λ set

$$p(\lambda) := D((A+\lambda P)^{2n}) - D((A+\lambda P)^n)(A+\lambda P)^n - (A+\lambda P)^n D((A+\lambda P)^n).$$

Then, considering that D is additive and that λ is an integer number, we can write

$$p(\lambda) = f_0(A,P) + \lambda f_1(A,P) + \cdots + \lambda^{2n} f_{2n}(A,P),$$

with $f_i(A,P)$ $(i = 0,\ldots,2n)$ in $\mathscr{L}(X)$. But, written in this way, $p(\lambda)$ has a meaning for every real value of λ, and hence p can be seen as a formal polynomial in the indeterminate $\lambda \in \mathbb{R}$ with coefficients in $\mathscr{L}(X)$. Moreover it follows from (2.1) that $p(\lambda) = 0$ for every integer value of $\lambda \in \mathbb{R}$. Therefore $f_i(A,P) = 0$ for every $i = 0,\ldots,2n$. In particular, $f_{2n-2}(A,P) = 0$ and $f_{2n-1}(A,P) = 0$. After a straightforward but tedious computation, the above equalities reduce into

$$\begin{aligned} 2(2n-1)D(A^2) = (n-1)\big(&D(A^2)P + PD(A^2) + D(P)A^2 + A^2D(P)\big) \\ &+ 2n(D(A)A + AD(A)), \end{aligned} \tag{2.3}$$

and

$$2D(A) = D(A)P + PD(A) + D(P)A + AD(P), \tag{2.4}$$

respectively. Now, arguing as in the proof we are following, we realize that equalities (2.2), (2.3), and (2.4) lead to

$$D(A^2) = D(A)A + AD(A). \tag{2.5}$$

Now, since A is arbitrary in $\mathscr{A}$, and P lies in $\mathscr{A}$, and $\mathscr{A}$ is an ideal of $\mathscr{L}(X)$, it follows from (2.4) that the range of D is contained in $\mathscr{A}$. Then it follows from (2.5) that D is a Jordan derivation of $\mathscr{A}$ in the ring sense. Therefore, since $\mathscr{A}$ is a prime ring, it follows from a celebrated theorem of Herstein [16] (see also [17, Theorem 3.3]) that D is a derivation of $\mathscr{A}$ in the ring sense. Now the proof of the first conclusion in the proposition is concluded by applying Theorem 1.3(i).

Suppose that X is infinite-dimensional (respectively, that D is linear). Then, considering Proposition 2.2(ii), the fact that T lies in $\mathscr{L}(X)$ follows from Theorem 1.3(ii) (respectively, Theorem 1.1).

Finally, suppose that D is continuous. Then, by the bracket-free version of the above paragraph, to prove that T lies in $\mathscr{L}(X)$ we may additionally suppose that X is finite-dimensional. Then, by Theorem 1.4, $T = B + \widehat{d}$ for some $B \in \mathscr{L}(X)$ and some ring derivation d of $\mathbb{K}$, and hence $D(A) = [A,B] + [A,\widehat{d}]$ for every $A \in \mathscr{A}$. Therefore the mapping $A \to [A,\widehat{d}]$ is continuous, as D is so. But, regarding $\mathscr{A}$ as the algebra of all $m \times m$ matrices with entries in $\mathbb{K}$, it is easily seen that the above mapping is nothing other than the mapping $(\lambda_{i,j}) \to -\big(d(\lambda_{i,j})\big)$. Therefore d is continuous, so $d = 0$, and so $T = B \in \mathscr{L}(X)$. $\square$

In the particular case that D is in fact linear, the above proposition has been proved in [21, Corollary 2.2].

Now the main result on the mappings we are dealing with is the following.

THEOREM 2.4. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard Jordan ring on X, let $D : \mathscr{A} \to \mathscr{L}(X)$ be an additive mapping, and let n be a positive integer. Then the following conditions are equivalent:*

(i) *D satisfies* (2.1).

(ii) *D can be written as $D = D_1 + D_0$ where $D_1 : \mathscr{A} \to \mathscr{L}(X)$ is of the form $A \to [A,T]$ for some differential operator T on X, and $D_0 : \mathscr{A} \to \mathscr{L}(X)$ is an additive mapping satisfying* (2.1) *and vanishing on $\mathscr{F}(X)$.*

Now suppose that condition (ii) *is fulfilled. Then:*

(iii) *The writing $D = D_1 + D_0$ is unique.*

(iv) *If $n \geqslant 2$, then the possibility that $D_0 \neq 0$ cannot be discarded, even if X is a Hilbert space, $\mathscr{A}$ is a standard operator algebra on X which is norm-closed in $\mathscr{L}(X)$, D is linear and continuous, and the range of D is contained in $\mathscr{A}$.*

(v) *The differential operator T lies in $\mathscr{L}(X)$ whenever X is infinite-dimensional, or D is linear on $\mathscr{F}(X)$, or D is continuous.*

In relation to assertion (iv) above, it is worth mentioning that, actually, if condition (ii) is fulfilled, and if $n \geqslant 2$, then, according to [21, Example 3.5], the possibility that D_0 be discontinuous cannot be discarded, even if X is a Hilbert space, $\mathscr{A}$ is a standard operator algebra on X which is norm-closed in $\mathscr{L}(X)$, D is linear, and the range of D is contained in $\mathscr{A}$.

