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COMPLEX SYMMETRIC WEIGHTED COMPOSITION–DIFFERENTIATION OPERATORS OF ORDER n ON THE WEIGHTED BERGMAN SPACES

MAHBUBE MORADI, MAHSA FATEHI* AND CHRISTOPHER N. B. HAMMOND

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Abstract. We study the complex symmetry of weighted composition–differentiation operators of order n on the weighted Bergman spaces A^2_α. Several concrete examples are provided.

1. Preliminaries

Let $\mathbb{D}$ denote the open disk in the complex plane $\mathbb{C}$. For $\alpha > -1$, the *weighted Bergman space* A^2_α is the Hilbert space consisting of all analytic functions $f(z) = \sum_{j=0}^{\infty} a_j z^j$ on $\mathbb{D}$ such that $\|f\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2 < \infty$, where

$$\beta(j) = \|z^j\| = \sqrt{\frac{j!\Gamma(\alpha+2)}{\Gamma(j+\alpha+2)}}$$

for each non-negative integer j. The inner product of two functions in this space is given by the rule

$$\left\langle \sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} b_j z^j \right\rangle = \sum_{j=0}^{\infty} a_j \overline{b_j} \beta(j)^2.$$

It is well known that this space is a reproducing kernel Hilbert space; for any w in $\mathbb{D}$ and any non-negative integer m, there is a kernel function $K_w^{[m]}$ such that $\langle f, K_w^{[m]} \rangle = f^{(m)}(w)$ for each f in A^2_α. To simplify notation, we write K_w to denote $K_w^{[0]}$. In particular,

$$K_w(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}} = \sum_{j=0}^{\infty} \frac{\overline{w}^j z^j}{\beta(j)^2}$$

and

$$K_w^{[m]}(z) = \frac{(\alpha+2)\dots(\alpha+m+1)z^m}{(1-\overline{w}z)^{m+\alpha+2}} = \frac{m! z^m}{\beta(m)^2 (1-\overline{w}z)^{m+\alpha+2}}$$

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* Corresponding author.

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for $m \geqslant 1$ (note that from [2, p. 20], we can see that $K_w^{[m]}(z) = \frac{d^m K}{d\overline{w}^m}$, where $K(z,\overline{w}) = K_w(z)$ for each $z,w \in \mathbb{D}$ and $\beta(m)^2 = \frac{m!}{(\alpha+2)\dots(\alpha+m+1)}$). Moreover, for each non-negative integer m, we have

$$\|K_w^{[m]}\|^2 = \sum_{j=m}^{\infty} \frac{(|w|^2)^{j-m}}{\beta(j)^2} \left(\frac{j!}{(j-m)!} \right)^2$$

(note that $K_w^{[m]}(z) = \sum_{j=m}^{\infty} \frac{j!}{\beta(j)^2 (j-m)!} \overline{w}^{j-m} z^j$ by [2, Theorem 2.16]). Recall that H^∞ is the Banach space consisting of all bounded analytic functions defined on $\mathbb{D}$, with supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Let P_α denote the projection of $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. Given a function h in $L^\infty(\mathbb{D})$, the *Toeplitz operator* T_h on A^2_α is defined by the rule

$$T_h(f) = P_\alpha(hf)$$

for f in A^2_α. If h belongs to H^∞, it is easy to see that $T_h(f) = h \cdot f$. For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, the *composition operator* C_φ is defined by the rule

$$C_\varphi(f) = f \circ \varphi$$

for f in A^2_α. All Toeplitz operators and all composition operators are bounded on A^2_α. As a natural generalization of both of these classes, consider the operator $C_{\psi,\varphi}$ that takes f to $\psi \cdot (f \circ \varphi)$, where $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ are both analytic on $\mathbb{D}$. Such an operator is called a *weighted composition operator*.

For a positive integer n, we define the *differentiation operator of order n* on A^2_α by $D^{(n)}(f) = f^{(n)}$. None of these operators is bounded on A^2_α. Nevertheless, for many analytic self-maps φ, the operator $C_\varphi D^{(n)}$ is bounded on A^2_α. This class of operators was initially considered by Hibschweiler and Portnoy [9] and by Ohno [11], and has been studied further by other researchers (see [4], [5], and [12]). Ohno [11] characterized the boundedness and compactness of $C_\varphi D^{(1)}$ on the Hardy space; Stević [12] obtained analogous results for $C_\varphi D^{(n)}$ on the weighted Bergman spaces. We will write $D_{\varphi,n}$ to denote $C_\varphi D^{(n)}$, particularly when such an operator is bounded on A^2_α, referring to it as a *composition–differentiation operator of order n*. For an analytic function $\psi : \mathbb{D} \to \mathbb{C}$, the *weighted composition–differentiation operator of order n* on A^2_α is defined by the rule

$$D_{\psi,\varphi,n}(f) = \psi \cdot \left(f^{(n)} \circ \varphi \right).$$

Note that $D_{\psi,\varphi,n}$ is actually the product of the Toeplitz operator T_ψ and $D_{\varphi,n}$, whenever ψ belongs to H^∞ and $D_{\varphi,n}$ is bounded. To avoid trivial situations, we will assume throughout this paper that φ is not constant and that ψ is not identically 0.

A bounded linear operator T is called *complex symmetric* on a complex Hilbert space $\mathscr{H}$ if there exits a conjugation C (i.e., an antilinear isometric involution) such that $CT^*C = T$; for a particular conjugation C, we say that T is *C-symmetric*. Garcia and Putinar initiated the study of complex symmetric operators on Hilbert spaces of analytic functions (see [7] and [8]). Complex symmetric weighted composition operators have been considered in [3], [6], [10], and [13]. In this paper, we use the symbol J to denote the specific conjugation $(Jf)(z) = \overline{f(\overline{z})}$.

Any complex number z can be represented $z = |z|e^{i\theta}$, where $0 \leqslant \theta < 2\pi$. We write $\operatorname{Arg}(z)$ to denote this value of θ, taking $\operatorname{Arg}(0) = 0$.

2. Complex symmetric operators $D_{\psi,\varphi,n}$

For an analytic $\varphi\colon \mathbb{D} \to \mathbb{D}$ and $\alpha > -1$, the generalized Nevanlinna counting function $N_{\varphi,\alpha+2}$ is defined by the rule

$$N_{\varphi,\alpha+2}(w) = \sum_{\varphi(z)=w} \left(\ln(1/|z|)\right)^{\alpha+2},$$

where w belongs to $\mathbb{D}\setminus\{\varphi(0)\}$. The next proposition provides necessary and sufficient conditions for $D_{\varphi,n}$ to be bounded and compact.

