

DILATIONS OF LINEAR MAPS ON VECTOR SPACES

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Abstract. Dilation of linear maps on vector spaces has been recently introduced by Bhat, De, and Rakshit. This notion is a variant of vector space dilation introduced by Han, Larson, Liu, and Liu. We derive vector space versions of Wold decomposition, Halmos dilation, N-dilation, inter-twining lifting theorem and a variant of Ando dilation. It is noted further that unlike a kind of uniqueness of Halmos dilation of strict contractions on Hilbert spaces, vector space version of Halmos dilation cannot be characterized.

1. Introduction

Using functional calculus and Weierstrass polynomial approximation theorem, Halmos in 1950 proved an interesting result that every contraction on a Hilbert space can be lifted to unitary.

THEOREM 1.1. [9] (Halmos dilation) *Let \mathcal{H} be a Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a contraction. Then the operator*

$$U := \begin{pmatrix} T & \sqrt{I - TT^*} \\ \sqrt{I - T^*T} & -T^* \end{pmatrix}$$

is unitary on $\mathcal{H} \oplus \mathcal{H}$. In other words,

$$T = P_{\mathcal{H}}U|_{\mathcal{H}},$$

where $P_{\mathcal{H}} : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$ is the orthogonal projection onto \mathcal{H} .

Three years later, Sz. Nagy extended the result of Halmos which reads as follows.

THEOREM 1.2. [24] (Sz. Nagy dilation) *Let \mathcal{H} be a Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a contraction. Then there exists a Hilbert space \mathcal{K} which contains \mathcal{H} isometrically and a unitary $U : \mathcal{K} \rightarrow \mathcal{K}$ such that*

$$T^n = P_{\mathcal{H}}U^n|_{\mathcal{H}}, \quad \forall n = 1, 2, \dots,$$

where $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{K}$ is the orthogonal projection onto \mathcal{H} .

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Following the Theorem 1.2, extension of contractions on Hilbert spaces became an active area of research, known as dilation theory. Some standard references for this theory are [26, 17, 2, 22]. This study of contractions boosted the study of other classes of operators not only on Hilbert spaces, but also on Banach spaces. In a recent paper [3], Bhat, De, and Rakshit abstracted the key ingredients in Halmos and Sz. Nagy dilation theorem and set up a set theoretic version of dilation theory.

DEFINITION 1.3. [3] Let A be a (non empty) set and $h : A \rightarrow A$ be a map. An injective power dilation of h is a quadruple (B, i, v, p) , where B is a set, $i : A \rightarrow B$, $v : B \rightarrow B$ are injective maps, $p : B \rightarrow B$ is an idempotent map such that $p(B) = i(A)$ and

$$i(h^n(a)) = p(v^n(i(a))), \quad \forall a \in A, \forall n \in \mathbb{Z}_+.$$

A dilation (B, i, v, p) of h is said to be minimal if

$$B = \cup_{n=0}^{\infty} v^n(i(A)).$$

It is a simple observation that for Hilbert spaces, every operator cannot be dilated to unitary operator. However, the following is a surprising result for sets derived in [3].

THEOREM 1.4. [3] *Every map $h : A \rightarrow A$ admits a minimal injective power dilation.*

Bhat, De, and Rakshit succeeded in obtaining fundamental theorems of dilations such as Wold decomposition, Halmos dilation, Sz. Nagy dilation, inter-twining lifting theorem, Sarason's lemma, Ando dilation and BCL (Berger, Coburn and Lebow) theorem. Definition 1.3 allowed the authors of [3] to introduce the dilation of linear maps on vector spaces and showed that every linear map admits a minimal injective power dilation (see Section 2 for definition).

In this paper, we give the abstract study of dilation initiated by Bhat, De, and Rakshit for vector spaces. We follow a similar development as done in [3]. We first derive Wold decomposition, followed by Halmos dilation. After that we derive an N-dilation result which is motivated from the construction of Egervary. Followed by this, we derive inter-twining lifting theorem. At present exact analogue of Ando dilation is not known but a variant of that is given. Before ending the introduction, we note that there is another vector space approach of dilation theory by Han, Larson, Liu, and Liu [13] which is motivated from the Naimark dilation theorem [7, 19, 20], dilation theory of frames [16, 10, 11, 12, 6, 15, 14] and Stinespring dilation theorem [23]. Dilation results presented by Han, Larson, Liu, and Liu in [13] try to dilate a unital linear map from an algebra to the algebra of operators on a vector space to a unital homomorphism from the same algebra to the algebra of operators on a bigger vector space. The dilation presented in this paper is completely different than of dilation given in [13] for the following reasons. First, we are dilating a linear map from vector space (and not a unital linear map from an algebra) to injective or bijective linear operators (not a homomorphism from an algebra). Second, there is no power dilation derived in [13] but we derive power dilation. Third, there are no inter-twining lifting and commuting dilation derived in [13] but we consider them.