Proof of Theorem 2.4. The implication (ii)$\Rightarrow$(i) in the theorem is not difficult to verify. Indeed, the set $\mathscr{S}$ of all additive mappings $D:\mathscr{A}\to\mathscr{L}(X)$ satisfying (2.1) is an additive subgroup of the additive group of all additive mappings from $\mathscr{A}$ to $\mathscr{L}(X)$, and contains both all mappings from $\mathscr{A}$ to $\mathscr{L}(X)$ which are of the form $A\to[A,T]$ for some differential operator T and all additive mappings $D:\mathscr{A}\to\mathscr{L}(X)$ such that $D(A^n)=0$ for every $A\in\mathscr{A}$. But, in view of Proposition 2.2(i), if $D\in\mathscr{S}$ vanishes on $\mathscr{F}(X)$, then $D(A^n)=0$ for every $A\in\mathscr{A}$.

Suppose that condition (i) in the theorem is fulfilled, i.e. D satisfies (2.1). Then, restricting D to $\mathscr{F}(X)$, and applying the first conclusion in Proposition 2.3, we realize that there exists a differential operator T on X such that $D(A)=[A,T]$ for every $A\in\mathscr{F}(X)$. Moreover, for every $A\in\mathscr{A}$, we have that $[A,T]\in\mathscr{L}(X)$. (Indeed, this follows from the last conclusion in Proposition 2.3 if X is infinite dimensional, and from Fact 1.2 otherwise.) Now let $D_1:\mathscr{A}\to\mathscr{L}(X)$ be defined by $D_1(A)=[A,T]$ for every $A\in\mathscr{A}$, and set $D_0:=D-D_1:\mathscr{A}\to\mathscr{L}(X)$. Then clearly both D_1 and D_0 are additive mappings satisfying (2.1), $D=D_1+D_0$, D_1 is of the form $A\to[A,T]$ for some differential operator T on X, and D_0 vanishes on $\mathscr{F}(X)$. Thus condition (ii) in the theorem is fulfilled.

Now that the equivalence (i)$\Leftrightarrow$(ii) has been shown, let us prove properties (iii), (iv), and (v).

Property (iii) follows from Proposition 2.2(ii), whereas property (iv) follows from [21, Example 2.4].

Suppose that X is infinite-dimensional, or that D is linear on $\mathscr{F}(X)$, or that D is continuous. Then, considering Proposition 2.2(ii) and that D_0 vanishes on $\mathscr{F}(X)$, the fact that T lies in $\mathscr{L}(X)$ follows by restricting D to $\mathscr{F}(X)$ and applying the second (and last) conclusion in Proposition 2.3. Thus property (v) has been proved. □

COROLLARY 2.5. *Let X, $\mathscr{A}$, D, and n be as in Theorem* 2.4. *Suppose that D satisfies* (2.1) *and that the additive subgroup of $\mathscr{A}$ generated by the set $\{A^n : A\in\mathscr{A}\}$ is equal to $\mathscr{A}$. Then the conclusions in Theorem* 1.5 *holds.*

Proof. Since D satisfies (2.1), we can write $D=D_1+D_0$ as in condition (ii) in Theorem 2.4. Then, by Proposition 2.2(i), $D_0(A^n)=0$ for every $A\in\mathscr{A}$. Therefore, since the additive subgroup of $\mathscr{A}$ generated by the set $\{A^n : A\in\mathscr{A}\}$ is equal to $\mathscr{A}$, we conclude that $D_0=0$ □

Now the proof of Theorem 1.5, already announced in Section 1, goes straightforwardly.

Proof of Theorem 1.5. Take $n=1$ in Corollary 2.5. □

As another outstanding application of Theorem 2.4, we can prove the following proposition, which gives light on the mapping D_0 in that theorem.

PROPOSITION 2.6. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard operator Jordan ring on X, and let n be a positive integer. An additive mapping*

$D : \mathscr{A} \to \mathscr{L}(X)$ satisfies (2.1) and vanishes on $\mathscr{F}(X)$ if and only if $D(A^n) = 0$ for every $A \in \mathscr{A}$.

Proof. The 'only if' part was already proved in Proposition 2.2(i).

Suppose that $D(A^n) = 0$ for every $A \in \mathscr{A}$. Then clearly D satisfies (2.1). To prove that D vanishes on $\mathscr{F}(X)$, we divide the argument in several steps.