PROPOSITION 2.1. [12, Theorem 9] *Let φ be an analytic self-map of $\mathbb{D}$, with n in $\mathbb{N}$ and $\alpha > -1$.*

a) An operator $D_{\varphi,n}\colon A^2_\alpha \to A^2_\alpha$ is bounded if and only if

$$N_{\varphi,\alpha+2}(w) = O\left(\left(\ln(1/|w|)\right)^{\alpha+2+2n}\right).$$

b) An operator $D_{\varphi,n}\colon A^2_\alpha \to A^2_\alpha$ is compact if and only if

$$N_{\varphi,\alpha+2}(w) = o\left(\left(\ln(1/|w|)\right)^{\alpha+2+2n}\right), \quad \text{as } |w| \to 1^-.$$

Since $\ln(1/|w|)$ is comparable to $1-|w|$ as $|w| \to 1^-$, the following characterization holds in the case where φ is univalent on $\mathbb{D}$.

COROLLARY 2.2. *Let φ be a univalent self-map of $\mathbb{D}$, with n in $\mathbb{N}$ and $\alpha > -1$.*

a) An operator $D_{\varphi,n}$ is bounded on A^2_α if and only if

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|)^{\alpha+2}}{(1-|\varphi(w)|)^{\alpha+2+2n}} < \infty.$$

b) An operator $D_{\varphi,n}$ is compact on A^2_α if and only if

$$\lim_{|w|\to 1} \frac{(1-|w|)^{\alpha+2}}{(1-|\varphi(w)|)^{\alpha+2+2n}} = 0.$$

Note that Corollary 2.2 shows that if $D_{\varphi,n}$ is bounded, then φ does not have finite angular derivative at any point on $\partial\mathbb{D}$ (see [2, Theorem 2.44]). Moreover, we infer from Corollary 2.2 that an operator $D_{\varphi,n}$ is bounded if $\|\varphi\|_\infty < 1$ and so $D_{\psi,\varphi,n}$ is bounded on A^2_α whenever ψ belongs to H^∞. We will employ the following lemma frequently.

LEMMA 2.3. *If an operator $D_{\psi,\varphi,n}$ is bounded on A^2_α, then*

$$D^*_{\psi,\varphi,n}(K_w) = \overline{\psi(w)}K^{[n]}_{\varphi(w)}.$$

Proof. Observe that

$$\langle f, D^*_{\psi,\varphi,n}(K_w)\rangle = \langle D_{\psi,\varphi,n}f, K_w\rangle = \psi(w)f^{(n)}(\varphi(w)) = \left\langle f, \overline{\psi(w)}K^{[n]}_{\varphi(w)}\right\rangle$$

for any f in A^2_α. Our result follows from the fact that the span of the kernel functions K_w is dense in A^2_α. $\square$

Throughout this paper, we set $t = (\alpha+2)(\alpha+3)\ldots(\alpha+n+1)$, which will be appeared several times in this paper. We will now make a few observations about J-symmetric operators $D_{\psi,\varphi,n}$, which will be used in the proof of Theorem 2.7.

PROPOSITION 2.4. *If an operator $D_{\psi,\varphi,n}$ is J-symmetric on A^2_α, the following conditions hold:*

- (i) $\psi^{(m)}(0) = 0$ *for each* $0 \leqslant m < n$*;*
- (ii) $\psi^{(n)}(0) \neq 0$*;*
- (iii) $\psi(w) \neq 0$ *for any* w *in* $\mathbb{D}\setminus\{0\}$*;*
- (iv) *the map* φ *is univalent.*

Proof. Suppose that $D_{\psi,\varphi,n}$ is J-symmetric. Observe that

$$JD_{\psi,\varphi,n}(K_0) = 0. \tag{2.1}$$

Lemma 2.3 shows that

$$D^*_{\psi,\varphi,n}J(K_0) = \overline{\psi(0)}K^{[n]}_{\varphi(0)}. \tag{2.2}$$

Since $D_{\psi,\varphi,n}$ is J-symmetric, it follows from (2.1) and (2.2) that $\psi(0) = 0$. Assume that $\psi^{(m)}(0) = 0$ for $m < n-1$. One can see that

$$JD_{\psi,\varphi,n}K^{[m+1]}_0 = 0. \tag{2.3}$$

On the other hand, for any f in A^2_α, we obtain

$$\begin{aligned}
\left\langle f, D^*_{\psi,\varphi,n}JK^{[m+1]}_0\right\rangle &= \left\langle f, D^*_{\psi,\varphi,n}K^{[m+1]}_0\right\rangle \\
&= \left\langle D_{\psi,\varphi,n}f, K^{[m+1]}_0\right\rangle \\
&= \left(\psi\cdot(f^{(n)}\circ\varphi)\right)^{(m+1)}(0) \\
&= \sum_{j=0}^{m+1}\binom{m+1}{j}\psi^{(m+1-j)}(0)\left(f^{(n)}\circ\varphi\right)^{(j)}(0)
\end{aligned}$$

$$
\begin{aligned}
&= \psi^{(m+1)}(0) f^{(n)}(\varphi(0)) \\
&\qquad + \sum_{j=1}^{m+1} \binom{m+1}{j} \psi^{(m+1-j)}(0) \big(f^{(n)} \circ \varphi\big)^{(j)}(0) \\
&= \psi^{(m+1)}(0) f^{(n)}(\varphi(0)) \\
&= \big\langle f, \overline{\psi^{(m+1)}(0)} K_{\varphi(0)}^{[n]} \big\rangle, \qquad (2.4)
\end{aligned}
$$

so

$$
D_{\psi,\varphi,n}^* J K_0^{[m+1]} = D_{\psi,\varphi,n}^* K_0^{[m+1]} = \overline{\psi^{(m+1)}(0)} K_{\varphi(0)}^{[n]}. \qquad (2.5)
$$

If $D_{\psi,\varphi,n}$ is J-symmetric, then (2.3) and (2.5) imply that $\psi^{(m+1)}(0) = 0$. By the same idea as in (2.4), we have

$$
D_{\psi,\varphi,n}^* J K_0^{[n]} = D_{\psi,\varphi,n}^* K_0^{[n]} = \overline{\psi^{(n)}(0)} K_{\varphi(0)}^{[n]}, \qquad (2.6)
$$

since $\psi^{(m)}(0) = 0$ for any $m < n$. Because

$$
J D_{\psi,\varphi,n} K_0^{[n]} = t n! J(\psi) \qquad (2.7)
$$

and ψ is not identically 0, it follows from (2.6) and (2.7) that $\psi^{(n)}(0) \neq 0$. Now suppose that $\psi(w) = 0$ for some w in $\mathbb{D}$. Lemma 2.3 shows that $D_{\psi,\varphi,n}^* J(K_{\overline{w}}) = 0$. Moreover,

$$
J D_{\psi,\varphi,n}(K_{\overline{w}}) = \frac{t \overline{w^n} J(\psi)}{\big(1 - \overline{w} J(\varphi)\big)^{n+\alpha+2}}.
$$

Since $D_{\psi,\varphi,n}$ is J-symmetric and ψ is not identically zero, we observe that $w = 0$.

