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A NORM INEQUALITY FOR SOME SPECIAL FUNCTIONS

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Abstract. Let A,B be invertible positive operators on a complex separable Hilbert space $\mathscr{H}$ and X be an operator on $\mathscr{H}$ associated with a norm ideal corresponding to a unitarily invariant norm $|||\cdot|||$. We shall prove that

$$|||\Gamma(A)X - X\Gamma(B)||| \leqslant c(m,M)|||AX - XB|||$$

for all unitarily invariant norms $|||\cdot|||$, where $c(m,M)$ is a function of $m = min\{||A||,||B||\}$ and $M = \max\{||A||,||B||\}$, and Γ denotes the Gamma function. Further if f is a Bernstein function, we shall prove that

$$|||f(A)X - Xf(B)||| \leqslant f'(m)|||AX - XB|||.$$

This inequality supplements and unify all the results proved by a number of authors for operator monotone functions.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex separable Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called self-adjoint if $A^* = A$. A self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax,x\rangle \geqslant 0$ for all $x \in \mathscr{H}$ and is called strictly positive if $\langle Ax,x\rangle > 0$ for all nonzero $x \in \mathscr{H}$. The set of all self-adjoint operators in $\mathbb{B}(\mathscr{H})$ is denoted by $\mathbb{B}(\mathscr{H})_s$, the set of all positive operators shall be denoted by $\mathbb{B}(\mathscr{H})_+$ and the set of all strictly positive operators shall be denoted by $\mathbb{B}(\mathscr{H})_+^+$. For $A,B \in \mathbb{B}(\mathscr{H})_s, A \geqslant B$ $(A > B)$ means $A-B$ is positive (strictly positive). A norm $|||\cdot|||$ on $\mathbb{B}(\mathscr{H})$ is called unitarily invariant or symmetric if

$$|||UAV||| = |||A|||$$

for all $A \in \mathbb{B}(\mathscr{H})$ and for all unitary operators $U,V \in \mathbb{B}(\mathscr{H})$. The most basic unitarily invariant norms are the Ky-Fan norms and Schatten p-norms defined respectively as

$$||A||_{(k)} = \sum_{j=1}^{k} \sigma_j(A) \qquad (k = 1,2,\ldots),$$

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and

$$||A||_p = \left(\sum_{j=1}^{\infty}(\sigma_j(A))^p\right)^{1/p} \qquad (1 \leqslant p < \infty),$$

where $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \dots$ are the singular values of A. We shall consider a norm ideal $(\mathscr{I}, |||\cdot|||)$ of $\mathbb{B}(\mathscr{H})$ with respect to a unitarily invariant norm $|||\cdot|||$. For convenience we shall write $(\mathscr{I}, |||\cdot|||)$ as $\mathscr{I}$. By I we mean identity operator in $\mathbb{B}(\mathscr{H})$. For $A, B \in \mathbb{B}(\mathscr{H})$, we shall denote by $m = min\{||A||, ||B||\}$ and by $M = \max\{||A||, ||B||\}$ throughout. Here $||\cdot||$ denotes the operator norm on $\mathbb{B}(\mathscr{H})$.

A nonnegative and infinitely differentiable function f on $(0, \infty)$ is called completely monotone if $(-1)^k f^{(k)}(x) \geqslant 0$ and is called Bernstein function if $(-1)^{k-1} f^{(k)}(x) \geqslant 0$ for all $x \in (0, \infty)$, $k = 1, 2, \cdots$, see [11, 13]. Here $f^{(k)}$, $k = 1, 2, \cdots$ denotes the kth derivative of f. One should note that a completely monotone function is decreasing and convex whereas a Bernstein function is increasing and concave.

Every $A \in \mathbb{B}(\mathscr{H})_s$ admits spectral decomposition

$$A = \int \lambda dE_\lambda$$

where E_λ is a spectral measure. Let f be a real valued function defined on an interval J and let $A \in \mathbb{B}(\mathscr{H})_s$ has its spectrum in J. Then $f(A)$ is defined by

$$f(A) = \int f(\lambda) dE_\lambda \ .$$

The function f is called operator monotone if $A \geqslant B$ implies $f(A) \geqslant f(B)$ for $A, B \in \mathbb{B}(\mathscr{H})_s$ with spectrum in J.