Step 1. By restricting D to $\mathscr{F}(X)$ and applying the implication (i)$\Rightarrow$(ii) in [21, Proposition 2.3], we realize that D vanishes on $\mathscr{F}(X)$ whenever D is linear on $\mathscr{F}(X)$, so in particular whenever D is of the form $A \to [A,T]$ for some $T \in \mathscr{L}(X)$.

Step 2. Since D satisfies (2.1), Theorem 2.4 applies, and hence we can write $D = D_1 + D_0$ as in condition (ii) in that theorem, so that D_0 satisfies (2.1) and vanishes on $\mathscr{F}(X)$. Then for $A \in \mathscr{A}$ we have $D_0(A^n) = 0$ (by the 'only if' part of the current proposition), and hence $0 = D(A^n) = D_1(A^n)$. Thus D_1 is in the same situation than D. Moreover D_1 is of the form $A \to [A,T]$ for some differential operator T on X.

Step 3. Suppose that X is infinite-dimensional. Then, invoking again Theorem 2.4, and considering the two steps above, we realize that D_1 vanishes on $\mathscr{F}(X)$.

Step 4. Suppose that X is finite-dimensional. We identify X with $\mathbb{K}^m$, for a suitable positive integer m, and $\mathscr{A} = \mathscr{F}(X)$ with the algebra of all $m \times m$ matrices with entries in $\mathbb{K}$. Then, by Step 2, $D(A^n) = 0$ for every $A \in \mathscr{A}$, and moreover, by Theorem 1.4, the differential operator T in that step can be written as $T = B + \widehat{d}$ for some $B \in \mathscr{L}(X)$ and some ring derivation d of $\mathbb{K}$. Therefore for $A \in \mathscr{A}$ we have $D_1(A) = [A,B] + [A,\widehat{d}]$, and hence $[A^n,B] + [A^n,\widehat{d}] = D_1(A^n) = 0$, which implies $t([A^n,\widehat{d}]) = 0$, where t denotes the usual trace on $\mathscr{A}$. Now let λ be in $\mathbb{K}$, and take $A = \mathrm{Diag}\{\lambda,0,\ldots,0\}$. Then $A^n = \mathrm{Diag}\{\lambda^n,0,\ldots,0\}$, $[A^n,\widehat{d}] = -\mathrm{Diag}\{d(\lambda^n),0,\ldots,0\}$, and hence

$$d(\lambda^n) = t\,(\mathrm{Diag}\{d(\lambda^n),0,\ldots,0\}) = -t\,([A^n,\widehat{d}]) = 0.$$

But, given $\mu \in \mathbb{K}$, there is $\lambda \in \mathbb{K}$ such that $\mu = \lambda^n$ or $\mu = -\lambda^n$. It follows that $d = 0$. Then $D_1(\cdot) = [\cdot,B]$ with $B \in \mathscr{L}(X)$, and hence, according to Step 1, D_1 vanishes on $\mathscr{F}(X)$.

Concluding step. According to Steps 3 and 4, in any case D_1 vanishes on $\mathscr{F}(X)$. Therefore, since $D = D_1 + D_0$ and (as recalled in Step 2) D_0 also vanishes on $\mathscr{F}(X)$, the same happens for D. □

Taking $n = 1$ in Proposition 2.2(i), we obtain the following

COROLLARY 2.7. *Let X, $\mathscr{A}$, and D be as in a Theorem 1.5, and suppose that D vanishes on $\mathscr{F}(X)$. Then $D = 0$.*

In the case of the above corollary, the reference to [21], given in the proof of Proposition 2.2(i) for general values of n, can be avoided. To realize this, we first prove the following.

LEMMA 2.8. *Let $X \neq 0$ be a Banach space over $\mathbb{K}$, and let $F \in \mathscr{L}(X)$ be such that*

$$G \bullet F = 0 \ \text{ for every } \ G \in \mathscr{F}(X). \tag{2.6}$$

Then $F = 0$.

Proof. Take $(y,f) \in X \times X'$ such that $f(y) = 1$. Then, for each $x \in X$, (2.6) yields $(x \otimes f) \bullet F = 0$, which reads as $F(x) \otimes f = -x \otimes F'(f)$, which implies

$$F(x) = (F(x) \otimes f)(y) = -(x \otimes F'(f))(y) = -f(F(y))x.$$

Therefore, setting $\lambda := -f(F(y)) \in \mathbb{K}$, we obtain that $F = \lambda I_X$, where I_X denotes the identity operator on X. Now (2.6) yields $\lambda \mathscr{F}(X) = 0$, hence $\lambda = 0$ and $F = 0$, as desired. □