Now assume that $D_{\psi,\varphi,n}$ is J-symmetric and that there exist distinct points w_1 and w_2 in $\mathbb{D}$ with $\varphi(w_1) = \varphi(w_2)$. (If either w_1 or w_2 is zero, the open mapping theorem allows us to find a pair of distinct nonzero points w_3 and w_4 in $\mathbb{D}$ with $\varphi(w_3) = \varphi(w_4)$. Hence we may assume that w_1 and w_2 are both nonzero.) One can easily see that the kernel of $D_{\psi,\varphi,n}$ consists of the set of all polynomials with degree less than n. Lemma 2.3 implies that

$$
\begin{aligned}
D_{\psi,\varphi,n}^* J\big(\psi(w_2) K_{\overline{w_1}} - \psi(w_1) K_{\overline{w_2}}\big) &= D_{\psi,\varphi,n}^* \big(\overline{\psi(w_2)} K_{w_1} - \overline{\psi(w_1)} K_{w_2}\big) \\
&= \overline{\psi(w_1)\psi(w_2)} K_{\varphi(w_1)}^{[n]} - \overline{\psi(w_1)\psi(w_2)} K_{\varphi(w_2)}^{[n]} = 0.
\end{aligned}
$$

Since $D_{\psi,\varphi,n}$ is J-symmetric, it follows that $\psi(w_2) K_{\overline{w_1}} - \psi(w_1) K_{\overline{w_2}}$ is a polynomial of degree less than n. Therefore

$$
\psi(w_2) \sum_{j=n}^{\infty} \frac{\Gamma(j+2+\alpha)(w_1)^j z^j}{j!\Gamma(\alpha+2)} - \psi(w_1) \sum_{j=n}^{\infty} \frac{\Gamma(j+2+\alpha)(w_2)^j z^j}{j!\Gamma(\alpha+2)} = 0.
$$

Thus $\psi(w_2) {w_1}^m = \psi(w_1) {w_2}^m$ for each $m \geqslant n$. We observe that

$$
\psi(w_1) w_2^{n+1} = \psi(w_2) w_1^{n+1} = \psi(w_2) w_1^n w_1 = \psi(w_1) w_2^n w_1,
$$

so $w_1 = w_2$. Consequently φ must be univalent. $\square$

REMARK 2.5. We can follow the outline of the proof of Proposition 2.4 to see that an analogue of Proposition 2.4 holds for any normal operator $D_{\psi,\varphi,n}$.

If

$$\varphi(z) = \frac{az+b}{cz+d}$$

is a nonconstant linear fractional self-map of $\mathbb{D}$, then the map

$$\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$$

also takes $\mathbb{D}$ into itself (see [1, Lemma 1]). Recall that $\|\sigma\|_\infty < 1$ whenever $\|\varphi\|_\infty < 1$, in which case both $D_{\varphi,n}$ and $D_{\sigma,n}$ are bounded operators on A^2_α. Cowen [1] determined the adjoint of C_φ acting on the Hardy space H^2. Similarly, the second and third authors investigated the adjoints of certain weighted composition–differentiation operators $D_{\psi,\varphi,1}$ on H^2 (see [4, Theorem 1]). Our next result shows that an analogue of [4, Theorem 1] holds in the context of the weighted Bergman spaces A^2_α. Recall that $t = (\alpha+2)(\alpha+3)\ldots(\alpha+n+1)$.

PROPOSITION 2.6. *For the linear fractional self-maps φ and σ described above, it follows that*

$$D^*_{K^{[n]}_{\sigma(0)},\varphi,n} = D_{K^{[n]}_{\varphi(0)},\sigma,n}.$$

Proof. We know that

$$K^{[n]}_{\varphi(0)}(z) = \frac{tz^n}{\left(1-\overline{(b/d)}z\right)^{n+\alpha+2}} = \frac{t\overline{d^{n+\alpha+2}}z^n}{(\overline{d}-\overline{b}z)^{n+\alpha+2}}$$

and

$$K^{[n]}_{\sigma(0)}(z) = \frac{tz^n}{(1+(c/d)z)^{n+\alpha+2}} = \frac{td^{n+\alpha+2}z^n}{(cz+d)^{n+\alpha+2}}.$$

We see that

$$\begin{aligned} D_{K^{[n]}_{\varphi(0)},\sigma,n}(K_w)(z) &= T_{K^{[n]}_{\varphi(0)}}\left(\frac{t\overline{w^n}}{\left(1-\overline{w}\sigma(z)\right)^{n+\alpha+2}}\right) \\ &= \frac{t^2\overline{d^{n+\alpha+2}w^n}z^n}{(-\overline{b}z+\overline{d}-\overline{wa}z+\overline{wc})^{n+\alpha+2}} \end{aligned} \tag{2.8}$$

(note that $D^{(n)}(K_w) = \frac{d^nK_w}{dz^n} = \frac{t\overline{w^n}}{\left(1-\overline{w}z\right)^{n+\alpha+2}}$). By Lemma 2.3, we obtain

$$\begin{aligned} D^*_{K^{[n]}_{\sigma(0)},\varphi,n}(K_w)(z) &= \frac{t\overline{d^{n+\alpha+2}w^n}}{(\overline{cw}+\overline{d})^{n+\alpha+2}}K^{[n]}_{\varphi(w)}(z) \\ &= \frac{t^2\overline{d^{n+\alpha+2}w^n}z^n}{\left(\overline{cw}+\overline{d}-(\overline{aw}+\overline{b})z\right)^{n+\alpha+2}}. \end{aligned} \tag{2.9}$$

Since the span of the reproducing kernel functions K_w is dense in A^2_α, the result follows from (2.8) and (2.9). $\square$

Now we give an example for Proposition 2.6.

EXAMPLE 1. Suppose that $\varphi(z) = \frac{i}{2}z + \frac{1}{3}$. We can see that $\sigma(z) = \frac{\frac{-i}{2}z}{\frac{1}{3}z+1}$, $\varphi(0) = 1/3$, and $\sigma(0) = 0$. Then by Proposition 2.6, we can see that $\frac{n!}{\beta(n)^2} D^*_{z^n,\varphi,n} = D_{K^{[n]}_{\frac{1}{3}},\sigma,n}$.

Some more examples for Proposition 2.6 will be seen in the proofs of Theorem 2.7 and Propositions 3.1 and 3.2.