When Hilbert space $\mathscr{H}$ is finite dimensional, van Hemmen and Ando [8] proved that if $A, B \in \mathbb{B}(\mathscr{H})_+$ are such that $A + B \geqslant cI$ for some $c > 0$ and f is a nonnegative operator monotone function on $[0, \infty)$, then

$$|||f(A) - f(B)||| \leqslant \left(\frac{f(c/2) - f(0)}{c/2}\right) |||A - B|||$$

for all unitarily invariant norms $|||\cdot|||$. Kittaneh and Kosaki [9] generalized this result to its commutator version by proving that, if $A, B \in \mathbb{B}(\mathscr{H})_+$ are such that $A \geqslant aI$, $B \geqslant bI$ for some $a, b > 0$ and $X \in \mathbb{B}(\mathscr{H})$, then for every nonnegative operator monotone function f on $(0, \infty)$

$$||f(A)X - Xf(B)||_p \leqslant c(a,b)||AX - XB||_p,$$

where $c(a,b) = \frac{f(a)-f(b)}{a-b}$ if $a \neq b$ and $c(a,b) = f'(a)$ if $a = b$. Bhatia [6] proved the above inequality for all unitarily invariant norms when $X = I$ and $b = a$, using Fréchet differential calculus. If the function f is completely monotone on $(0, \infty)$, it is proved in [4] that

$$|||f(A)X - Xf(B)||| \leqslant |f'(m)| \ |||AX - XB|||$$

for all operators $A,B \in \mathbb{B}(\mathscr{H})_+^+$, $X \in \mathbb{B}(\mathscr{H})$ and for all unitarily invariant norms $|||\cdot|||$. In [7], it is proved that if f is nonegative operator monotone function on $(0,\infty)$, then

$$|||f(A)X - Xf(B)||| \leqslant \max\{||f'(A)||, ||f'(B)||\}|||AX - XB|||.$$

Similar type of inequalities for the functions e^x and x^α can also be found in [12]. Our aim in this article is to prove that for $A,B \in \mathbb{B}(\mathscr{H})_+^+$ and $X \in \mathscr{I}$,

$$|||\Gamma(A)X - X\Gamma(B)||| \leqslant c(m,M)|||AX - XB|||$$

for all unitarily invariant norms $|||\cdot|||$, where Γ denotes the Gamma function and $c(m,M)$ is a constant depending upon A,B. Further for a Bernstein function f, we shall prove that

$$|||f(A)X - Xf(B)||| \leqslant f'(m)|||AX - XB|||.$$

At the end, as a remark, we shall demonstrate, how the above inequality includes a number of inequalities proved by several authors.

2. Main results

We begin this section by proving a norm inequality for the Gamma function. For this we need the following proposition.

PROPOSITION 2.1. *Let $A,B \in \mathbb{B}(\mathscr{H})_s$ and $X \in \mathscr{I}$. Then*

$$|||a^AX - Xa^B||| \leqslant |\log a| \ \max\{||a^A||, ||a^B||\} \ |||AX - XB|||$$

where $a > 0$, for all unitarily invariant norms $|||\cdot|||$.

Proof. Let $A,B \in \mathbb{B}(\mathscr{H})_s$ and $X \in \mathscr{I}$, then we have [1, 10]

$$|||e^AX - Xe^B||| \leqslant \frac{1}{2}|||e^A(AX - XB) + (AX - XB)e^B|||.$$

On replacing A with $\log a^A$ and B with $\log a^B$ in the above inequality, we get

$$|||e^{\log a^A}X - Xe^{\log a^B}||| \leqslant \frac{1}{2}|||e^{\log a^A}(\log a^AX - X\log a^B) + (\log a^AX - X\log a^B)e^{\log a^B}|||,$$

i.e.,

$$|||a^AX - Xa^B||| \leqslant \frac{1}{2}|||\log a(a^A(AX - XB) + (AX - XB)a^B)|||.$$