Actually the above lemma follows from a more general and deeper result asserting that, *if* $\mathscr{A}$ *is a unital associative prime algebra, if* I *is a nonzero ideal of* $\mathscr{A}$*, and if* $F \in \mathscr{A}$ *is such that* $G \bullet F = 0$ *for every* $G \in I$*, then* $F = 0$. Indeed, the assumption that $G \bullet F = 0$ for every $G \in I$, can be read as that $FG\mathbf{1} + \mathbf{1}GF = 0$ for every $G \in I$. Therefore, since the associative algebra $\mathscr{A}$ is prime, it follows from [2, Lemma 6.1.2(i)] (which can be read as that 'elementary operators' [12] on $\mathscr{A}$ vanishing on I, actually vanish on $\mathscr{A}$) that $FG\mathbf{1} + \mathbf{1}GF = 0$ for every $G \in \mathscr{A}$. By taking $G = \mathbf{1}$, we obtain $F = 0$, as desired.

New proof of Corollary 2.7. Let A be in $\mathscr{A}$, and let B be in $\mathscr{F}(X)$. Then we have

$$0 = D(A \bullet B) = D(A) \bullet B + A \bullet D(B) = D(A) \bullet B.$$

Since B is arbitrary in $\mathscr{F}(X)$, it follows from Lemma 2.8 that $D(A) = 0$. But A is arbitrary in $\mathscr{A}$. □

In the linear case, Corollary 2.5 has a converse. Indeed, noticing that assertion (2.1) in the current paper is nothing other than assertion (1.2) in [21], and considering Proposition 2.2(i), Propositions 2.9 and 2.10 immediately below are proved by arguing verbatim as in the proofs of [21, Proposition 3.3] and [21, Proposition 3.6], respectively, by simply replacing [21, Theorem 1.2] with the linear version of Theorem 2.4 in both proofs.

PROPOSITION 2.9. *Let* X *be a Banach space over* $\mathbb{K}$*, let* $\mathscr{A}$ *be a standard operator Jordan algebra on* X*, and let* n *be a positive integer. Then the following conditions are equivalent:*

(i) $\mathscr{A}$ *is equal to the linear hull of the set* $\{A^n : A \in \mathscr{A}\}$.

(ii) *Each linear mapping* $D : \mathscr{A} \to \mathscr{L}(X)$ *satisfying* (2.1) *for every* $A \in \mathscr{A}$ *is of the form* $A \to [A,B]$ *for some* $B \in \mathscr{L}(X)$.

(iii) *Each linear mapping* $D : \mathscr{A} \to \mathscr{A}$ *satisfying* (2.1) *for every* $A \in \mathscr{A}$ *is of the form* $A \to [A,B]$ *for some* $B \in \mathscr{L}(X)$.

PROPOSITION 2.10. *Let* X*,* $\mathscr{A}$*, and* n *be as in Proposition* 2.9*, and consider the following conditions:*

(i) *The linear hull of the set* $\{A^n : A \in \mathscr{A}\}$ *is dense in* $\mathscr{A}$.

(ii) *Each continuous linear mapping* $D : \mathscr{A} \to \mathscr{L}(X)$ *satisfying* (2.1) *is of the form* $A \to [A,B]$ *for some* $B \in \mathscr{L}(X)$.

(iii) *Each continuous linear mapping* $D : \mathscr{A} \to \mathscr{A}$ *satisfying* (2.1) *is of the form* $A \to [A,B]$ *for some* $B \in \mathscr{L}(X)$.

(iv) *Every linear mapping* $D : \mathscr{A} \to \mathscr{A}$ *satisfying* (2.1) *is continuous.*

(v) *The linear hull of the set* $\{A^n : A \in \mathscr{A}\}$ *is closed in* $\mathscr{A}$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), *and* (iv) $\Rightarrow$ (v).

Now we can prove the following.

PROPOSITION 2.11. *Let* X, $\mathscr{A}$, D, *and* n *be as in Theorem* 2.4. *Suppose that assertion* (2.1) *holds, and that for each* $A \in \mathscr{A}$ *there is a projection* $P \in \mathscr{A}$ *such that* $AP = PA = A$ *(which happens for instance if the identity operator on* X *belongs to* $\mathscr{A}$*). Then there exists* $B \in \mathscr{L}(X)$ *such that* $D(A) = [B,T]$ *for every* $A \in \mathscr{A}$.