Our next theorem completely describes the J-symmetric operators $D_{\psi,\varphi,n}$.

THEOREM 2.7. *A bounded operator $D_{\psi,\varphi,n}$ is J-symmetric on A^2_α if and only if*

$$\psi(z) = \frac{a}{tn!} K^{[n]}_{\overline{c}}(z) = \frac{az^n}{n!(1-cz)^{n+\alpha+2}}$$

and

$$\varphi(z) = c + \frac{bz}{1-cz},$$

where $a = \psi^{(n)}(0)$ and $b = \varphi'(0)$ are both nonzero complex number and $c = \varphi(0)$ belongs to $\mathbb{D}$.

Proof. Suppose that $D_{\psi,\varphi,n}$ is J-symmetric. By (2.6), (2.7), and Proposition 2.4, we conclude that $J(\psi) = \frac{\overline{\psi^{(n)}(0)}}{tn!} K^{[n]}_{\varphi(0)}$ and so $\psi = \frac{\psi^{(n)}(0)}{tn!} K^{[n]}_{\overline{\varphi(0)}} = \frac{\psi^{(n)}(0)z^n}{n!\left(1-\varphi(0)z\right)^{n+\alpha+2}}$, where $\psi^{(n)}(0) \neq 0$. By the general Leibniz rule, we can see that

$$\psi^{(n+1)}(z) = \frac{\psi^{(n)}(0)}{n!} \sum_{k=0}^{n+1} \binom{n+1}{k} \left(z^n\right)^{(n+1-k)} \left(\frac{1}{\left(1-\varphi(0)z\right)^{n+\alpha+2}}\right)^{(k)}.$$

Since $\left(z^n\right)^{(n+1-k)}(0) = 0$, when $0 \leqslant k \leqslant n+1$ and $k \neq 1$, we obtain

$$\psi^{(n+1)}(0) = (n+1)(n+\alpha+2)\varphi(0)\psi^{(n)}(0). \tag{2.10}$$

Observe that

$$\begin{aligned} JD_{\psi,\varphi,n}\left(K^{[n+1]}_0\right)(z) &= t(n+1)!(\alpha+n+2)J(\psi)(z)J(\varphi)(z) \\ &= \frac{t(n+1)(n+\alpha+2)\overline{\psi^{(n)}(0)}z^n}{\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+2}} J(\varphi)(z). \end{aligned} \tag{2.11}$$

By the proof of (2.4), we can see that for any $f \in A^2_\alpha$, we obtain

$$\begin{aligned}\left\langle f, D^*_{\psi,\varphi,n} K_0^{[n+1]} \right\rangle &= \psi^{(n+1)}(0) f^{(n)}(\varphi(0)) \\ &\quad + \sum_{j=1}^{n+1} \binom{n+1}{j} \psi^{(n+1-j)}(0) \big(f^{(n)} \circ \varphi\big)^{(j)}(0) \\ &= \psi^{(n+1)}(0) f^{(n)}(\varphi(0)) + (n+1)\psi^{(n)}(0) \big(f^{(n)} \circ \varphi\big)^{(1)}(0) \\ &\quad + \sum_{j=2}^{n+1} \binom{n+1}{j} \psi^{(n+1-j)}(0) \big(f^{(n)} \circ \varphi\big)^{(j)}(0) \\ &= \psi^{(n+1)}(0) f^{(n)}(\varphi(0)) + (n+1)\psi^{(n)}(0)\varphi'(0) f^{(n+1)}(\varphi(0)) \qquad (2.12)\end{aligned}$$

(note that Proposition 2.4(i) implies that $\psi^{(n+1-j)}(0) = 0$ for each $2 \leqslant j \leqslant n+1$). Hence by (2.12), we have

$$D^*_{\psi,\varphi,n}\big(K_0^{[n+1]}\big)(z) = \overline{\psi^{(n+1)}(0)} K^{[n]}_{\varphi(0)}(z) + (n+1)\overline{\psi^{(n)}(0)\varphi'(0)} K^{[n+1]}_{\varphi(0)}(z). \qquad (2.13)$$

Therefore by (2.10) and (2.13), we observe that

$$\begin{aligned}D^*_{\psi,\varphi,n} J\big(K_0^{[n+1]}\big)(z) &= D^*_{\psi,\varphi,n}\big(K_0^{[n+1]}\big)(z) \\ &= \overline{\psi^{(n+1)}(0)} K^{[n]}_{\varphi(0)}(z) + (n+1)\overline{\psi^{(n)}(0)\varphi'(0)} K^{[n+1]}_{\varphi(0)}(z) \\ &= \frac{t(n+1)(n+\alpha+2)\overline{\varphi(0)\psi^{(n)}(0)} z^n}{\big(1-\overline{\varphi(0)}z\big)^{n+\alpha+2}} \\ &\quad + \frac{t(n+1)(n+\alpha+2)\overline{\psi^{(n)}(0)\varphi'(0)} z^{n+1}}{\big(1-\overline{\varphi(0)}z\big)^{n+\alpha+3}}. \qquad (2.14)\end{aligned}$$

Because $D_{\psi,\varphi,n}$ is J-symmetric, it follows from (2.11) and (2.14) that

$$J(\varphi)(z) = \overline{\varphi(0)} + \frac{\overline{\varphi'(0)}z}{1-\overline{\varphi(0)}z},$$

and so

$$\varphi(z) = \varphi(0) + \frac{\varphi'(0)z}{1-\varphi(0)z},$$

with $\varphi'(0) \neq 0$ because φ is nonconstant.

Conversely, take ψ and φ as in the statement of the theorem. For each f in A^2_α, we have

$$JD_{\psi,\varphi,n}(f)(z) = J(\psi)(z) J(f^{(n)}(\varphi(z))) = J(\psi)(z)\overline{f^{(n)}(\varphi(\overline{z}))}. \qquad (2.15)$$

On the other hand, by Proposition 2.6, we see that

$$D^*_{\psi,\varphi,n} J = \frac{\overline{a}}{n!t} D^*_{K^{[n]}_{\sigma(0)},\varphi,n} J = \frac{\overline{a}}{n!t} D_{K^{[n]}_{\varphi(0)},\sigma,n} J.$$

Thus

$$D^*_{\psi,\varphi,n}J(f)(z) = \frac{\overline{a}}{n!t}K^{[n]}_{\varphi(0)}(z)\overline{f^{(n)}(\overline{\sigma(z)})} = J(\psi)(z)\overline{f^{(n)}(\varphi(\overline{z}))}. \tag{2.16}$$

Therefore, by (2.15) and (2.16), the operator $D_{\psi,\varphi,n}$ is J-symmetric. □

We infer from the paragraph after Corollary 2.2, from [10, Lemma 4.8], and from the proof of [10, Theorem 4.10] that an operator $D_{\psi,\varphi,n}$ from Theorem 2.7 is bounded on A^2_α whenever $2|c+\overline{c}(b-c^2)| < 1-|b-c^2|^2$.