Consequently,

$$\begin{aligned}
|||a^AX - Xa^B||| &\leqslant \frac{1}{2}|\log a| \ (|||a^A(AX - XB)||| + |||(AX - XB)a^B|||) \\
&\leqslant \frac{|\log a|}{2}(||a^A|| \ |||AX - XB||| + |||AX - XB||| \ ||a^B||) \\
&\leqslant \frac{|\log a|}{2}2(\max\{||a^A||, ||a^B||\}) \ |||AX - XB||| \\
&= |\log a| \ \max\{||a^A||, ||a^B||\} \ |||AX - XB|||.
\end{aligned}$$

The first inequality in the above inequalities follows from the triangle inequality for norms and the second follows from the well known inequality

$$|||ABC||| \leqslant ||A||\ |||B|||\ ||C||$$

for all $A,B,C \in \mathbb{B}(\mathscr{H})$. This completes the proof of the proposition. □

THEOREM 2.2. *Let* $A,B \in \mathbb{B}(\mathscr{H})_+^+$ *and* $X \in \mathscr{I}$. *Then*

$$|||\Gamma(A)X - X\Gamma(B)||| \leqslant c(m,M)|||AX - XB||| \tag{2.1}$$

for all unitarily invariant norms $|||\cdot|||$, *where*

$$c(m,M) = \int_1^\infty \log t\ e^{-t}t^{M-1}dt - \int_0^1 \log t\ e^{-t}t^{m-1}dt.$$

Proof. For inequality (2.1) to be valid, $c(m,M)$ must be finite. We claim that the integrals

$$\int_1^\infty \log t\ e^{-t}t^{M-1}dt \quad \text{and} \quad \int_0^1 (-\log t)e^{-t}t^{m-1}dt$$

are finite. The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt,\ \ x > 0,\ \ t > 0.$$

For the first integral in $c(m,M)$, note that $\log t < t$ for all $t > 0$. Therefore

$$\log t\ e^{-t}t^{M-1} \leqslant te^{-t}t^{M-1}$$

for $1 < t < \infty$. Therefore

$$0 \leqslant \int_1^\infty \log t\ e^{-t}t^{M-1}dt \leqslant \int_1^\infty t\ e^{-t}t^{M-1}dt. \tag{2.2}$$

But,

$$\begin{aligned} \int_1^\infty t\ e^{-t}t^{M-1}dt &= \int_1^\infty e^{-t}t^M dt \\ &\leqslant \int_0^\infty e^{-t}t^M dt = \Gamma(M+1). \end{aligned}$$

So inequality (2.2) implies that

$$\int_1^\infty \log t\ e^{-t}t^{M-1}dt$$

is finite. For the second integral, we have $e^{-t} < 1$ for $t > 0$. Therefore

$$(-\log t)e^{-t}t^{m-1} \leqslant (-\log t)t^{m-1},$$

for $0 < t < 1$. Then

$$0 \leqslant \int_0^1 (-\log t) e^{-t} t^{m-1} dt \leqslant \int_0^1 (-\log t) t^{m-1} dt. \tag{2.3}$$

Integrating the integral

$$\int_0^1 (-\log t) t^{m-1} dt$$

by parts, it turns out to be equal to $\frac{1}{m^2}$. So from inequality (2.3) we conclude that

$$\int_0^1 (-\log t) e^{-t} t^{m-1} dt$$

is finite. This establishes our claim and hence $c(m,M)$ is finite. Now we proceed to prove inequality (2.1). Note that

$$\Gamma(x) = \int_0^\infty \frac{e^{-t}}{t} f_t(x) dt$$

where $f_t(x) = t^x$. We shall first prove the inequality (2.1) for all functions f_t, $t > 0$. By Proposition 2.1 with $a = t$, we have

$$\begin{aligned}|||t^A X - X t^B||| &\leqslant |\log t| \ \max\{||t^A||, ||t^B||\} \ |||AX - XB||| \\ &= |\log t| \ |||AX - XB||| \begin{cases} t^M, & \text{if } t \geqslant 1 \\ t^m, & \text{if } 0 < t < 1 \end{cases}.\end{aligned}$$