Proof. Let $\mathscr{L}$ denote the linear hull of the set $\{A^n : A \in \mathscr{A}\}$. In view of Proposition 2.9, it is enough to show that $\mathscr{A} = \mathscr{L}$. Since this is obviously true if $n = 1$, we suppose that $n > 1$. Let A be in $\mathscr{A}$ and, according to our assumption, let $P \in \mathscr{A}$ be a projection such that $AP = PA = A$. Then for every real number λ we have

$$L_\lambda := \sum_{i=1}^{n-1} \binom{n}{i} \lambda^i A^i \big(= (\lambda A + P)^n - \lambda^n A^n - P\big) \in \mathscr{L}. \tag{2.7}$$

Set $\chi := (L_1, \ldots, L_{n-1}) \in \mathscr{L}^{n-1}$, $\xi := \left(\binom{n}{1}A^1, \ldots, \binom{n}{n-1}A^{n-1}\right) \in \mathscr{A}^{n-1}$, and

$$M := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}.$$

Then, writing χ and ξ as matrix columns, it follows from (2.7) that $M\xi = \chi$. Therefore, since M is a Vandermonde matrix, and $\chi \in \mathscr{L}^{n-1}$, we have that $\xi = M^{-1}\chi \in \mathscr{L}^{n-1}$, i.e. $\binom{n}{i}A^i \in \mathscr{L}$ for every $i = 1, \ldots, n-1$. In particular $A \in \mathscr{L}$. Since A is arbitrary in $\mathscr{A}$, the inclusion $\mathscr{A} \subseteq \mathscr{L}$ holds. But the converse inclusion is obvious. □

In the case that $\mathscr{A}$ is in fact a standard operator algebra, the above proposition was proved in [21, Proposition 3.1] by other methods.

Now we go back to standard operator Jordan rings.

THEOREM 2.12. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a standard operator Jordan ring on X, and let $D:\mathscr{A}\to\mathscr{L}(X)$ be an additive mapping satisfying*

$$D(A^3)=D(A^2)\bullet A+A^2\bullet D(A) \text{ for every } A\in\mathscr{A}. \tag{2.8}$$

Then the conclusions in Theorem 1.5 holds.

Proof. Let A be in $\mathscr{F}(X)$. Then arguing verbatim as in the first part of the proof of [31, Theorem 5], we obtain $D(A^2)=2A\bullet D(A)$. Since A is arbitrary in $\mathscr{F}(X)$, it follows from Proposition 2.3 that there exists a differential operator T on X such that $D(A)=[A,T]$ for every $A\in\mathscr{F}(X)$, and that T lies in $\mathscr{L}(X)$ whenever X is infinite-dimensional, or D is linear, or D is continuous. Now, as in the proof of the implication (i)⇒(ii) in Theorem 2.4, we realize that $[A,T]$ lies in $\mathscr{L}(X)$ for every $A\in\mathscr{A}$. Let $D_1:\mathscr{A}\to\mathscr{L}(X)$ be defined by $D_1(A)=[A,T]$ for every $A\in\mathscr{A}$, and set $D_0:=D-D_1:\mathscr{A}\to\mathscr{L}(X)$. Then clearly D_0 is an additive mapping satisfying (2.8) and vanishing on $\mathscr{F}(X)$. Therefore, to conclude the proof it is enough to show that D_0 vanishes on the whole $\mathscr{A}$. But this is verified by arguing verbatim as in the last part of the proof of [31, Theorem 5]. □

As noticed in [21, Remark 1.4], condition (1.3) in Theorem 1.5 implies condition (2.8) in Theorem 2.12 above. Therefore Theorem 1.5, which was already derived almost straightforwardly from Theorem 2.4, also follows from Theorem 2.12.

Theorem 1.5 and Theorem 2.12 were proved in [30, Theorem 2] and [31, Theorem 5], respectively, in the case that $\mathscr{A}$ is a standard operator algebra and that D is linear. Now, to conclude this section, let us say that some other results in [30] concerning standard operator algebras (like Theorems 1, 3, and 4 of that paper) could have appropriate variants in the more general setting of standard operator Jordan rings and of additive mappings. Actually we feel that, when standard operator Jordan rings replace standard operator algebras, and additivity of mappings replaces linearity, the results just quoted survive with the appropriate changes in their formulations and proofs.

3. Standard operator Jordan algebras have minimum norm topology

As proved by Dales in [13], *standard operator algebras on any real or complex Banach space have minimum norm topology.* According to [13], this fact is a very old result of M. Eidelheit [14], which may even go back to S. Mazur before 1939.

As the main result in this section, we are going to prove that the result just commented remains true when standard operator Jordan algebras replace standard operator algebras.

To this end, we recall that, according to [3, Definition 27.1], by a *pairing* over $\mathbb{K}$ we mean a triple $(X,Y,\langle\cdot,\cdot\rangle)$ where X, Y are vector spaces over $\mathbb{K}$ and $\langle\cdot,\cdot\rangle$ is a non-degenerate bilinear form on $X\times Y$; i.e. the conditions $y\in Y$ and $\langle X,y\rangle=0$ imply $y=0$, and the conditions $x\in X$ and $\langle x,Y\rangle=0$ imply $x=0$.