By an idea similar to one stated in the proof of [3, Proposition 2.1] (see also [13, Theorem 4.1]), we remark that $C_{\psi,\varphi}$ is unitary and J-symmetric on A^2_α if and only if either

$$\psi(z) = \frac{\alpha\left(1-|p|^2\right)^{\frac{\alpha+2}{2}}}{(1-\overline{p}z)^{\alpha+2}} \tag{2.17}$$

and

$$\varphi(z) = \frac{\overline{p}}{p}\frac{p-z}{1-\overline{p}z}, \tag{2.18}$$

where p belongs to $\mathbb{D}\setminus\{0\}$ and $|\alpha| = 1$, or $\psi \equiv \mu$ and $\varphi(z) = \lambda z$, where $|\mu| = |\lambda| = 1$. In the case that $p \neq 0$, we denote the linear functional transformations in (2.17) and (2.18) by ψ_p and φ_p respectively. Invoking [3, Lemma 2.2], we observe that $C_{\lambda z}J$ and $C_{\psi_p,\varphi_p}J$ are conjugations. Next we will characterize the complex symmetric operators $D_{\psi,\varphi,n}$ with conjugations $C_{\lambda z}J$ and $C_{\psi_p,\varphi_p}J$.

THEOREM 2.8. *Suppose that*

$$\tilde{\varphi}(z) = c + \frac{bz}{1-cz}$$

and that

$$\tilde{\psi}(z) = \frac{az^n}{n!(1-cz)^{n+\alpha+2}},$$

where a and b belong to $\mathbb{C}\setminus\{0\}$ and c belongs to $\mathbb{D}$. Assume that $D_{\tilde{\psi},\tilde{\varphi},n}$ is bounded on A^2_α.

(1) *For $p \neq 0$, an operator $D_{\psi,\varphi,n}$ on A^2_α is complex symmetric with conjugation $C_{\psi_p,\varphi_p}J$ if and only if $\varphi = \tilde{\varphi}\circ\varphi_p$ and $\psi = \psi_p\cdot(\tilde{\psi}\circ\varphi_p)$ for some $\tilde{\varphi}$ and $\tilde{\psi}$.*

(2) *For $|\mu| = |\lambda| = 1$, an operator $D_{\psi,\varphi,n}$ on A^2_α is complex symmetric with conjugation $C_{\mu,\lambda z}J$ if and only if $\psi(z) = \mu\tilde{\psi}(\lambda z)$ and $\varphi(z) = \tilde{\varphi}(\lambda z)$ for some $\tilde{\varphi}$ and $\tilde{\psi}$.*

Proof. (1) Let $p \neq 0$ and suppose that $D_{\psi,\varphi,n}$ is $C_{\psi_p,\varphi_p}J$-symmetric. As mentioned in the paragraph preceding the statement of Theorem 2.8, the operator $C^*_{\psi_p,\varphi_p}$ is unitary and J-symmetric, so it is not difficult to see that $C^*_{\psi_p,\varphi_p}$ is $C_{\psi_p,\varphi_p}J$-symmetric. Then [3, Proposition 2.3] implies that $C^*_{\psi_p,\varphi_p}D_{\psi,\varphi,n}$ is J-symmetric. It follows from Theorem 2.7 that there is a J-symmetric operator $D_{\tilde{\psi},\tilde{\varphi},n}$ so that $D_{\psi,\varphi,n} = C_{\psi_p,\varphi_p}D_{\tilde{\psi},\tilde{\varphi},n}$. Hence we observe that $\varphi = \tilde{\varphi}\circ\varphi_p$ and $\psi = \psi_p\cdot(\tilde{\psi}\circ\varphi_p)$.

Conversely, suppose that $\varphi = \tilde{\varphi} \circ \varphi_p$ and $\psi = \psi_p \cdot (\tilde{\psi} \circ \varphi_p)$ for some $\tilde{\varphi}$ and $\tilde{\psi}$. Then $D_{\psi,\varphi,n} = C_{\psi_p,\varphi_p} D_{\tilde{\psi},\tilde{\varphi},n}$. Since the weighted composition operator C_{ψ_p,φ_p} is unitary and J-symmetric and the operator $D_{\tilde{\psi},\tilde{\varphi},n}$ is J-symmetric (see Theorem 2.7), the operator $D_{\psi,\varphi,n}$ is $C_{\psi_p,\varphi_p}J$-symmetric by [3, Proposition 2.3].

(2) The result follows immediately from the technique demonstrated in the proof of part (1). □

In the following example, we give some complex symmetric weighted composition–differentiation operators.

EXAMPLE 2. a) Suppose that

$$\varphi(z) = \frac{1}{4} + \frac{1+2iz}{4+\frac{i}{2}+(2i-1)z}$$

and

$$\psi(z) = \frac{3^{\alpha+2}\left(z-\frac{i}{2}\right)^n}{4^{\alpha+1}n!\left(1+\frac{i}{8}+\left(\frac{i}{2}-\frac{1}{4}\right)z\right)^{n+\alpha+2}}.$$

We can see that $\varphi = \tilde{\varphi} \circ \varphi_p$ and $\psi = \psi_p \cdot (\tilde{\psi} \circ \varphi_p)$, where $p = i/2$, $\varphi_p(z) = \frac{z-i/2}{1+iz/2}$, $\tilde{\varphi}(z) = \frac{1}{4} + \frac{iz/2}{1-z/4}$, $\psi_p(z) = \frac{(3/4)^{\alpha+2}}{(1+iz/2)^{\alpha+2}}$, and $\tilde{\psi}(z) = \frac{4z^n}{n!(1-z/4)^{n+\alpha+2}}$. It is easy to see that $\|\tilde{\varphi}\|_\infty < 1$ and so $D_{\tilde{\psi},\tilde{\varphi},n}$ is bounded on A^2_α. Invoking Theorem 2.7, the operator $D_{\tilde{\psi},\tilde{\varphi},n}$ is J-symmetric and it satisfies the conditions of Proposition 2.4. Theorem 2.8 shows that $D_{\psi,\varphi,n}$ is $C_{\psi_p,\varphi_p}J$-symmetric.

b) Suppose that $\varphi(z) = \frac{1}{5} + \frac{\left((i/10)+1/100\right)iz}{1-iz/5}$ and $\psi(z) = \frac{e^{i\pi/4}i^n z^n}{n!(1-iz/5)^{n+\alpha+2}}$. We can see that $\varphi(z) = \tilde{\varphi}(\lambda z)$ and $\psi(z) = \mu\tilde{\psi}(\lambda z)$, where $\tilde{\varphi}(z) = \frac{1}{5} + \frac{\left((i/10)+1/100\right)z}{1-z/5}$, $\tilde{\psi}(z) = \frac{z^n}{n!(1-z/5)^{n+\alpha+2}}$, $\lambda = i$, and $\mu = e^{i\pi/4}$. One can see that $D_{\tilde{\psi},\tilde{\varphi},n}$ is bounded and by Theorem 2.7, it is J-symmetric. Theorem 2.8 implies that $D_{\psi,\varphi,n}$ is $C_{e^{i\pi/4},iz}J$-symmetric.

