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REFINING SOME INEQUALITIES ON 2×2 BLOCK ACCRETIVE MATRICES

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Abstract. We obtain some matrix inequalities on the off-diagonal blocks of 2×2 block accretive matrices and the geometric mean of its diagonal blocks. They improve some results of Liu et al. [Operators and Matrices, **15**, 2(2021), 581–587] and refine an inequality of Yang et al. [Journal of Inequalities and Applications (2020) 2020:90].

1. Introduction

Let M_n be the space of $n\times n$ complex matrices with the identity matrix I. For $X\in M_n$, $X\geqslant(>)0$ means X is a positive semidefinite (definite) matrix. For any $X\in M_n$, the singular values $s_j(X)$, which are the eigenvalues of $|X|=(X^*X)^{\frac{1}{2}}$, are arranged in nonincreasing order as $s_1(X)\geqslant s_2(X)\geqslant\cdots\geqslant s_n(X)$. For $X,Y\in M_n$, if

$$\prod_{i=1}^{k}s_i(X)\leqslant\prod_{i=1}^{k}s_i(Y)$$

for all $k=1,2,\cdots,n$, then we say the singular values of X are weakly log majorized by the singular values of Y and we write $S(X)\prec_{wlog}S(Y)$. More information on majorization can be found in [9]. When $X,Y\in M_n$ with $X,Y>0$, the geometric mean $X\sharp Y$ is defined by

$$X\sharp Y=X^{\frac{1}{2}}(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}X^{\frac{1}{2}}.$$

Note this definition could be extended to positive semidefinite matrices X,Y by a limit:

$$X\sharp Y=\lim_{\varepsilon\downarrow 0}(X+\varepsilon I)\sharp(Y+\varepsilon I).$$

More information on the geometric mean can be found in [1, Chapter 4].

A matrix $X\in M_n$ is called accretive if $\mathscr{R}X=\frac{X+X^*}{2}\geqslant 0$. Recently Drury [10] defined the geometric mean for two accretive matrices $X,Y\in M_n$ by

$$X\sharp Y:=\left(\frac{2}{\pi}\int_0^{\infty}(vX+v^{-1}Y)^{-1}\frac{dv}{v}\right)^{-1}. \tag{1}$$

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A weighted version of (1) was given by Raissouli et al. in [11]. It is noted that if X,Y are accretive, then so is $X\sharp Y$.

Let $X=\begin{pmatrix} A & W \\ Y^* & B \end{pmatrix}\in M_2(M_n)$, and $X^\tau=\begin{pmatrix} A & Y^* \\ W & B \end{pmatrix}$. We say X is PPT (i.e., positive partial transpose) if both X and X^τ are positive semidefinite. In [12], the authors introduced the notion of APT (i.e., accretive partial transpose). We say X is APT if both X and X^τ are accretive.

Clearly, the class of APT matrices includes the class of PPT.

Moreover, in [12], Liu et al. proved the following results.

THEOREM 1. *Let* $H=\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}\in M_2(M_n)$ *be APT. Then*

$$S\left(\frac{X+Y}{2}\right)\prec_{wlog} S(A\sharp B). \tag{2}$$

and

$$|X+Y|\leqslant \mathscr{R}(A\sharp B+U(A\sharp B)U), \tag{3}$$

for some unitary matrix $U\in M_n$.

Let Hua matrix be given by

$$\begin{pmatrix} (I-X^*X)^{-1} & (I-Y^*X)^{-1} \\ (I-X^*Y)^{-1} & (I-Y^*Y)^{-1} \end{pmatrix}, \tag{4}$$

where $X,Y\in M_{m\times n}$ are strictly contractive, i.e, $\|X\|,\|Y\|<1$. Lin and Wolkowicz in [6] proved the following inequality for every unitarily invariant norm $\|\cdot\|_u$.

$$2\|(I-X^*Y)^{-1}\|_u\leqslant \|(I-X^*X)^{-1}+(I-Y^*Y)^{-1}\|_u.$$

Since Hua matrix is PPT, Yang et.al. in [7] gave a generalization of the above inequality.

$$\|(I-X^*Y)^{-1}\|_u\leqslant \|(I-X^*X)^{-1}\sharp(I-Y^*Y)^{-1}\|_u. \tag{5}$$

Moreover, Liu et al. in [12] also proved the following inequalities related to $X\sharp X^*$

$$X\sharp X^*\geqslant \mathscr{R}X; \tag{6}$$

and

$$\|X\sharp X^*\|_u\geqslant \|X\|_u, \tag{7}$$

where $X\in M_n$ is accretive.

In this paper, using a result in [5], we present some new related inequalities, which are refinements of (2), (3), (5) and (7) respectively.