For every vector space Z over $\mathbb{K}$, let us denote by $L(Z)$ the associative algebra over $\mathbb{K}$ of all linear operators on Z. Now let $(X,Y,\langle\cdot,\cdot\rangle)$ be a pairing over $\mathbb{K}$. Following [3, Definition 27.4], operators $S\in L(X)$, $T\in L(Y)$ are said to be adjoint with respect to $\langle\cdot,\cdot\rangle$ if

$$\langle S(x),y\rangle = \langle x,T(y)\rangle \text{ for all } (x,y)\in X\times Y.$$

To each $S\in L(X)$ corresponds at most one $T\in L(Y)$ such that S,T are adjoint with respect to $\langle\cdot,\cdot\rangle$; this operator T, if it exists, is called the *adjoint* of S with respect to $\langle\cdot,\cdot\rangle$, and is denoted by $S^\sharp$. Similarly each $T\in L(Y)$ has at most one adjoint $T^\sharp\in L(X)$ with respect to $\langle\cdot,\cdot\rangle$. The set of all linear operators on X which have adjoints with respect to $\langle\cdot,\cdot\rangle$ is a subalgebra of $L(X)$, which is denoted by $L(X,Y,\langle\cdot,\cdot\rangle)$. The subset of $L(X,Y,\langle\cdot,\cdot\rangle)$ consisting of those operators in $L(X,Y,\langle\cdot,\cdot\rangle)$ which have finite-dimensional range is an ideal of $L(X,Y,\langle\cdot,\cdot\rangle)$, which is denoted by $F(X,Y,\langle\cdot,\cdot\rangle)$.

A pairing $(X,Y,\langle\cdot,\cdot\rangle)$ over $\mathbb{K}$ is said to be a *Banach pairing* if X and Y are Banach spaces over $\mathbb{K}$, and the bilinear form $\langle\cdot,\cdot\rangle$ is continuous. Note that, if $(X,Y,\langle\cdot,\cdot\rangle)$ is a Banach pairing over $\mathbb{K}$, then $x\to\langle x,\cdot\rangle$ becomes a natural continuous linear embedding from X to Y', and that, thanks to the closed graph theorem, the inclusion $L(X,Y,\langle\cdot,\cdot\rangle)\subseteq\mathscr{L}(X)$ holds.

Now we invoke the following.

PROPOSITION 3.1. [22, Proposition 3.1] *For a Banach pairing $(X,Y,\langle\cdot,\cdot\rangle)$ over $\mathbb{K}$ the following conditions are equivalent:*

(i) *The natural continuous linear embedding $X\to Y'$ is in fact a topological embeding.*

(ii) *All Jordan subalgebras of $L(X,Y,\langle\cdot,\cdot\rangle)$ containing $F(X,Y,\langle\cdot,\cdot\rangle)$ have minimum norm topology.*

(iii) *All subalgebras of $L(X,Y,\langle\cdot,\cdot\rangle)$ containing $F(X,Y,\langle\cdot,\cdot\rangle)$ have minimum norm topology.*

Now let X be a Banach space over $\mathbb{K}$, and let $\langle\cdot,\cdot\rangle$ stand for the natural bilinear form on $X\times X'$, i.e. $\langle x,f\rangle := f(x)$. Then, as pointed out in [3, Example 27.3] or [19, Example IV.10.5], $(X,X',\langle\cdot,\cdot\rangle)$ becomes a Banach pairing over $\mathbb{K}$, and we have $L(X,X',\langle\cdot,\cdot\rangle)=\mathscr{L}(X)$ and $F(X,X',\langle\cdot,\cdot\rangle)=\mathscr{F}(X)$. Moreover the natural embedding $X\to X''$ associated to $\langle\cdot,\cdot\rangle$ is nothing other than the natural inclusion $X\subseteq X''$. Therefore it is enough to apply the implication (i)$\Rightarrow$(iii) in Proposition 3.1 to obtain the Dales-Eidelheit-Mazur result reviewed at the beginning of this section. Analogously, applying the implication (i)$\Rightarrow$(ii) in Proposition 3.1, we obtain the following.

THEOREM 3.2. *Standard operator Jordan algebras on any Banach space over $\mathbb{K}$ have minimum norm topology.*

To take profit from Theorem 3.2, the next lemma shall be useful.

LEMMA 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ (possibly non-associative) normed algebras over $\mathbb{K}$, and let $\Phi:\mathscr{A}\to\mathscr{B}$ be a surjective algebra homomorphism. Suppose that $\mathscr{B}$ has minimum norm topology, and that $\ker(\Phi)$ is closed in $\mathscr{A}$. Then Φ is continuous.*

Proof. Since $\ker(\Phi)$ is closed in $\mathscr{A}$, we can see $\mathscr{A}/\ker(\Phi)$ as a normed algebra under the quotient norm. Then, translating the norm of $\mathscr{A}/\ker(\Phi)$ to $\mathscr{B}$ by means of the natural algebra isomorphism $\mathscr{A}/\ker(\Phi) \simeq \mathscr{B}$, and considering that $\mathscr{B}$ has minimum norm topology, the continuity of Φ follows. □

Now, combining Lemma 3.3 with [8, Lemma 4.4.21(i)], we obtain the following.