2. Dilations of linear maps

Let \mathcal{H} be a Hilbert space. We recall that an operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is called a shift if $\bigcap_{n=0}^{\infty} T^n(\mathcal{H}) = \{0\}$. Classical Wold decomposition is the following.

THEOREM 2.1. [26] (Wold decomposition) *Let T be an isometry on a Hilbert space \mathcal{H} . Then \mathcal{H} decomposes uniquely as $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$, where \mathcal{H}_u and \mathcal{H}_s are T -reducing subspaces of \mathcal{H} , $T|_{\mathcal{H}_u} : \mathcal{H}_u \rightarrow \mathcal{H}_u$ is a unitary and $T|_{\mathcal{H}_s} : \mathcal{H}_s \rightarrow \mathcal{H}_s$ is a shift.*

We note that the definition of shift of an operator does not use the Hilbert space structure. Thus it can be formulated for vector spaces without modifications.

DEFINITION 2.2. Let \mathcal{V}, \mathcal{W} be vector spaces and $T : \mathcal{V} \rightarrow \mathcal{W}$ be a linear map on V . The map T is said to be a shift if $\bigcap_{n=0}^{\infty} T^n(\mathcal{V}) = \{0\}$.

We now have the vector space version of Wold decomposition.

THEOREM 2.3. (Wold decomposition for vector spaces) *Let T be an injective linear map on a vector space \mathcal{V} . Then \mathcal{V} decomposes as $\mathcal{V} = \mathcal{V}_b \oplus \mathcal{V}_s$, where \mathcal{V}_b is a T -invariant subspace of \mathcal{V} , $T|_{\mathcal{V}_b} : \mathcal{V}_b \rightarrow \mathcal{V}_b$ is a bijection and $T|_{\mathcal{V}_s} : \mathcal{V}_s \rightarrow \mathcal{V}$ is a shift.*

Proof. Define $\mathcal{V}_b := \bigcap_{n=0}^{\infty} T^n(\mathcal{V})$ and let \mathcal{V}_s be a vector space complement of \mathcal{V}_b in \mathcal{V} . We clearly have $\mathcal{V} = \mathcal{V}_b \oplus \mathcal{V}_s$. Now $T(\mathcal{V}_b) = T(\bigcap_{n=0}^{\infty} T^n(\mathcal{V})) \subseteq \bigcap_{n=0}^{\infty} T^n(\mathcal{V}) = \mathcal{V}_b$. Thus \mathcal{V}_b is a T -invariant subspace of \mathcal{V} . We now try to show that $T|_{\mathcal{V}_b}$ is a bijection. Since T is already injective, it suffices to show that $T|_{\mathcal{V}_b}$ is surjective. Let $y \in \mathcal{V}_b$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in \mathcal{V} such that $y = Tx_1 = T^2x_2 = T^3x_3 = \dots$. Since T is injective, we then have $x_1 = Tx_2 = T^2x_2 = \dots$. Therefore $y = Tx_1$ and $x_1 \in \mathcal{V}_b$. Thus $T|_{\mathcal{V}_b}$ is surjective. We are now left with proving that $T|_{\mathcal{V}_s}$ is a shift. Let $y \in \bigcap_{n=0}^{\infty} (T|_{\mathcal{V}_s})^n(\mathcal{V}_s) \subseteq (\bigcap_{n=0}^{\infty} T^n(\mathcal{V})) \cap \mathcal{V}_s = \mathcal{V}_b \cap \mathcal{V}_s$. Hence $y = 0$ which completes the proof. \square

Since vector space complements are not unique, note that, we do not have uniqueness in Wold decomposition for vector spaces. We now derive Halmos dilation for linear maps on vector spaces.

THEOREM 2.4. (Halmos dilation for vector spaces) *Let \mathcal{V} be a vector space and $T : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map. Then the operator*

$$U := \begin{pmatrix} T & I \\ I & 0 \end{pmatrix}$$

is invertible on $\mathcal{V} \oplus \mathcal{V}$. In other words,

$$T = P_{\mathcal{V}}U|_{\mathcal{V}},$$

where $P_{\mathcal{V}} : \mathcal{V} \oplus \mathcal{V} \rightarrow \mathcal{V} \oplus \mathcal{V}$ is the first coordinate projection onto \mathcal{V} .

