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A CRITERION OF LOCAL DERIVATIONS ON THE SEVEN–DIMENSIONAL SIMPLE MALCEV ALGEBRA

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Abstract. In the present paper we give a matrix form of local derivations of the complex finite dimensional simple (non-Lie) Malcev algebra $\mathbb{M}$, and a direct proof of the statement that every 2-local derivation of $\mathbb{M}$ is a derivation. We have some description of local and 2-local derivations of complex finite-dimensional semisimple binary Lie algebras.

Introduction

The present paper is devoted to local and 2-local derivations of Malcev algebras. The history of local derivations began in the paper of Kadison [14]. Kadison proved that every continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation. A similar notion of 2-local derivations was introduced by Šemrl. He proved that any 2-local derivation of the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H is a derivation [22]. After his works, numerous new results related to the description of local and 2-local derivations of associative algebras have appeared. For example, the papers [1, 5, 6, 17, 18, 20] are devoted to local and 2-local derivations of associative algebras.

The study of local and 2-local derivations of nonassociative algebras was initiated in the papers of Ayupov and Kudaybergenov (for the case of Lie algebras, see [7, 8]). In particular, they proved that there are no nontrivial local and 2-local derivations on semisimple finite-dimensional Lie algebras. In the paper [10] one can find examples of 2-local derivations on nilpotent Lie algebras which are not derivations. After the cited works, the study of local and 2-local derivations was continued for Leibniz algebras [9] and Jordan algebras [2], [3]. Local and 2-local automorphisms were also studied in many cases. For example, local and 2-local automorphisms on Lie algebras have been studied in [7, 11].

The variety of Malcev algebras is a generalization of the variety of Lie algebras [21]. It is closely related to other classes of nonassociative structures: it is a proper subvariety of binary Lie algebras, under the multiplication $ab-ba$ an alternative algebra is a Malcev algebra. Moreover, they have connections to various classes of algebraic

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systems such as Moufang loops, Poisson-Malcev algebras, etc. The study of generalizations of derivations of simple Malcev algebras was initiated by Filippov in [13] and continued in some papers of Kaygorodov and Popov [15, 16]. In [4] Sh.Ayupov, A.Elduque and K.Kudaybergenov obtain descriptions of local and 2-local derivations of the seven dimensional simple non-Lie Malcev algebras over fields of characteristic $\neq 2,3$.

In the present paper, we continue the study of generalizations of derivations of simple Malcev algebras. Namely, we give a matrix form of local derivations of the finite dimensional simple (non-Lie) Malcev algebra $\mathbb{M}_7$ over algebraically closed field $\mathbb{F}$ of characteristic zero, and a direct proof of the statement that every 2-local derivation of $\mathbb{M}_7$ is a derivation. As a corollary we have some description of local and 2-local derivations of complex finite dimensional semisimple binary Lie algebras.

1. Preliminaries

Malcev algebras are anticommutative algebras satisfying the following identity:

$$J(x,y,xz) = J(x,y,z)x,$$

where $J(x,y,z) = (xy)z + (yz)x + (zx)y$ is the *Jacobiator* of x,y,z.

From [19] it follows that there is only one complex finite-dimensional simple non-Lie Malcev algebra. It is the seven-dimensional algebra $\mathbb{M}_7$. In the case of the algebraically closed field $\mathbb{F}$ of characteristic zero $\mathbb{M}_7$ has a basis $\{x,y,z,x',y',z',h\}$, and the multiplication table in this basis is as follows:

$$hx = 2x,\ hy = 2y,\ hz = 2z,\ hx' = -2x',\ hy' = -2y',\ hz' = -2z',$$

$$xx' = h,\ yy' = h,\ zz' = h,$$

$$xy = 2z',\ yz = 2x',\ zx = 2y',\ x'y' = -2z,\ y'z' = -2x,\ z'x' = -2y.$$

Let $\mathbb{M}$ be an algebra. A linear map $D\colon \mathbb{M} \to \mathbb{M}$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for any two elements $x,\ y \in \mathbb{M}$. A linear map $D\colon \mathbb{M} \to \mathbb{M}$ is called an inner derivation if it is a derivation and belongs to the subalgebra of $\mathfrak{gl}(\mathbb{M})$ generated by left and right multiplication operators.

THEOREM 1.1. *Let $\mathbb{M}$ be a Malcev algebra. Then any inner derivation can be written as follows:*

$$\sum (R_{xy} + R_x R_y - R_y R_x),$$

where R_a, $a \in \mathbb{M}$, is a right multiplication operator, i.e., $R_a(b) = ab$, $b \in \mathbb{M}$. Moreover, each derivation of $\mathbb{M}_7$ is inner.