Therefore,

$$\begin{aligned}|||\Gamma(A)X - X\Gamma(B)||| &= \left|\left|\left| \int_0^\infty \frac{e^{-t}}{t} f_t(A) dt \ X - X \int_0^\infty \frac{e^{-t}}{t} f_t(B) dt \right|\right|\right| \\ &= \left|\left|\left| \int_0^\infty \frac{e^{-t}}{t} (f_t(A)X - X f_t(B)) dt \right|\right|\right| \\ &\leqslant \int_0^\infty \left|\left|\left| \frac{e^{-t}}{t} (f_t(A)X - X f_t(B)) \right|\right|\right| dt \\ &= \int_0^\infty \frac{e^{-t}}{t} \ |||f_t(A)X - X f_t(B)||| dt \\ &\leqslant |||AX - XB||| \ \left(\int_0^1 \frac{e^{-t}}{t} |\log t| \ t^m dt + \int_1^\infty \frac{e^{-t}}{t} |\log t| \ t^M dt \right) \\ &= |||AX - XB||| \ \left(\int_0^1 |\log t| \ e^{-t} t^{m-1} dt + \int_1^\infty \log t \ e^{-t} t^{M-1} dt \right) \\ &= |||AX - XB||| \ \left(\int_1^\infty \log t \ e^{-t} t^{M-1} dt - \int_0^1 \log t \ e^{-t} t^{m-1} dt \right) \\ &= c(m,M) |||AX - XB|||.\end{aligned}$$

This completes the proof. $\square$

COROLLARY 2.3. *Let $A,B \in \mathbb{B}(\mathscr{H})^{+}_{+} \cap \mathscr{I}$. Then*

$$|||\Gamma(A)\Gamma(B) - \Gamma(B)\Gamma(A)||| \leqslant c^2(m,M)|||AB - BA|||$$

for all unitarily invariant norms $|||\cdot|||$.

Proof. Taking $B = A$ and $X = \Gamma(B)$ in Theorem 2.2., we obtain

$$\begin{aligned} |||\Gamma(A)\Gamma(B) - \Gamma(B)\Gamma(A)||| &\leqslant c(m,M)|||A\Gamma(B) - \Gamma(B)A||| \\ &= c(m,M)|||\Gamma(B)A - A\Gamma(B)||| \\ &\leqslant c^2(m,M)|||BA - AB||| \\ &= c^2(m,M)|||AB - BA||| \end{aligned}$$

where the last inequality is obtained by taking $A = B$ and $X = A$ in Theorem 2.2. This completes the proof of the corollary. □

Next we state and outline a proof for a similar norm inequality for the Bernstein function.

THEOREM 2.4. *Let f be a Bernstein function. Then*

$$|||f(A)X - Xf(B)||| \leqslant f'(m)|||AX - XB|||$$

for all $A,B \in \mathbb{B}(\mathscr{H})^{+}_{+}, X \in \mathscr{I}$ and all unitarily invariant norms $|||\cdot|||$.

Proof. It is known that the Bernstein function admits the integral representation

$$f(x) = \alpha + \beta x + \int_0^\infty (1 - e^{-tx})d\mu(t), \qquad x,\ t > 0,$$

where μ is a positive measure on $(0,\infty)$ and $\alpha,\beta \geqslant 0$ (see [11]). Therefore

$$\begin{aligned} |||f(A)X - Xf(B)||| &= \left|\left|\left|\beta(AX - XB) + \int_0^\infty (Xe^{-tB} - e^{-tA}X)d\mu(t)\right|\right|\right| \\ &\leqslant \beta|||(AX - XB)||| + \int_0^\infty |||e^{-tA}X - Xe^{-tB}|||d\mu(t). \end{aligned} \tag{2.4}$$

It follows by taking $a = e^{-t}$ in Proposition 2.1 that

$$\begin{aligned} |||e^{-tA}X - Xe^{-tB}||| &\leqslant |\log e^{-t}|\ \max\{||e^{-tA}||, ||e^{-tB}||\}|||AX - XB||| \\ &= |-t|\ \max\{||e^{-tA}||, ||e^{-tB}||\}|||AX - XB||| \\ &= \max\{||t\ e^{-tA}||, ||t\ e^{-tB}||\}|||AX - XB||| \\ &= te^{-mt}|||AX - XB|||. \end{aligned} \tag{2.5}$$

Using (2.5) in (2.4) we get the desired inequality. □

We state the following corollary the proof for which is similar to the proof of Corollary 2.3.