Proof. It suffices to produce inverse map for U . A direct calculation says that

$$V := \begin{pmatrix} 0 & I \\ I & -T \end{pmatrix}$$

is the inverse of U . \square

In the sequel, any invertible operator of the form

$$\begin{pmatrix} T & B \\ C & D \end{pmatrix},$$

where $B, C, D: \mathcal{V} \rightarrow \mathcal{V}$ are linear operators, will be called as a Halmos dilation of T . Now we observe that Halmos dilation is not unique. Using the theory of block matrices [18] we can produce a variety of Halmos dilations for a given operator. Following are some classes of Halmos dilations.

- (i) If $T: \mathcal{V} \rightarrow \mathcal{V}$ is an invertible linear map and the linear operators $B, C, D: \mathcal{V} \rightarrow \mathcal{V}$ are such that $D - CT^{-1}B$ is invertible, then the operator

$U := \begin{pmatrix} T & B \\ C & D \end{pmatrix}$ is a Halmos dilation of T on $\mathcal{V} \oplus \mathcal{V}$ whose inverse is

$$\begin{pmatrix} T^{-1} + T^{-1}B(D - CT^{-1}B)^{-1} & -T^{-1}B(D - CT^{-1}B)^{-1} \\ -D^{-1}C(D - CT^{-1}B)^{-1} & (D - CT^{-1}B)^{-1} \end{pmatrix}.$$

- (ii) If $D: \mathcal{V} \rightarrow \mathcal{V}$ is an invertible linear map and the linear operators $B, C: \mathcal{V} \rightarrow \mathcal{V}$ are such that $T - BD^{-1}C$ is invertible, then the operator

$\begin{pmatrix} T & B \\ C & D \end{pmatrix}$ is a Halmos dilation of T on $\mathcal{V} \oplus \mathcal{V}$ whose inverse is

$$\begin{pmatrix} (T - BD^{-1}C)^{-1} & -(T - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(T - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(T - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}.$$

- (iii) If $B: \mathcal{V} \rightarrow \mathcal{V}$ is an invertible linear map and the linear operators $C, D: \mathcal{V} \rightarrow \mathcal{V}$ are such that $C - DB^{-1}T$ is invertible, then the operator

$\begin{pmatrix} T & B \\ C & D \end{pmatrix}$ is a Halmos dilation of T on $\mathcal{V} \oplus \mathcal{V}$ whose inverse is

$$\begin{pmatrix} -(C - DB^{-1}T)^{-1}DB^{-1} & (C - DB^{-1}T)^{-1} \\ B^{-1} + B^{-1}T(C - DB^{-1}T)^{-1}DB^{-1} & -B^{-1}T(C - DB^{-1}T)^{-1} \end{pmatrix}.$$

(iv) If $C : \mathcal{V} \rightarrow \mathcal{V}$ is an invertible linear map and the linear operators $B, D : \mathcal{V} \rightarrow \mathcal{V}$ are such that $B - TC^{-1}D$ is invertible, then the operator

$\begin{pmatrix} T & B \\ C & D \end{pmatrix}$ is a Halmos dilation of T on $\mathcal{V} \oplus \mathcal{V}$ whose inverse is

$$\begin{pmatrix} -C^{-1}D(B - TC^{-1}D)^{-1} & C^{-1} + C^{-1}D(B - TC^{-1}D)^{-1}TC^{-1} \\ (B - TC^{-1}D)^{-1} & -(B - TC^{-1}D)^{-1}TC^{-1} \end{pmatrix}.$$

Recently, Bhat and Mukherjee [4] proved that there is certain kind of uniqueness of Halmos dilation for strict contractions in Hilbert spaces. Result reads as follows.

THEOREM 2.5. [4] *Let \mathcal{H} be a finite dimensional Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a strict contraction. Then Halmos dilation of T on $\mathcal{H} \oplus \mathcal{H}$ is unitarily equivalent to*

$$\left(\begin{array}{cc} T & \sqrt{I - TT^*}W \\ \sqrt{I - T^*T} & -T^*W \end{array} \right), \quad \text{for some unitary operator } W : \mathcal{H} \rightarrow \mathcal{H}.$$

We next derive a negative result to Theorem 2.5 for Halmos dilation in vector spaces.