2. Main results

Before we give our first main results, we need the following lemmas.

LEMMA 1. [5] *If $X, Y \in M_n$ with $X, Y \geqslant 0$, then*

$$\prod_{j=1}^{k} s_j^2(X \sharp Y) \leqslant \prod_{j=1}^{k} s_j(X) s_j(Y), \quad k = 1, \cdots, n. \tag{8}$$

LEMMA 2. [5] *Let $\begin{pmatrix} A & Y \\ Y^* & B \end{pmatrix} \in M_2(M_n)$ be PPT and let $Y = U|Y|$ be the polar decomposition of Y. Then*

$$|Y| \leqslant (A \sharp B) \sharp (U^*(A \sharp B) U), \tag{9}$$

and

$$|Y^*| \leqslant (A \sharp B) \sharp (U^*(A \sharp B) U). \tag{10}$$

LEMMA 3. [1, 2] *Let A, B, C, D be positive semidefinite. Then*

(i) $A \sharp B = B \sharp A$;

(ii) $A \leqslant C$ *and* $B \leqslant D \Rightarrow A \sharp B \leqslant C \sharp D$;

(iii) $A \sharp B \leqslant \frac{A+B}{2}$;

(iv) $A \sharp B = A^{\frac{1}{2}} V B^{\frac{1}{2}}$ *for some unitary V.*

The following lemma about geometric mean can been found in [8].

LEMMA 4. [8] *Let $A, B \in M_n$ be accretive. Then*

$$(\mathscr{R}A) \sharp (\mathscr{R}B) \leqslant \mathscr{R}(A \sharp B). \tag{11}$$

LEMMA 5. [9, p. 63] *If $H \in M_n$, then*

$$s_j(\mathscr{R}H) \leqslant s_j(H), \quad j = 1, \cdots, n. \tag{12}$$

LEMMA 6. (Weyl's Monotonicity Theorem) [2, p. 63] *If $A, B \in M_n$ with $0 \leqslant A \leqslant B$, then*

$$s_j(A) \leqslant s_j(B), \quad j = 1, \cdots, n. \tag{13}$$

LEMMA 7. (Fan Dominance Theorem) [2, p. 93] *If $A, B \in M_n$ with*

$$\|A\|_{(k)} \leqslant \|B\|_{(k)} \quad \text{for } k = 1, 2, \cdots, n,$$

then

$$\|A\|_u \leqslant \|B\|_u, \tag{14}$$

where $\|A\|_{(k)} = \sum_{i=1}^{k} s_i(A), \ 1 \leqslant k \leqslant n$.

Now, we are in the position to state our first main result which is recited in the following.

THEOREM 2. *Let* $T = \begin{pmatrix} A & X \\ Y^* & B \end{pmatrix} \in M_2(M_n)$ *be APT and let* $X+Y = U|X+Y|$ *be the polar decomposition of* $X+Y$. *Then*

$$\left|\frac{X+Y}{2}\right| \leqslant \mathscr{R}(A\sharp B)\sharp\mathscr{R}(U^*(A\sharp B)U) \leqslant \frac{\mathscr{R}(A\sharp B + U^*(A\sharp B)U)}{2}. \tag{15}$$

and

$$\left|\frac{X^*+Y^*}{2}\right| \leqslant \mathscr{R}(A\sharp B)\sharp\mathscr{R}(U^*(A\sharp B)U) \leqslant \frac{\mathscr{R}(A\sharp B + U^*(A\sharp B)U)}{2}. \tag{16}$$

Proof. Since

$$T = \begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$$

is APT, we have that

$$\mathscr{R}T = \begin{pmatrix} \mathscr{R}A & (X+Y)/2 \\ (X^*+Y^*)/2 & \mathscr{R}B \end{pmatrix}$$

is PPT. From Lemma 2, we get

$$\begin{aligned}
\left|\frac{X+Y}{2}\right| &\leqslant (\mathscr{R}A\sharp\mathscr{R}B)\sharp(U^*(\mathscr{R}A\sharp\mathscr{R}B)U) \quad \text{(by (9))} \\
&\leqslant \mathscr{R}(A\sharp B)\sharp\mathscr{R}(U^*(A\sharp B)U) \quad \text{(by Lemma 4 and Lemma 3 (ii))} \\
&\leqslant \frac{\mathscr{R}(A\sharp B) + \mathscr{R}(U^*(A\sharp B)U)}{2} \quad \text{(by Lemma 3 (iii))} \\
&= \frac{\mathscr{R}(A\sharp B + U^*(A\sharp B)U)}{2}.
\end{aligned}$$

We also have

$$\left|\frac{X^*+Y^*}{2}\right| \leqslant (\mathscr{R}A\sharp\mathscr{R}B)\sharp(U^*(\mathscr{R}A\sharp\mathscr{R}B)U) \quad \text{(by (10))} \leqslant \frac{\mathscr{R}(A\sharp B + U^*(A\sharp B)U)}{2}. \quad \square$$

REMARK 1. It is clear that (15) is a refinement of (3).