3. Some examples of complex symmetric operators

In this section, we will show that the complex symmetric operators $D_{\psi,\varphi,n}$ we have already identified include all the self-adjoint operators $D_{\psi,\varphi,n}$ and some of the normal operators $D_{\psi,\varphi,n}$. The next proposition provides a characterization of self-adjoint weighted composition–differentiation operators of order n on A^2_α.

PROPOSITION 3.1. *A bounded operator $D_{\psi,\varphi,n}$ is self-adjoint on A^2_α if and only if*

$$\psi(z) = \frac{az^n}{n!(1-\overline{c}z)^{n+\alpha+2}} = \frac{a}{tn!}K_c^{[n]}(z)$$

and

$$\varphi(z) = c + \frac{bz}{1-\overline{c}z},$$

where $a = \psi^{(n)}(0)$ *and* $b = \varphi'(0)$ *are both nonzero real numbers and* $c = \varphi(0)$ *belongs to* $\mathbb{D}$. *Furthermore, for the self-adjoint operator* $D_{\psi,\varphi,n}$, *one of the following holds:*

(i) *If* $c = 0$, *then* $D_{\psi,\varphi,n}$ *is* J*-symmetric.*

(ii) *If* $c \neq 0$, *then* $D_{\psi,\varphi,n}$ *is* $C_{e^{-2i\theta}z}J$*-symmetric, where* $\theta = Arg(c)$.

Proof. Suppose that $D_{\psi,\varphi,n}$ is self-adjoint on A^2_α. By (2.4) and Remark 2.5, we have $D^*_{\psi,\varphi,n}K_0^{[n]} = \overline{\psi^{(n)}(0)}K^{[n]}_{\varphi(0)}$. Moreover, we can see that $D_{\psi,\varphi,n}K_0^{[n]}(z) = D_{\psi,\varphi,n}(tz^n)$ $= tn!\psi(z)$. Since $D_{\psi,\varphi,n}$ is self-adjoint, we conclude that

$$\psi(z) = \frac{\overline{\psi^{(n)}(0)}}{tn!}K^{[n]}_{\varphi(0)}(z) = \frac{\overline{\psi^{(n)}(0)}z^n}{n!\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+2}}. \tag{3.1}$$

Differentiating both sides of (3.1) n times with respect to z, we obtain

$$\begin{aligned}\psi^{(n)}(z) &= \frac{\overline{\psi^{(n)}(0)}}{n!}\sum_{j=0}^{n}\binom{n}{j}\left(z^n\right)^{(n-j)}\left(\frac{1}{\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+2}}\right)^{(j)}\\ &= \frac{\overline{\psi^{(n)}(0)}}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{n!}{j!}z^j\left(\frac{1}{\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+2}}\right)^{(j)}\end{aligned} \tag{3.2}$$

(note that $\left(z^n\right)^{(t)} = \frac{n!}{(n-t)!}z^{n-t}$ for each t with $0 \leqslant t \leqslant n$). It follows from (3.2) that $\psi^{(n)}(0) = \overline{\psi^{(n)}(0)}$, and so $\psi^{(n)}(0)$ is real. Moreover, note that $\psi^{(n)}(0) \neq 0$ since ψ is not identically 0. On the other hand, differentiating both sides of (3.1) $n+1$ times with respect to z yields

$$\psi^{(n+1)}(0) = (n+1)(n+\alpha+2)\overline{\varphi(0)}\psi^{(n)}(0). \tag{3.3}$$

We can see that

$$\begin{aligned}D_{\psi,\varphi,n}\left(K_0^{[n+1]}\right)(z) &= D_{\psi,\varphi,n}\left(t(n+\alpha+2)z^{n+1}\right)\\ &= \frac{t(n+1)(n+\alpha+2)\psi^{(n)}(0)z^n}{\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+2}}\varphi(z).\end{aligned} \tag{3.4}$$

Furthermore, by the idea from (2.4) and the fact that $\psi^{(m)}(0) = 0$ for each $m < n$ (see Remark 2.5), we have

$$\begin{aligned}D^*_{\psi,\varphi,n}\left(K_0^{[n+1]}\right)(z) &= \overline{\psi^{(n+1)}(0)}K^{[n]}_{\varphi(0)}(z) + (n+1)\overline{\psi^{(n)}(0)\varphi'(0)}K^{[n+1]}_{\varphi(0)}(z)\\ &= \frac{t\overline{\psi^{(n+1)}(0)}z^n}{\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+2}}\\ &\quad + \frac{(n+1)\overline{\psi^{(n)}(0)\varphi'(0)}t(n+\alpha+2)z^{n+1}}{\left(1-\overline{\varphi(0)}z\right)^{n+\alpha+3}}.\end{aligned} \tag{3.5}$$

Since $D_{\psi,\varphi,n}$ is self-adjoint, by combining (3.3), (3.4), and (3.5), we see that

$$\varphi(z) = \varphi(0) + \frac{\overline{\varphi'(0)}z}{1-\overline{\varphi(0)}z}. \tag{3.6}$$

Differentiating both sides of (3.6) with respect to z and taking $z=0$, we observe that $\varphi'(0)$ is real. In addition, because φ is not constant, we see that $\varphi'(0) \neq 0$.

For the converse, take φ and ψ as in the statement of the proposition and suppose that $D_{\psi,\varphi,n}$ is bounded on A^2_α. Proposition 2.6 dictates that

$$D^*_{\psi,\varphi,n} = \frac{\overline{a}}{tn!} D^*_{K^{[n]}_{\sigma(0)},\varphi,n} = \frac{\overline{a}}{tn!} D_{K^{[n]}_{\varphi(0)},\sigma,n} = D_{\psi,\varphi,n}.$$

Thus $D_{\psi,\varphi,n}$ is self-adjoint.