THEOREM 2.6. *Let \mathcal{V} be a finite dimensional vector space and $T : \mathcal{V} \rightarrow \mathcal{V}$ be a trace non-zero linear operator. Then there are Halmos dilations of T which are not similar.*

Proof. Note that

$$\begin{pmatrix} T & T - I \\ T + I & T \end{pmatrix}$$

is an invertible operator and hence is a Halmos dilation of T . It is now enough to show that the matrices

$$\begin{pmatrix} T & T - I \\ T + I & T \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T & I \\ I & 0 \end{pmatrix}$$

are not similar. Since \mathcal{V} is finite dimensional, we can use the property of trace map to conclude that these matrices are not similar. \square

It was Egervary [8] who observed that Halmos dilation of contraction can be extended finitely so that power of dilation will be dilation of power of contraction. This can be formally stated as follows.

THEOREM 2.7. [8] (N-dilation) *Let \mathcal{H} be a Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a contraction. Let N be a natural number. Then the operator*

$$U := \begin{pmatrix} T & 0 & 0 \cdots 0 & \sqrt{I - TT^*} \\ \sqrt{I - T^*T} & 0 & 0 \cdots 0 & -T^* \\ 0 & I & 0 \cdots 0 & 0 \\ 0 & 0 & I \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 \cdots I & 0 \end{pmatrix}_{(N+1) \times (N+1)}$$

is unitary on $\oplus_{k=1}^{N+1} \mathcal{H}$ and

$$T^k = P_{\mathcal{H}} U^k|_{\mathcal{H}}, \quad \forall k = 1, \dots, N,$$

where $P_{\mathcal{H}} : \oplus_{k=1}^{N+1} \mathcal{H} \rightarrow \oplus_{k=1}^{N+1} \mathcal{H}$ is the orthogonal projection onto \mathcal{H} .

We now derive vector space version of Theorem 2.7.

THEOREM 2.8. (N-dilation for vector spaces) *Let \mathcal{V} be a vector space and $T : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map. Let N be a natural number. Then the operator*

$$U := \begin{pmatrix} T & 0 & 0 \cdots 0 & I \\ I & 0 & 0 \cdots 0 & 0 \\ 0 & I & 0 \cdots 0 & 0 \\ 0 & 0 & I \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 \cdots I & 0 \end{pmatrix}_{(N+1) \times (N+1)}$$

is invertible on $\oplus_{k=1}^{N+1} \mathcal{V}$ and

$$T^k = P_{\mathcal{V}} U^k|_{\mathcal{V}}, \quad \forall k = 1, \dots, N, \tag{1}$$

where $P_{\mathcal{V}} : \oplus_{k=1}^{N+1} \mathcal{V} \rightarrow \oplus_{k=1}^{N+1} \mathcal{V}$ is the first coordinate projection onto \mathcal{V} .

Proof. A direct calculation of power of U gives Equation (1). To complete the proof, now we need show that U is invertible. Define

$$V := \begin{pmatrix} 0 & I & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & I & 0 \cdots 0 & 0 \\ 0 & 0 & 0 & I \cdots 0 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots 0 & I \\ I & -T & 0 & 0 \cdots 0 & 0 \end{pmatrix}_{(N+1) \times (N+1)}$$

where 0 is in the $(0,0)$ position (which is underlined), satisfies $UV = VU = I$ and hence U is invertible which completes the proof. \square

An important observation associated with Theorems 2.8 and 2.9 is that the dilation is not optimal, i.e., even if the given operator is invertible, then also U is not same as T . To overcome this, next we move on with the definition of dilation given by Bhat, De, and Rakshit [3].

DEFINITION 2.10. [3] Let \mathcal{V} be a vector space and $T : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map. A linear injective dilation of T is a quadruple (\mathcal{W}, I, U, P) , where \mathcal{W} is a vector space, and $I : \mathcal{V} \rightarrow \mathcal{W}$ is an injective linear map, $U : \mathcal{W} \rightarrow \mathcal{W}$ is an injective linear map, $P : \mathcal{W} \rightarrow \mathcal{W}$ is an idempotent linear map such that $P(\mathcal{W}) = I(\mathcal{V})$ and

$$\text{(Dilation equation)} \quad IT^n x = PU^n Ix, \quad \forall n \in \mathbb{Z}_+, \forall x \in \mathcal{V}.$$

A dilation (\mathcal{W}, I, U, P) of T is said to be minimal if

$$\mathcal{W} = \text{span}\{U^n Ix : n \in \mathbb{Z}_+, x \in \mathcal{V}\}.$$

An easier way to remember the dilation equation is the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{W} & \xrightarrow{U^n} & \mathcal{W} & \xrightarrow{P} & \mathcal{W} \\ & \swarrow I & & & \uparrow I \\ & & \mathcal{V} & \xrightarrow{T^n} & \mathcal{V} \end{array}$$

In [3] vector space analogous of Sz. Nagy dilation result was proved.