The following result follows from Theorem 2 and Lemma 5.

COROLLARY 1. *Let* $T=\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}\in M_2(M_n)$ *be APT and let* $X+Y=U|X+Y|$ *be the polar decomposition of* $X+Y$. *Then for* $i=1,2,\cdots,n$

$$\begin{aligned} s_i\left(\frac{X+Y}{2}\right) &\leqslant s_i\left((\mathscr{R}A\sharp\mathscr{R}B)\sharp(U^*(\mathscr{R}A\sharp\mathscr{R}B)U)\right) \\ &\leqslant s_i\left(\frac{\mathscr{R}(A\sharp B+U^*(A\sharp B)U)}{2}\right)\leqslant s_i\left(\frac{A\sharp B+U^*(A\sharp B)U}{2}\right). \end{aligned} \tag{17}$$

COROLLARY 2. *Let* $T=\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}\in M_2(M_n)$ *be APT and let* $X+Y=U|X+Y|$ *be the polar decomposition of* $X+Y$. *Then*

$$\begin{aligned} S\left(\frac{X+Y}{2}\right) &\prec_{wlog} S\left((\mathscr{R}A\sharp\mathscr{R}B)\sharp(U^*(\mathscr{R}A\sharp\mathscr{R}B)U)\right) \\ &\prec_{wlog} S(A\sharp B). \end{aligned} \tag{18}$$

Proof. Using the first inequality in (17), for $k=1,2,\cdots,n$ we have

$$\begin{aligned} &\prod_{i=1}^{k} s_i\left(\frac{X+Y}{2}\right) \\ &\leqslant \prod_{i=1}^{k} s_i((\mathscr{R}A\sharp\mathscr{R}B)\sharp(U^*(\mathscr{R}A\sharp\mathscr{R}B)U)) \quad \text{(by (17))} \\ &\leqslant \prod_{i=1}^{k} s_i^{\frac{1}{2}}(\mathscr{R}A\sharp\mathscr{R}B)\, s_i^{\frac{1}{2}}(U^*(\mathscr{R}A\sharp\mathscr{R}B)U) \quad \text{(by Lemma 1)} \\ &= \prod_{i=1}^{k} s_i^{\frac{1}{2}}(\mathscr{R}A\sharp\mathscr{R}B)\, s_i^{\frac{1}{2}}(\mathscr{R}A\sharp\mathscr{R}B) \\ &= \prod_{i=1}^{k} s_i(\mathscr{R}A\sharp\mathscr{R}B) \\ &\leqslant \prod_{i=1}^{k} s_i(\mathscr{R}(A\sharp B)) \quad \text{(by Lemma 4)} \\ &\leqslant \prod_{i=1}^{k} s_i(A\sharp B) \quad \text{(by Lemma 5).} \quad \square \end{aligned}$$

REMARK 2. Inequality (18) gives a refinement of (2).

THEOREM 3. *If Hua matrix is as in (4) and let* $(I-A^*B)^{-1}=U|(I-A^*B)^{-1}|$ *be the polar decomposition of* $(I-A^*B)^{-1}$, *then for* $i=1,\cdots,n$

$$\begin{aligned} &s_i\left((I-A^*B)^{-1}\right) \\ &\leqslant s_i\left(\left((I-A^*A)^{-1}\sharp(I-B^*B)^{-1}\right)\sharp\left(U^*(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}U\right)\right) \\ &\leqslant \frac{1}{2}s_i\left(\left((I-A^*A)^{-1}\sharp(I-B^*B)^{-1}\right)+\left(U^*(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}U\right)\right). \end{aligned} \tag{19}$$

Proof. Since Hua matrix is PPT, then by Lemma 2 and Lemma 3 (iii), we have

$$\left|(I-A^*B)^{-1}\right| \leqslant \left((I-A^*A)^{-1}\sharp(I-B^*B)^{-1}\right)\sharp\left(U^*(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}U\right)$$
$$\leqslant \frac{1}{2}\left((I-A^*A)^{-1}\sharp(I-B^*B)^{-1}+U^*(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}U\right),$$

which implies the desired result by Lemma 6. □

From Theorem 3, we obtain the following result for unitarily invariant norm which is a refinement of (5).