We infer from Theorem 2.7 that the operator $D_{\psi,\varphi,n}$ is J-symmetric when $c=0$. Now take $c \neq 0$ and set $\tilde{\psi}(z) = \frac{ae^{2ni\theta}z^n}{n!(1-cz)^{n+\alpha+2}}$ and $\tilde{\varphi}(z) = c + \frac{be^{2i\theta}z}{1-cz}$. From Theorem 2.7, the operator $D_{\tilde{\psi},\tilde{\varphi},n}$ is J-symmetric. By [3, Lemma 2.2] and [3, Proposition 2.3], we observe that $C_{e^{-2i\theta}z}D_{\tilde{\psi},\tilde{\varphi},n}$ is $C_{e^{-2i\theta}z}J$-symmetric. (As stated in the paragraph preceding Theorem 2.8, the composition operator $C_{e^{-2i\theta}z}$ is unitary and J-symmetric.) A direct computation shows that $C_{e^{-2i\theta}z}D_{\tilde{\psi},\tilde{\varphi},n} = D_{\psi,\varphi,n}$, so the result follows. □

Now we will characterize those operators $D_{\psi,\varphi,n}$ on A^2_α that are normal in the case where $\varphi(0)=0$.

PROPOSITION 3.2. *Suppose that an operator $D_{\psi,\varphi,n}$ is bounded on A^2_α and that $\varphi(0)=0$. Then $D_{\psi,\varphi,n}$ is normal if and only if $\psi(z)=az^n$ and $\varphi(z)=bz$, where a belongs to $\mathbb{C}\setminus\{0\}$ and b belongs to $\mathbb{D}\setminus\{0\}$. Moreover, in this case $D_{\psi,\varphi,n}$ is J-symmetric.*

Proof. Assume that $D_{\psi,\varphi,n}$ is normal on A^2_α. We can see that

$$\|D_{\psi,\varphi,n}K^{[n]}_0\|^2 = \left\| \left(\frac{n!}{\beta(n)}\right)^2 \psi \right\|^2 = \left(\frac{n!}{\beta(n)}\right)^4 \sum_{j=0}^{\infty} \left(\frac{\beta(j)}{j!}\right)^2 |\psi^{(j)}(0)|^2. \tag{3.7}$$

On the other hand, by (2.4) and Remark 2.5, we observe that

$$\|D^*_{\psi,\varphi,n}K^{[n]}_0\|^2 = \|\overline{\psi^{(n)}(0)}K^{[n]}_0\|^2 = |\psi^{(n)}(0)|^2 \left(\frac{n!}{\beta(n)}\right)^2. \tag{3.8}$$

Because $D_{\psi,\varphi,n}$ is normal, by Remark 2.5, (3.7), and (3.8), we conclude that

$$|\psi^{(n)}(0)|^2 \left(\frac{n!}{\beta(n)}\right)^2 = \left(\frac{n!}{\beta(n)}\right)^4 \sum_{j=n}^{\infty} \left(\frac{\beta(j)}{j!}\right)^2 |\psi^{(j)}(0)|^2. \tag{3.9}$$

Remark 2.5 implies that $\psi^{(n)}(0) \neq 0$, so (3.9) shows that $\psi^{(j)}(0) = 0$ for each $j > n$. Since Remark 2.5 also shows that $\psi^{(j)}(0) = 0$ for any $j < n$, the map ψ must have the form $\psi(z) = az^n$ for some a in $\mathbb{C} \setminus \{0\}$. We have

$$D_{\psi,\varphi,n}\left(K_0^{[n+1]}\right)(z) = \left(\frac{(n+1)!}{\beta(n+1)}\right)^2 \psi(z)\varphi(z) \tag{3.10}$$
$$= \left(\frac{(n+1)!}{\beta(n+1)}\right)^2 az^n\varphi(z).$$

On the other hand, by using (2.4) and the fact that $\psi^{(m)}(0) = 0$ for each $m \neq n$, we see that

$$D^*_{\psi,\varphi,n}\left(K_0^{[n+1]}\right)(z) = (n+1)\overline{\psi^{(n)}(0)\varphi'(0)}K_0^{[n+1]}(z)$$
$$= \overline{a\varphi'(0)}\left(\frac{(n+1)!}{\beta(n+1)}\right)^2 z^{n+1}.$$
$$= \overline{a\varphi'(0)}(n+1)!K_0^{[n+1]}(z), \tag{3.11}$$

so $K_0^{[n+1]}$ is an eigenvalue for $D^*_{\psi,\varphi,n}$ corresponding to eigenvalue $\overline{a\varphi'(0)}(n+1)!$. Therefore

$$D_{\psi,\varphi,n}K_0^{[n+1]} = a\varphi'(0)(n+1)!K_0^{[n+1]}. \tag{3.12}$$

Since $D_{\psi,\varphi,n}$ is normal on A^2_α, by (3.10) and (3.12), we see that

$$a\varphi'(0)(n+1)!K_0^{[n+1]}(z) = \left(\frac{(n+1)!}{\beta(n+1)}\right)^2 az^n\varphi(z).$$

Thus $\varphi(z) = \varphi'(0)z$. Because φ is not identically 0, we conclude that $\varphi(z) = bz$ for some b in $\mathbb{D} \setminus \{0\}$.

For the converse, take ψ and φ as in the statement of the proposition and assume that $D_{\psi,\varphi,n}$ is bounded on A^2_α. Proposition 2.6 implies that $D^*_{az^n,bz,n} = D_{\overline{a}z^n,\overline{b}z,n}$. After some computation, we see that

$$D_{az^n,bz,n}D^*_{az^n,bz,n}(f)(z) = D_{az^n,bz,n}D_{\overline{a}z^n,\overline{b}z,n}(f)(z)$$
$$= D_{az^n,bz,n}\left(\overline{a}z^n f^{(n)}(\overline{b}z)\right)$$
$$= |a|^2 z^n \sum_{j=0}^{n} \binom{n}{j}\frac{n!}{j!}|b|^{2j}z^j f^{(n+j)}\left(|b|^2 z\right) \tag{3.13}$$

for each f in A^2_α; similarly,

$$D^*_{az^n,bz,n}D_{az^n,bz,n}(f)(z) = |a|^2 z^n \sum_{j=0}^{n} \binom{n}{j}\frac{n!}{j!}|b|^{2j}z^j f^{(n+j)}\left(|b|^2 z\right). \tag{3.14}$$

Hence (3.13) and (3.14) show that $D_{\psi,\varphi,n}$ is normal. Furthermore, Theorem 2.7 shows that $D_{\psi,\varphi,n}$ is J-symmetric. □

Next we describe the conditions under which the analytic functions φ and ψ from Proposition 3.1 induce a normal operator $D_{\psi,\varphi,n}$.