THEOREM 2.11. [3] Every linear map $T : \mathcal{V} \rightarrow \mathcal{V}$ admits minimal injective linear dilation.

Proof. We reproduce the proof given by Bhat, De, and Rakshit [3] for the sake of future use. Define

$$\mathcal{W} := \{(x_n)_{n=0}^\infty : x_n \in \mathcal{V}, \forall n \in \mathbb{Z}_+, x_n \neq 0 \text{ only for finitely many } n's\}.$$

Clearly \mathcal{W} is a vector space w.r.t. natural operations. Now define

$$\begin{aligned} I : \mathcal{V} \ni x &\mapsto (x, 0, \dots) \in \mathcal{W}, \\ U : \mathcal{W} \ni (x_n)_{n=0}^\infty &\mapsto (0, x_0, \dots) \in \mathcal{W}, \\ P : \mathcal{W} \ni (x_n)_{n=0}^\infty &\mapsto \sum_{n=0}^\infty IT^n x_n \in \mathcal{W}. \end{aligned}$$

Then (\mathcal{W}, I, U, P) is a minimal injective linear dilation of T . \square

We call the dilation (\mathcal{W}, I, U, P) constructed in Theorem 2.11 as standard dilation of T . We next consider inter-twining lifting theorem. For contractions acting on Hilbert spaces this says that any operator which intertwines contractions can be lifted so that the lifted operator intertwines dilation operator.

THEOREM 2.12. [25] (inter-twining lifting theorem) *Let $T_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1, T_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be contractions, where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces. Let $V_1 : \mathcal{K}_1 \rightarrow \mathcal{K}_1, V_2 : \mathcal{K}_2 \rightarrow \mathcal{K}_2$ be minimal isometric dilations of T_1, T_2 , respectively. Assume that $S : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is a bounded linear operator such that $T_1 S = S T_2$. Then there exists a bounded linear operator $R : \mathcal{K}_2 \rightarrow \mathcal{K}_1$ such that $V_1 R = R V_2, P_{\mathcal{H}_1} R_{\mathcal{H}_2^\perp} = 0, P_{\mathcal{H}_1} R_{\mathcal{H}_2} = S$ and $\|R\| = \|S\|$. Conversely if $R : \mathcal{K}_2 \rightarrow \mathcal{K}_1$ is a bounded linear operator such that $V_1 R = R V_2$ and $P_{\mathcal{H}_1} R_{\mathcal{H}_2^\perp} = 0$, then $S := P_{\mathcal{H}_1} R_{\mathcal{H}_2}$ satisfies $T_1 S = S T_2$.*

THEOREM 2.13. (inter-twining lifting theorem for vector spaces) *Let $\mathcal{V}_1, \mathcal{V}_2$ be vector spaces, $T_1 : \mathcal{V}_1 \rightarrow \mathcal{V}_1, T_2 : \mathcal{V}_2 \rightarrow \mathcal{V}_2$ be linear maps. Let $(\mathcal{W}_1, I_1, U_1, P_1), (\mathcal{W}_2, I_2, U_2, P_2)$ be standard dilations of T_1, T_2 , respectively. If $S : \mathcal{V}_2 \rightarrow \mathcal{V}_1$ is a linear map such that $T_1 S = S T_2$, then there exists a linear map $R : \mathcal{W}_2 \rightarrow \mathcal{W}_1$ such that*

$$U_1 R = R U_2, \quad R P_2 = P_1 R, \quad R I_2 = I_1 S. \tag{3}$$

Conversely if $R : \mathcal{W}_2 \rightarrow \mathcal{W}_1$ is a linear map such that $U_1 R = R U_2, R P_2 = P_1 R$, then there exists a linear map $S : \mathcal{V}_2 \rightarrow \mathcal{V}_1$ such that

$$R I_2 = I_1 S, \quad T_1 S = S T_2. \tag{4}$$

Proof. Define $R : \mathcal{W}_2 \ni (x_n)_{n=0}^\infty \mapsto (Sx_n)_{n=0}^\infty \in \mathcal{W}_1$. We now verify three equalities in Equation (3). Let $(x_n)_{n=0}^\infty \in \mathcal{W}_2$. Then

$$\begin{aligned} U_1 R(x_n)_{n=0}^\infty &= U_1 (Sx_n)_{n=0}^\infty = (0, Sx_0, Sx_1, \dots), \\ R U_2(x_n)_{n=0}^\infty &= R(0, x_0, x_1, \dots) = (0, Sx_0, Sx_1, \dots), \end{aligned}$$