COROLLARY 2.5. *Let f be a Bernstein function and $A,B \in \mathbb{B}(\mathscr{H})_+^+ \cap \mathscr{I}$. Then*

$$|||f(A)f(B) - f(B)f(A)||| \leqslant (f'(m))^2|||AB - BA|||$$

for all unitarily invariant norms $|||\cdot|||$.

One may observe that a function may not be Bernstein function but the inverse (if exists) of the function may be a Bernstein function. Examples of such functions include $x^r, r \geqslant 1$, $e^x - 1$.

THEOREM 2.6. *Let a function $f : [0,\infty) \to (0,\infty)$ be such that its inverse function f^{-1} (if exists) is Bernstein function. Then for all $A,B \in \mathbb{B}(\mathscr{H})_+^+$ and $X \in \mathscr{I}$,*

$$f'(m)|||AX - XB||| \leqslant |||f(A)X - Xf(B)|||$$

for all unitarily invariant norms $|||\cdot|||$.

Proof. Since f^{-1} is Bernstein function, we have from Theorem 2.4,

$$|||f^{-1}(A)X - Xf^{-1}(B)||| \leqslant (f^{-1})'(m)|||AX - XB|||.$$

On replacing A by $f(A)$ and B by $f(B)$, one gets

$$|||AX - XB||| \leqslant (f^{-1})'(f(m))|||f(A)X - Xf(B)|||.$$

That $(f^{-1})'(f(m))$ is equal to $(f'(m))^{-1}$ follows from $(f^{-1} \circ f)(x) = x$ for all x. □

COROLLARY 2.7. *Let $A,B \in \mathbb{B}(\mathscr{H})_+^+$ and $X \in \mathscr{I}$. Then*

$$rm^{r-1}|||AX - XB||| \leqslant |||A^rX - XB^r|||, \quad r \geqslant 1$$

for all unitarily invariant norms $|||\cdot|||$.

Proof. The inverse function $x^{\frac{1}{r}}$ of $x^r, r \geqslant 1$ is operator monotone and hence is Bernstein function. Therefore, Theorem 2.6 gives the desired inequality. □

In case when $\mathscr{H}$ is finite dimensional, a weaker version of the following corollary $(m = 0)$ is proved in [12].

COROLLARY 2.8. *Let $A,B \in \mathbb{B}(\mathscr{H})_+^+$ and $X \in \mathscr{I}$. Then*

$$e^m|||AX - XB||| \leqslant |||e^AX - Xe^B|||$$

for all unitarily invariant norms $|||\cdot|||$.

Proof. The inverse function $\log(x+1)$ of e^x-1 is operator monotone and hence is a Bernstein function. Therefore, Theorem 2.6 gives the desired inequality. □

REMARK. From the definition of completely monotone function and the Bernstein function given in Section 1, it follows that a function f is a Bernstein function if its derivative is completely monotone. If f is nonnegative operator monotone function on $(0,\infty)$ then f admits the integral representation

$$f(x)=\alpha+\beta x+\int_0^\infty\left(\frac{t}{t^2+1}-\frac{1}{x+t}\right)d\mu(t),$$

where α is a real number, $\beta\geqslant 0$ and μ is a positive measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{1}{t^2+1}d\mu(t)<\infty$$

(see [5]). From this integral representation, it follows that the derivative of an operator monotone function is completely monotone and hence it is a Bernstein function. Consequently, we see that Theorem 2.4. supplements and unifies the results proved by van Hemmen and Ando [8], Kittaneh and Kosaki [9], Bhatia [6] and A. G. Ghazanfari [7]. It is further remarked that the inequalities for log-convex functions are also studied in [2, 3].

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