PROPOSITION 3.4. *Let $\mathscr{A}$ be a Jordan-admissible normed $\mathscr{Q}$-algebra over $\mathbb{K}$, let $\mathscr{B}$ be normed J-semisimple Jordan-admissible algebra over $\mathbb{K}$ having minimum norm topology, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a surjective algebra homomorphism. Then Φ is continuous.*

We refer the reader to [8, Definition 2.4.9, paragraph immediately before Lemma 4.4.21, and Definition 4.4.12] for the meaning of a Jordan-admissible algebra, of a Jordan-admissible normed $\mathscr{Q}$-algebra, and of a Jordan-admissible J-semisimple algebra, respectively. We only note that the class of all Jordan-admissible algebras contains both all associative algebras and all Jordan algebras, that an associative algebra is J-semisimple if and only if it is semisimple in the classical sense of Jacobson (see [8, Definition 3.6.12]), that complete normed Jordan-admissible algebras are normed $\mathscr{Q}$-algebras [8, Fact 4.4.15], and that the converse is not true. Thus, for example, every (possibly non-closed) ideal of a Jordan-admissible complete normed algebra is a normed $\mathscr{Q}$-algebra.

Asociative normed $\mathscr{Q}$-algebras have become a classical topic in the theory of associative normed algebras, whose study goes back to Kaplansky [20] (see [8, §3.6.61] for additional information). Without enjoying their name, Jordan normed $\mathscr{Q}$-algebras first appeared in Viola Devapakkiam's paper [29].

Since standard operator algebras are semisimple, it follows from Proposition 3.4, the above comments, and the Dales-Eidelheit-Mazur result reviewed at the beginning of this section, that *surjective algebra homomorphisms from associative normed $\mathscr{Q}$-algebras to standard operator algebras are continuous.*

To obtain the Jordan version of the above result, we need some auxiliary results.

Let X be a Banach space over $\mathbb{K}$ and, for $F \in \mathscr{L}(X)$, let us denote by $F' \in \mathscr{L}(X')$ the transpose of F and let (x, f) be in $X \otimes X'$. Then it is clear that $F \circ (x \otimes f) = F(x) \otimes f$ and that $(x \otimes f) \circ F = x \otimes F'(f)$ for every $F \in \mathscr{L}(X)$, so that in particular we have

$$(x_1 \otimes f_1) \circ (x_2 \otimes f_2) = f_1(x_2)(x_1 \otimes f_2) \text{ for all } x_1, x_2 \in X \text{ and } f_1, f_2 \in X'.$$

We also recall that, given an element a in a (possibly non-associative and non-unital) ring $\mathscr{A}$, the operator U_a on $\mathscr{A}$ is defined by

$$U_a(b) = a(ab + ba) - a^2 b \text{ for all } b \in \mathscr{A}.$$

Clearly, for any ideal I of $\mathscr{A}$, we have that $U_a(b) \in I$ whenever either $a \in I$ or $b \in I$. On the other hand, if $\mathscr{A}$ is a *Jordan subring* of an associative algebra $\mathscr{B}$ over $\mathbb{K}$ (i.e. an additive subgroup of $\mathscr{B}$ such that $a_1 \bullet a_2 := \frac{1}{2}(a_1 a_2 + a_2 a_1)$ lies in $\mathscr{A}$ whenever a_1 and a_2 are in $\mathscr{A}$), and if we consider $\mathscr{A}$ as a Jordan ring under the product $\bullet$, then, for each $a \in \mathscr{A}$, the operator U_a on the ring $(\mathscr{A}, \bullet)$ satisfies that $U_a(b) = aba$ for every $b \in \mathscr{A}$, where juxtaposition means the associative product of $\mathscr{B}$. Indeed, in this case we have

$$\begin{aligned} U_a(b) &= a \bullet (a \bullet b + b \bullet a) - (a \bullet a) \bullet b \\ &= \frac{1}{2}[a(ab+ba) + (ab+ba)a] - \frac{1}{2}[a^2 b + ba^2] = aba. \end{aligned}$$

LEMMA 3.5. *Let $X \neq 0$ be a Banach space over $\mathbb{K}$, and let $\mathscr{A}$ be a standard operator Jordan ring on X. Then $\mathscr{F}(X)$ is the smallest nonzero (ring) ideal of $\mathscr{A}$.*

Proof. Let I be a nonzero ring ideal of $\mathscr{A}$. We want to show that $\mathscr{F}(X) \subseteq I$. Take $F \in I$ and $x_0 \in X$ such that $F(x_0) \neq 0$, and let $f_0 \in X'$ with $f_0(F(x_0)) = 1$. Let Γ denote the set of all nonzero elements of $\mathbb{K}$ having a square root in $\mathbb{K}$. Then for $\rho \in \Gamma$ we have

$$\begin{aligned} 0 \neq x_0 \otimes f_0 &= (x_0 \otimes f_0) \circ F \circ (x_0 \otimes f_0) = \rho\big((\sqrt{\rho}^{-1} x_0) \otimes f_0\big) \circ F \circ \big((\sqrt{\rho}^{-1} x_0) \otimes f_0\big) \\ &= \rho U_{(\sqrt{\rho}^{-1} x_0) \otimes f_0}(F) \in \rho I. \end{aligned}$$