$$\begin{aligned} R P_2(x_n)_{n=0}^\infty &= R \left(\sum_{n=0}^\infty I_2 T_2^n x_n \right) = \sum_{n=0}^\infty R I_2 T_2^n x_n \\ &= \sum_{n=0}^\infty R(T_2^n x_n, 0, 0, \dots) = \sum_{n=0}^\infty (S T_2^n x_n, 0, 0, \dots), \end{aligned}$$

$$\begin{aligned} P_1 R(x_n)_{n=0}^\infty &= P_1 (Sx_n)_{n=0}^\infty = \sum_{n=0}^\infty I_1 T_1^n Sx_n \\ &= \sum_{n=0}^\infty I_1 S T_2^n x_n = \sum_{n=0}^\infty (S T_2^n x_n, 0, 0, \dots), \end{aligned}$$

$$R I_2 x = R(x, 0, 0, \dots) = (Sx, 0, 0, \dots), \quad I_1 Sx = (Sx, 0, 0, \dots).$$

We now consider the converse part. For this, first we have to define linear map S . Let $y \in \mathcal{V}_2$. Now $R P_2(y, 0, \dots) = P_1 R(y, 0, \dots) \in I_1(\mathcal{V}_1)$ and I_1 is injective implies that there exists a unique $x \in \mathcal{V}_2$ such that $R P_2(y, 0, \dots) = P_1 R(y, 0, \dots) = I_1(x)$. We now define $Sy := x$. Then S is well-defined and linear. Let $y \in \mathcal{V}_2$ and $x \in \mathcal{V}_2$ be such that $Sy = x$.

Then $I_1Sy = RP_2(y, 0, \dots) = RI_2y$. Thus we verified first equality in (4). We are left with verification of second equality. We now calculate

$$RP_2U_2(x, 0, \dots) = RP_2(0, x, 0, \dots) = RI_2T_2x \tag{5}$$

and

$$P_1U_1R(x, 0, \dots) = P_1RU_2(x, 0, \dots) = P_1R(0, x, 0, \dots) \tag{6}$$

$$= RP_2(0, x, 0, \dots) = RI_2T_2x, \quad \forall x \in \mathcal{V}_2. \tag{7}$$

Given conditions produce

$$RP_2U_2 = P_1RU_2 = P_1U_1R. \tag{8}$$

Equation (8) says that (5) and (6) are equal which completes the proof. \square

Sz. Nagy’s dilation theorem brings us to the question of dilating more than one operators which are commuting. After a decade of work of Sz. Nagy, Ando derived the following result.

THEOREM 2.14. [1] (Ando dilation theorem) *Let \mathcal{H} be a Hilbert space and $T_1, T_2 : \mathcal{H} \rightarrow \mathcal{H}$ be commuting contractions. Then there exist a Hilbert space \mathcal{K} which contains \mathcal{H} isometrically and a pair of commuting unitaries $U_1, U_2 : \mathcal{K} \rightarrow \mathcal{K}$ such that*

$$T_1^n T_2^m = P_{\mathcal{H}} U_1^n U_2^m|_{\mathcal{H}}, \quad \forall n, m = 1, 2, \dots,$$

where $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{K}$ is the orthogonal projection onto \mathcal{H} .

It is known that Ando dilation theorem cannot be extended for more than two commuting contractions [5]. However, it is a surprising result obtained by Bhat, De, and Rakshit [3] that for set theoretic consideration, Ando dilation holds for arbitrary number functions. We do not know Ando dilation for linear maps on vector spaces but have a variant of it which is given in the following theorem. On the other hand, we note that any number of doubly commuting contractions on a Hilbert space can be dilated to commuting unitaries [22].

THEOREM 2.15. *Let \mathcal{V} be a vector space and $T, S : \mathcal{V} \rightarrow \mathcal{V}$ be commuting linear maps. Then there are dilations (\mathcal{W}, I, U_1, P) and (\mathcal{W}, I, U_2, P) of T, S respectively, such that*

$$(0_c U) = \begin{pmatrix} 0_r \\ V \end{pmatrix}$$

and

$$IT^n S^m x = PU^n V^m Ix, \quad \forall n, m \in \mathbb{Z}_+, \forall x \in \mathcal{V},$$

where 0_c denotes the infinite column matrix of zero vectors and 0_r denotes the infinite row matrix of zero vectors.