PROPOSITION 3.3. *Suppose that $D_{\psi,\varphi,n}$ is a bounded operator, with*

$$\psi(z) = \frac{az^n}{n!(1-\overline{c}z)^{n+\alpha+2}}$$

and

$$\varphi(z) = c + \frac{bz}{1-\overline{c}z},$$

where $a = \psi^{(n)}(0)$ and $b = \varphi'(0)$ are both nonzero complex numbers and $c = \varphi(0)$ belongs to $\mathbb{D}$. The operator $D_{\psi,\varphi,n}$ is normal on A^2_α if and only if either b belongs to $\mathbb{R}\setminus\{0\}$ or $c = 0$. Moreover, when $D_{\psi,\varphi,n}$ is normal, one of the following holds:

(i) *If $c = 0$, then $D_{\psi,\varphi,n}$ is J-symmetric.*

(ii) *If $c \neq 0$, then $D_{\psi,\varphi,n}$ is $C_{e^{-2i\theta}z}J$-symmetric, where $\theta = Arg(c)$.*

Proof. If b belongs to $\mathbb{R}\setminus\{0\}$ or $c = 0$, Propositions 3.1 and 3.2 imply that $D_{\psi,\varphi,n}$ is normal.

For the converse, suppose that b and c belong to $\mathbb{C}\setminus\mathbb{R}$. We have

$$D_{\psi,\varphi,n}\big(K_{\frac{1}{2}}\big)(z) = \frac{t\psi(z)}{2^n\big(1-\frac{1}{2}\varphi(z)\big)^{n+\alpha+2}} = \frac{a}{2^n n!(1-c/2)^{n+\alpha+2}}K^{[n]}_{p_1}(z),$$

where $p_1 = c + \frac{\overline{b}/2}{1-\overline{c}/2}$. On the other hand, by Lemma 2.3, we see that

$$D^*_{\psi,\varphi,n}\big(K_{\frac{1}{2}}\big)(z) = \overline{\psi(1/2)}K^{[n]}_{\varphi(1/2)}(z) = \frac{\overline{a}}{2^n n!(1-c/2)^{n+\alpha+2}}K^{[n]}_{p_2}(z),$$

where $p_2 = c + \frac{b/2}{1-\overline{c}/2}$.

If $D_{\psi,\varphi,n}$ were normal, then

$$\begin{aligned}\big\|D_{\psi,\varphi,n}\big(K_{\frac{1}{2}}\big)\big\|^2 &= \left|\frac{a}{2^n n!(1-c/2)^{n+\alpha+2}}\right|^2 \|K^{[n]}_{p_1}\|^2 \\ &= \left|\frac{a}{2^n n!(1-c/2)^{n+\alpha+2}}\right|^2 \sum_{j=n}^{\infty}\frac{(|p_1|^2)^{j-n}}{\beta(j)^2}\left(\frac{j!}{(j-n)!}\right)^2\end{aligned}$$

would equal

$$\begin{aligned}\big\|D^*_{\psi,\varphi,n}\big(K_{\frac{1}{2}}\big)\big\|^2 &= \left|\frac{a}{2^n n!(1-c/2)^{n+\alpha+2}}\right|^2 \|K^{[n]}_{p_2}\|^2 \\ &= \left|\frac{a}{2^n n!(1-c/2)^{n+\alpha+2}}\right|^2 \sum_{j=n}^{\infty}\frac{(|p_2|^2)^{j-n}}{\beta(j)^2}\left(\frac{j!}{(j-n)!}\right)^2.\end{aligned}$$

Therefore $|p_1|^2 = |p_2|^2$. Thus

$$\left(c+\frac{\overline{b}}{2-\overline{c}}\right)\left(\overline{c}+\frac{b}{2-c}\right) = \left(c+\frac{b}{2-\overline{c}}\right)\left(\overline{c}+\frac{\overline{b}}{2-c}\right)$$

$$|c|^2+\frac{bc}{2-c}+\frac{\overline{bc}}{2-\overline{c}}+\frac{|b|^2}{|2-c|^2}=|c|^2+\frac{c\overline{b}}{2-c}+\frac{b\overline{c}}{2-\overline{c}}+\frac{|b|^2}{|2-c|^2}$$
$$\frac{c(b-\overline{b})}{2-c}=\frac{\overline{c}(b-\overline{b})}{2-\overline{c}}$$
$$\frac{c}{2-c}=\frac{\overline{c}}{2-\overline{c}}.$$

Then $b=\overline{b}$ or $c=\overline{c}$, which is a contradiction. If $D_{\psi,\varphi,n}$ were normal, with b belonging to $\mathbb{C}\setminus\mathbb{R}$ and c belonging to $\mathbb{R}\setminus\{0\}$, a similar argument would show that $\|D^*_{\psi,\varphi,n}K_{\frac{i}{2}}\|\neq\|D_{\psi,\varphi,n}K_{\frac{i}{2}}\|$, which is also a contradiction. The rest of the proof is obtained by an argument similar to that of Proposition 3.1. □

Now, we give examples for the results of this section.

EXAMPLE 3. Suppose that $\varphi_1(z)=\frac{1}{4}+\frac{iz/2}{1-z/4}$, $\varphi_2(z)=\frac{e^{\frac{i\pi}{3}}}{4}+\frac{z/2}{1-\frac{e^{\frac{-i\pi}{3}}}{4}z}$, $\psi_1(z)=\frac{3z^n}{n!(1-z/4)^{n+\alpha+2}}$, and $\psi_2(z)=\frac{3z^n}{n!\left(1-\frac{e^{\frac{-i\pi}{3}}}{4}z\right)^{n+\alpha+2}}$. It is easy to see that $\|\varphi_1\|_\infty<1$ and $\|\varphi_2\|_\infty<1$ and so $D_{\psi_1,\varphi_1,n}$ and $D_{\psi_2,\varphi_2,n}$ are bounded on A^2_α. By Theorem 2.7, the operator $D_{\psi_1,\varphi_1,n}$ is J-symmetric. Proposition 3.3 implies that $D_{\psi_1,\varphi_1,n}$ is not normal. We observe that $D_{\psi_2,\varphi_2,n}$ is self-adjoint and $C_{e^{\frac{-2\pi}{3}i}z}J$-symmetric by Proposition 3.1.

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Mahbube Moradi
Department of Mathematics
Shiraz Branch, Islamic Azad University
Shiraz, Iran
e-mail: mmoradi8524@gmail.com

Mahsa Fatehi
Department of Mathematics
Shiraz Branch, Islamic Azad University
Shiraz, Iran
e-mail: fatehimahsa@yahoo.com
Mahsa.Fatehi@iau.ac.ir

Christopher N. B. Hammond
Department of Mathematics and Statistics
Connecticut College
New London, CT 06320, USA
e-mail: cnham@conncoll.edu