Therefore the set $J := \bigcap_{\rho \in \Gamma} \rho I$ is a nonzero ring ideal of $\mathscr{A}$ and a vector subspace of $\mathscr{L}(X)$, and is contained in I. Thus, to prove that $\mathscr{F}(X) \subseteq I$, there is no loss of generality if we suppose that I itself is a vector subspace of $\mathscr{L}(X)$. Then for every $(x, f) \in X \times X'$ we have

$$f_0(x) f(x_0)(x \otimes f) = (x \otimes f) \circ (x_0 \otimes f_0) \circ (x \otimes f) = U_{x \otimes f}(x_0 \otimes f_0) \in I,$$

hence $x \otimes f \in I$ whenever $f_0(x) \neq 0$ and $f(x_0) \neq 0$, and in particular $F(x_0) \otimes F'(f_0) \in I$.

Given $x \in X$, take $\lambda \in \mathbb{R}$ such that $f_0\big(x + \lambda F(x_0)\big) \neq 0$, and note that

$$x \otimes F'(f_0) = \big(x + \lambda F(x_0)\big) \otimes (f_0 \circ F) - \lambda\big(F(x_0) \otimes F'(f_0)\big) \in I.$$

Analogously, given $f \in X'$, take $\mu \in \mathbb{R}$ such that $\big(f + \mu F'(f_0)\big)(x_0) \neq 0$, and note that

$$F(x_0) \otimes f = F(x_0) \otimes \big(f + \mu F'(f_0)\big) - \mu\big(F(x_0) \otimes F'(f_0)\big) \in I.$$

As a consequence, given $(x, f) \in X \times X'$, taking $\lambda, \mu \in \mathbb{R}$ such that

$$f_0\big(x + \lambda F(x_0)\big) \neq 0 \ \text{ and } \ \big(f + \mu(f_0 \circ F)\big)(x_0) \neq 0,$$

we have

$$\begin{aligned} x \otimes f =& \big(x + \lambda F(x_0)\big) \otimes \big(f + \mu(f_0 \circ F)\big) - \lambda(F(x_0) \otimes f) \\ & - \mu\big(x \otimes (f_0 \circ F)\big) - \lambda\mu\big(F(x_0) \otimes (f_0 \circ F)\big) \in I. \end{aligned}$$

Finally, it is enough to apply [8, Fact 1.4.13] to conclude that $\mathscr{F}(X) \subseteq I$. $\square$

For the proof of the following fact, Definition 4.4.12 of [8] should be considered again.

FACT 3.6. *Standard operator Jordan algebras are J-semisimple.*

Proof. Assume otherwise that there are a Banach space X over $\mathbb{K}$, and a standard operator Jordan algebra $\mathscr{A}$ on X which is not J-semisimple. Then, clearly, we would have $X \neq 0$. Therefore, since J-Rad$(\mathscr{A})$ is an ideal of $\mathscr{A}$, it would follow from Lemma 3.5 that $\mathscr{F}(X)$ would be contained in J-Rad$(\mathscr{A})$. This would imply that $\mathscr{F}(X) = \text{J-Rad}(\mathscr{F}(X))$. But, since $\mathscr{F}(X)$ is an associative algebra, the equality above would read as that $\mathscr{F}(X)$ is a radical algebra in the sense of Jacobson, a fact which is obviously non true. □

Now, combining Theorem 3.2, Proposition 3.4, and Fact 3.6, we obtain the following.

THEOREM 3.7. *Let X be a Banach space over $\mathbb{K}$, let $\mathscr{A}$ be a Jordan normed $\mathscr{Q}$-algebra over $\mathbb{K}$, let $\mathscr{B}$ be a standard operator Jordan algebra on X, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a surjective algebra homomorphism. Then Φ is continuous.*

REMARK 3.8. A normed algebra $\mathscr{A}$ is said to have *minimality of norm topology* if every continuous algebra norm on $\mathscr{A}$ is equivalent to the natural norm. Clearly, minimum norm topology implies minimality of norm topology. According to [8, Theorem 4.4.23], *surjective algebra homomorphisms, from complex Jordan-admissible normed $\mathscr{Q}$-algebras to complete normed J-semisimple Jordan-admissible complex algebras having minimality of norm topology, are continuous.* This variant of Proposition 3.4 (whose associative forerunner can be found in [23]) is much deeper, as its proof involves arguments of Aupetit in [1], which were revolutionary at their time.

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