Proof. We extend the construction in the proof of Theorem 2.11. Define

$$\mathcal{W} := \left\{ \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty \times \infty} : x_{n,m} \in \mathcal{V}, \forall n, m \in \mathbb{Z}_+, x_{n,m} \neq 0 \right. \\ \left. \text{only for finitely many } (n, m)'s \right\}.$$

Then \mathcal{W} becomes a vector space with respect to natural operations. We now define the following four linear maps:

$$I : \mathcal{V} \ni x \mapsto \begin{pmatrix} x & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{W}$$

$$U : \mathcal{W} \ni \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{W}$$

$$V : \mathcal{W} \ni \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mapsto \begin{pmatrix} 0 & x_{0,0} & x_{0,1} & \cdots \\ 0 & x_{1,0} & x_{1,1} & \cdots \\ 0 & x_{2,0} & x_{2,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{W}$$

$$P : \mathcal{W} \ni \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mapsto \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} IT^n S^m x_{n,m} \in \mathcal{W}.$$

We then have

$$(0_c U) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ 0 & x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ 0 & x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ 0 & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0_r \\ V \end{pmatrix}.$$

Now $PU^nIx = IT^n x$, $PV^nIx = IS^m x$, $\forall x \in \mathcal{V}$, $\forall n, m \in \mathbb{Z}_+$. Hence (\mathcal{W}, I, U_1, P) and (\mathcal{W}, I, U_2, P) are dilations of T, S , respectively. A calculation now shows that $IT^n S^m x = PU^n V^m Ix, \forall n, m \in \mathbb{Z}_+, \forall x \in \mathcal{V}$. \square

We end by presenting a generalization of Theorem 2.15.

THEOREM 2.16. *Let $n \in \mathbb{N}$ and $T_1, \dots, T_n : \mathcal{V} \rightarrow \mathcal{V}$ be commuting linear maps. Then there are dilations $(\mathcal{W}, I, U_1, P), \dots, (\mathcal{W}, I, U_n, P)$ of T_1, \dots, T_n , respectively, such that*

$$IT_1^{j_1} \dots T_n^{j_n} x = PU_1^{j_1} \dots U_n^{j_n} Ix, \quad \forall j_1, \dots, j_n \in \mathbb{Z}_+, \forall x \in \mathcal{V} \quad (9)$$

and U_1, \dots, U_n commute.

Proof. Define

$$\mathcal{W} := \bigoplus_{(i_1, \dots, i_n) \in \mathbb{Z}_+^n} \mathcal{V}_{(i_1, \dots, i_n)}, \quad \mathcal{V}_{(i_1, \dots, i_n)} = \mathcal{V}, \quad \forall (i_1, \dots, i_n) \in \mathbb{Z}_+^n.$$

Now define the maps as follows.

$$I : \mathcal{V} \ni x \mapsto (x, 0, \dots, 0) \in \mathcal{W},$$

$$U_k : \mathcal{W} \ni (\{x_{i_1}^{(r)}\}_r, \dots, \{x_{i_k}^{(r)}\}_r, \dots, \{x_{i_n}^{(r)}\}_r) \mapsto (\{x_{i_1}^{(r)}\}_r, \dots, \{x_{i_{k+1}}^{(r)}\}_r, \dots, \{x_{i_n}^{(r)}\}_r) \in \mathcal{W},$$

$$\forall 1 \leq k \leq n,$$

$$P : \mathcal{W} \ni (\{x_{i_1}^{(r)}\}_r, \dots, \{x_{i_n}^{(r)}\}_r) \mapsto \sum_{(i_1, \dots, i_n) \in \mathbb{Z}_+^n} IU_1^{i_1} \dots U_n^{i_n} (\{x_{i_1}^{(r)}\}_r, \dots, \{x_{i_n}^{(r)}\}_r) \in \mathcal{W}.$$

Then each U_j commutes with U_k for $1 \leq j, k \leq n$, (\mathcal{W}, I, U_j, P) is a dilation of T_j for $1 \leq j \leq n$ and Equation (9) holds. \square

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REFERENCES

- [1] T. ANDÔ, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged), 24: 88–90, 1963.
- [2] WILLIAM ARVESON, *Dilation theory yesterday and today*, In A glimpse at Hilbert space operators, vol. 207 of Oper. Theory Adv. Appl., pages 99–123, Birkhäuser Verlag, Basel, 2010.
- [3] B. V. RAJARAMA BHAT, SANDIPAN DE, AND NARAYAN RAKSHIT, *A caricature of dilation theory*, Adv. Oper. Theory, 6 (4): Paper No. 63, 2021.
- [4] B. V. RAJARAMA BHAT AND MITHUN MUKHERJEE, *Two states*, To appear in Houston Journal of Mathematics, arXiv:1710.00180v3 [math.FA] 23 April 2020.
- [5] TIRTHANKAR BHATTACHARYYA, *Dilation of contractive tuples: a survey*, In Surveys in analysis and operator theory (Canberra, 2001), vol. 40 of Proc. Centre Math. Appl. Austral. Nat. Univ., pages 89–126, Austral. Nat. Univ., Canberra, 2002.