COROLLARY 3. *If Hua matrix is as in (4), then for every unitarily invariant norm* $\|\cdot\|_u$

$$\begin{aligned}&\left\|(I-A^*B)^{-1}\right\|_u\\ \leqslant\ &\left\|((I-A^*A)^{-1}\sharp(I-B^*B)^{-1})\sharp(U^*(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}U)\right\|_u\\ \leqslant\ &\frac{1}{2}\left\|\left((I-A^*A)^{-1}\sharp(I-B^*B)^{-1}\right)+\left(U^*(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}U\right)\right\|\\ \leqslant\ &\left\|(I-A^*A)^{-1}\sharp(I-B^*B)^{-1}\right\|_u,\end{aligned} \tag{20}$$

where $U\in M_n$ *is some unitary matrix.*

Proof. The first and the second inequalities follow from (19) and Lemma 7, and the third inequality follows from triangle inequality for the unitarily invariant norm $\|\cdot\|_u$. □

REMARK 3. Inequality (20) presents a refinement of (5).

3. Inequalities related to $X\sharp X^*$

In this section, we give a refinement of (7) and some related inequalities.

THEOREM 4. *If* $X\in M_n$ *is accretive and let* $X=U|X|$ *be the polar decomposition of* X, *then for every unitarily invariant norm* $\|\cdot\|_u$

$$\|X\|_u \leqslant \|(X\sharp X^*)\sharp(U^*(X\sharp X^*)U\|_u \leqslant \|X\sharp X^*\|_u. \tag{21}$$

Proof. Since $M=\begin{pmatrix} X\sharp X^* & X\\ X^* & X\sharp X^*\end{pmatrix}$ is positive semidefinite(see [12]), and $M^\tau = \begin{pmatrix} X\sharp X^* & X^*\\ X & X\sharp X^*\end{pmatrix}=\begin{pmatrix} X^*\sharp X & X^*\\ X & X^*\sharp X\end{pmatrix}$ is also positive semidefinite, M is PPT.
Therefore, by Lemma 2,

$$|X|\leqslant (X\sharp X^*)\sharp(U^*(X\sharp X^*)U),$$

which means

$$\begin{aligned}\|X\|_u &\leqslant \|(X\sharp X^*)\sharp(U^*(X\sharp X^*)U)\|_u\\ &\leqslant \frac{1}{2}\|(X\sharp X^*)+(U^*(X\sharp X^*)U)\|_u\\ &\leqslant \|X\sharp X^*\|_u. \quad \square\end{aligned}$$

REMARK 4. It is clear that (21) is a refinement of (7).

THEOREM 5. *If $X\in M_n$ is accretive and $X-X^*=U|X-X^*|$ is the polar decomposition of $X-X^*$, then*

$$|X-X^*|\leqslant (U^*(A\sharp B)U)\sharp(A\sharp B)\leqslant \frac{U^*(A\sharp B)U+A\sharp B}{2}. \tag{22}$$

where $A=X\sharp X^*+\mathscr{R}X, B=X\sharp X^*-\mathscr{R}X$.

Proof. Since $M=\begin{pmatrix} X\sharp X^* & X\\ X^* & X\sharp X^*\end{pmatrix}\geqslant 0$, we have

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left[\begin{pmatrix} I & I\\ -I & I\end{pmatrix}\right]\begin{pmatrix} X\sharp X^* & X\\ X^* & X\sharp X^*\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I\\ I & I\end{pmatrix}\right]\\ &=\begin{pmatrix} X\sharp X^*+\mathscr{R}X & X-X^*\\ X^*-X & X\sharp X^*-\mathscr{R}X\end{pmatrix}\geqslant 0.\end{aligned}$$

We also have

$$\begin{pmatrix} X\sharp X^*+\mathscr{R}X & X^*-X\\ X-X^* & X\sharp X^*-\mathscr{R}X\end{pmatrix}\geqslant 0.$$

Hence,

$$\begin{pmatrix} X\sharp X^*+\mathscr{R}X & X-X^*\\ X^*-X & X\sharp X^*-\mathscr{R}X\end{pmatrix}$$

is PPT.

By Lemma 2 and Lemma 3 (iii), the result follows. $\square$

COROLLARY 4. *If $X\in M_n$ is accretive and $X-X^*=U|X-X^*|$ is the polar decomposition of $X-X^*$, then*

$$\|X-X^*\|_u\leqslant \|(U^*(A\sharp B)U)\sharp(A\sharp B)\|_u\leqslant \|A\sharp B\|_u. \tag{23}$$

where $A=X\sharp X^*+\mathscr{R}X, B=X\sharp X^*-\mathscr{R}X$.

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