- [6] PETER G. CASAZZA, DEGUANG HAN, AND DAVID R. LARSON, *Frames for Banach spaces*, In The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), vol. 247 of Contemp. Math., pages 149–182, Amer. Math. Soc., Providence, RI, 1999.
- [7] WOJCIECH CZAJA, *Remarks on Naimark's duality*, Proc. Amer. Math. Soc., 136 (3): 867–871, 2008.
- [8] E. EGÉRVÁRY, *On the contractive linear transformations of n -dimensional vector space*, Acta Sci. Math. (Szeged), 15: 178–182, 1954.
- [9] PAUL R. HALMOS, *Normal dilations and extensions of operators*, Summa Brasil. Math., 2: 125–134, 1950.
- [10] DEGUANG HAN AND DAVID R. LARSON, *Frames, bases and group representations*, Mem. Amer. Math. Soc., 147 (697): x+94, 2000.
- [11] DEGUANG HAN, DAVID R. LARSON, BEI LIU, AND RUI LIU, *Dilations of frames, operator-valued measures and bounded linear maps*, In Operator methods in wavelets, tilings, and frames, vol. 626 of Contemp. Math., pages 33–53, Amer. Math. Soc., Providence, RI, 2014.
- [12] DEGUANG HAN, DAVID R. LARSON, BEI LIU, AND RUI LIU, *Operator-valued measures, dilations, and the theory of frames*, Mem. Amer. Math. Soc., 229 (1075): viii+84, 2014.
- [13] DEGUANG HAN, DAVID R. LARSON, BEI LIU, AND RUI LIU, *Structural properties of homomorphism dilation systems*, Chin. Ann. Math. Ser. B, 41 (4): 585–600, 2020.
- [14] DEGUANG HAN, DAVID R. LARSON, AND RUI LIU, *Dilations of operator-valued measures with bounded p -variations and framings on Banach spaces*, J. Funct. Anal., 274 (5): 1466–1490, 2018.
- [15] B. S. KASHIN AND T. YU. KULIKOVA, *A remark on the description of frames of general form*, Mat. Zametki, 72 (6): 941–945, 2002.
- [16] DAVID R. LARSON AND FRANCISZEK HUGON SZAFRANIEC, *Framings and dilations*, Acta Sci. Math. (Szeged), 79 (3–4): 529–543, 2013.
- [17] ELIAHU LEVY AND ORR MOSHE SHALIT, *Dilation theory in finite dimensions: the possible, the impossible and the unknown*, Rocky Mountain J. Math., 44 (1): 203–221, 2014.
- [18] TZON-TZER LU AND SHENG-HUA SHIOU, *Inverses of 2×2 block matrices*, Comput. Math. Appl., 43 (1–2): 119–129, 2002.
- [19] M. NEUMARK, *Spectral functions of a symmetric operator*, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR], 4: 277–318, 1940.
- [20] M. A. NEUMARK, *On a representation of additive operator set functions*, C. R. (Doklady) Acad. Sci. URSS (N.S.), 41: 359–361, 1943.
- [21] J. J. SCHÄFFER, *On unitary dilations of contractions*, Proc. Amer. Math. Soc., 6: 322, 1955.
- [22] ORR MOSHE SHALIT, *Dilation Theory: A Guided Tour*, In Operator theory, functional analysis and applications, vol. 282 of Oper. Theory Adv. Appl., pages 551–623, Birkhäuser/Springer, Cham, 2021.
- [23] W. FORREST STINESPRING, *Positive functions on C^* -algebras*, Proc. Amer. Math. Soc., 6: 211–216, 1955.
- [24] BÉLA SZ.-NAGY, *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. (Szeged), 15: 87–92, 1953.
- [25] BÉLA SZ.-NAGY AND CIPRIAN FOIAŞ, *The “lifting theorem” for intertwining operators and some new applications*, Indiana Univ. Math. J., 20 (10): 901–904, 1971.
- [26] BELA SZ.-NAGY, CIPRIAN FOIAS, HARI BERCOVICI, AND LASZLO KERCHY, *Harmonic analysis of operators on Hilbert space*, Universitext, Springer, New York, second edition, 2010.

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