Our principal tool for the description of local and 2-local derivations of $\mathbb{M}_7$ is the following Proposition.

PROPOSITION 1.2. *A linear map $D\colon \mathbb{M}_7 \to \mathbb{M}_7$ is a derivation if and only if the matrix of D in the standard basis has the following form:*

$$\begin{pmatrix} \alpha_x & \beta_x & \gamma_x & 0 & \gamma_h & -\beta_h & 2\beta_{z'} \\ \alpha_y & \beta_y & \gamma_y & -\gamma_h & 0 & \alpha_h & -2\alpha_{z'} \\ \alpha_z & \beta_z & -\alpha_x-\beta_y & \beta_h & -\alpha_h & 0 & 2\alpha_{y'} \\ 0 & -\alpha_{y'} & -\alpha_{z'} & -\alpha_x & -\alpha_y & -\alpha_z & -2\alpha_h \\ \alpha_{y'} & 0 & -\beta_{z'} & -\beta_x & -\beta_y & -\beta_z & -2\beta_h \\ \alpha_{z'} & \beta_{z'} & 0 & -\gamma_x & -\gamma_y & \alpha_x+\beta_y & -2\gamma_h \\ \alpha_h & \beta_h & \gamma_h & -\beta_{z'} & \alpha_{z'} & -\alpha_{y'} & 0 \end{pmatrix}.$$

Here the action of D corresponds to multiplying the matrix by a column on the right.

Proof. The proof is carried out by checking the derivation property on algebra $\mathbb{M}_7$. □

2. Local derivations of $\mathbb{M}_7$

Let $\mathbb{M}$ be an algebra. A linear map $\nabla\colon \mathbb{M} \to \mathbb{M}$ is called a local derivation if for any element $x \in \mathbb{M}$ there exists a derivation $D\colon \mathbb{M} \to \mathbb{M}$ such that $\nabla(x) = D(x)$.

THEOREM 2.1. *The following conditions are valid*

1. *a linear map $\nabla : \mathbb{M}_7 \to \mathbb{M}_7$ is a local derivation if and only if the matrix of ∇ in the standard basis has the following form:*

$$\begin{pmatrix} \alpha_x & \beta_x & \gamma_x & 0 & \bar{\gamma}_h & -\bar{\beta}_h & 2\bar{\beta}_{z'} \\ \alpha_y & \beta_y & \gamma_y & -\bar{\gamma}_h & 0 & \bar{\alpha}_h & -2\bar{\alpha}_{z'} \\ \alpha_z & \beta_z & -\Lambda & \bar{\beta}_h & -\bar{\alpha}_h & 0 & 2\bar{\alpha}_{y'} \\ 0 & -\alpha_{y'} & -\alpha_{z'} & -\alpha_x & -\alpha_y & -\alpha_z & -2\alpha_h \\ \alpha_{y'} & 0 & -\beta_{z'} & -\beta_x & -\beta_y & -\beta_z & -2\beta_h \\ \alpha_{z'} & \beta_{z'} & 0 & -\gamma_x & -\gamma_y & \Lambda & -2\gamma_h \\ \alpha_h & \beta_h & \gamma_h & -\bar{\beta}_{z'} & \bar{\alpha}_{z'} & -\bar{\alpha}_{y'} & 0 \end{pmatrix}.$$

2. *the local derivation $\nabla : \mathbb{M}_7 \to \mathbb{M}_7$ is a derivation if and only if*

$$\bar{\alpha}_h = \alpha_h, \ \ \bar{\alpha}_{y'} = \alpha_{y'}, \ \ \bar{\alpha}_{z'} = \alpha_{z'},$$

$$\bar{\beta}_{z'} = \beta_{z'}, \ \ \bar{\beta}_h = \beta_h, \ \ \bar{\gamma}_h = \gamma_h$$

and

$$\Lambda = \alpha_x + \beta_y.$$

Proof. Proof of (1): Let ∇ be an arbitrary local derivation on $\mathbb{M}_7$. By the definition for any $a \in \mathbb{M}_7$ there exists a derivation D_a on $\mathbb{M}_7$ such that

$$\nabla(a) = D_a(a).$$

By Proposition 1.2, the derivation D_a has the following matrix form:

$$A^a = \begin{pmatrix} \alpha_x^a & \beta_x^a & \gamma_x^a & 0 & \gamma_h^a & -\beta_h^a & 2\beta_{z'}^a \\ \alpha_y^a & \beta_y^a & \gamma_y^a & -\gamma_h^a & 0 & \alpha_h^a & -2\alpha_{z'}^a \\ \alpha_z^a & \beta_z^a & -\alpha_x^a-\beta_y^a & \beta_h^a & -\alpha_h^a & 0 & 2\alpha_{y'}^a \\ 0 & -\alpha_{y'}^a & -\alpha_{z'}^a & -\alpha_x^a & -\alpha_y^a & -\alpha_z^a & -2\alpha_h^a \\ \alpha_{y'}^a & 0 & -\beta_{z'}^a & -\beta_x^a & -\beta_y^a & -\beta_z^a & -2\beta_h^a \\ \alpha_{z'}^a & \beta_{z'}^a & 0 & -\gamma_x^a & -\gamma_y^a & \alpha_x^a+\beta_y^a & -2\gamma_h^a \\ \alpha_h^a & \beta_h^a & \gamma_h^a & -\beta_{z'}^a & \alpha_{z'}^a & -\alpha_{y'}^a & 0 \end{pmatrix}.$$

Let A be the matrix of ∇, then by choosing subsequently $a=x$, $a=y,\dots,a=h$, and using $\nabla(a)=D_a(a)$, it is easy to see that

$$A = \begin{pmatrix} \alpha_x^x & \beta_x^y & \gamma_x^z & 0 & \gamma_h^{y'} & -\beta_h^{z'} & 2\beta_{z'}^h \\ \alpha_y^x & \beta_y^y & \gamma_y^z & -\gamma_h^{x'} & 0 & \alpha_h^{z'} & -2\alpha_{z'}^h \\ \alpha_z^x & \beta_z^y & -\alpha_x^z-\beta_y^z & \beta_h^{x'} & -\alpha_h^{y'} & 0 & 2\alpha_{y'}^h \\ 0 & -\alpha_{y'}^y & -\alpha_{z'}^z & -\alpha_x^{x'} & -\alpha_y^{y'} & -\alpha_z^{z'} & -2\alpha_h^h \\ \alpha_{y'}^x & 0 & -\beta_{z'}^z & -\beta_x^{x'} & -\beta_y^{y'} & -\beta_z^{z'} & -2\beta_h^h \\ \alpha_{z'}^x & \beta_{z'}^y & 0 & -\gamma_x^{x'} & -\gamma_y^{y'} & \alpha_x^{z'}+\beta_y^{z'} & -2\gamma_h^h \\ \alpha_h^x & \beta_h^y & \gamma_h^z & -\beta_{z'}^{x'} & \alpha_{z'}^{y'} & -\alpha_{y'}^{z'} & 0 \end{pmatrix}.$$

From $\nabla(x+y)=\nabla(x)+\nabla(y)$ we have

$$\alpha_{y'}^{x+y}=\alpha_{y'}^x, \quad \alpha_{y'}^{x+y}=\alpha_{y'}^y, \quad \text{i.e. } \alpha_{y'}^y=\alpha_{y'}^x.$$

Analogously, from $\nabla(y+z)=\nabla(y)+\nabla(z)$ we deduce

$$\beta_{z'}^{y+z}=\beta_{z'}^y, \quad \beta_{z'}^{y+z}=\beta_{z'}^z, \quad \text{i.e. } \beta_{z'}^y=\beta_{z'}^z.$$

Similarly, we obtain

$$\alpha_x^x=\alpha_x^{x'},\ \ \alpha_y^x=\alpha_y^{y'},\ \ \alpha_z^x=\alpha_z^{z'},$$

$$\alpha_{y'}^h=\alpha_{y'}^{z'},\ \ \alpha_{z'}^x=\alpha_{z'}^z,\ \ \alpha_{z'}^{y'}=\alpha_{z'}^h,$$

$$\alpha_h^x=\alpha_h^h,\ \ \alpha_h^{z'}=\alpha_h^{y'},\ \ \beta_x^y=\beta_x^{x'},$$

$$\beta_y^y=\beta_y^{y'},\ \ \beta_z^y=\beta_z^{z'},\ \ \beta_h^y=\beta_h^h$$

$$\beta_h^{z'}=\beta_h^{x'},\ \ \gamma_x^z=\gamma_x^{x'},\ \ \gamma_y^z=\gamma_y^{y'},$$

$$\gamma_h^h=\gamma_h^z,\ \ \gamma_h^{x'}=\gamma_h^{y'},\ \ \beta_{z'}^h=\beta_{z'}^{x'},$$

$$\alpha_x^z+\beta_y^z=\alpha_x^{z'}+\beta_y^{z'}.$$

By these equalities we can represent the matrix A as the sum of the following two matrices:

$$A_1 = \begin{pmatrix} 0 & \beta_x^y & \gamma_x^z & 0 & 0 & 0 & 0 \\ \alpha_y^x & 0 & \gamma_y^z & 0 & 0 & 0 & 0 \\ \alpha_z^x & \beta_z^y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_y^x & -\alpha_z^x & 0 \\ 0 & 0 & 0 & -\beta_x^y & 0 & -\beta_z^y & 0 \\ 0 & 0 & 0 & -\gamma_x^z & -\gamma_y^z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A' = \begin{pmatrix} \alpha_x^x & 0 & 0 & 0 & \gamma_h^{x'} & -\beta_h^{x'} & 2\beta_{z'}^{x'} \\ 0 & \beta_y^y & 0 & -\gamma_h^{x'} & 0 & \alpha_h^{y'} & -2\alpha_{z'}^{y'} \\ 0 & 0 & -\alpha_x^z-\beta_y^z & \beta_h^{x'} & -\alpha_h^{y'} & 0 & 2\alpha_{y'}^{z'} \\ 0 & -\alpha_{y'}^x & -\alpha_{z'}^x & -\alpha_x^x & 0 & 0 & -2\alpha_h^x \\ \alpha_{y'}^x & 0 & -\beta_{z'}^y & 0 & -\beta_y^y & 0 & -2\beta_h^y \\ \alpha_{z'}^x & \beta_{z'}^y & 0 & 0 & 0 & \alpha_x^z+\beta_y^z & -2\gamma_h^z \\ \alpha_h^x & \beta_h^y & \gamma_h^z & -\beta_{z'}^{x'} & \alpha_{z'}^{y'} & -\alpha_{y'}^{z'} & 0 \end{pmatrix}.$$

Let D_1 be the linear operator defined by the matrix A_1. By Proposition 1.2, D_1 is a derivation. It follows that $\nabla' = \nabla - D_1$ is a new local derivation with the matrix A'.

Hence, we can represent the matrix A' as the sum of the following two matrices:

$$A_2 = \begin{pmatrix} \alpha_x^x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_y^y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_x^x-\beta_y^y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_x^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_y^y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_x^x+\beta_y^y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A'' = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_h^{x'} & -\beta_h^{x'} & 2\beta_{z'}^{x'} \\ 0 & 0 & 0 & -\gamma_h^{x'} & 0 & \alpha_h^{y'} & -2\alpha_{z'}^{y'} \\ 0 & 0 & \Lambda & \beta_h^{x'} & -\alpha_h^{y'} & 0 & 2\alpha_{y'}^{z'} \\ 0 & -\alpha_{y'}^x & -\alpha_{z'}^x & 0 & 0 & 0 & -2\alpha_h^x \\ \alpha_{y'}^x & 0 & -\beta_{z'}^y & 0 & 0 & 0 & -2\beta_h^y \\ \alpha_{z'}^x & \beta_{z'}^y & 0 & 0 & 0 & -\Lambda & -2\gamma_h^z \\ \alpha_h^x & \beta_h^y & \gamma_h^z & -\beta_{z'}^{x'} & \alpha_{z'}^{y'} & -\alpha_{y'}^{z'} & 0 \end{pmatrix},$$

where $\Lambda = \alpha_x^x + \beta_y^y - \alpha_x^z - \beta_y^z$.

Let D_2 be a linear operator defined by the matrix A_2. By Proposition 1.2, D_2 is a derivation. Then $\nabla'' = \nabla' - D_2$ is a local derivation.

Let

$$\bar{\alpha}_{y'} = \alpha_{y'}^x - \alpha_{y'}^{z'}, \ \ \bar{\alpha}_{z'} = \alpha_{z'}^{y'} - \alpha_{z'}^x, \ \ \bar{\alpha}_h = \alpha_h^{y'} - \alpha_h^x,$$

$$\bar{\beta}_{z'} = \beta_{z'}^y - \beta_{z'}^{x'}, \ \ \bar{\beta}_h = \beta_h^{x'} - \beta_h^y, \ \ \bar{\gamma}_h = \gamma_h^{x'} - \gamma_h^z.$$

Then we can represent the matrix A'' as the sum of the following two matrices:

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_h^z & -\beta_h^y & 2\beta_{z'}^y \\ 0 & 0 & 0 & -\gamma_h^z & 0 & \alpha_h^x & -2\alpha_{z'}^x \\ 0 & 0 & 0 & \beta_h^y & -\alpha_h^x & 0 & 2\alpha_{y'}^x \\ 0 & -\alpha_{y'}^x & -\alpha_{z'}^x & 0 & 0 & 0 & -2\alpha_h^x \\ \alpha_{y'}^x & 0 & -\beta_{z'}^y & 0 & 0 & 0 & -2\beta_h^y \\ \alpha_{z'}^x & \beta_{z'}^y & 0 & 0 & 0 & 0 & -2\gamma_h^z \\ \alpha_h^x & \beta_h^y & \gamma_h^z & -\beta_{z'}^y & \alpha_{z'}^x & -\alpha_{y'}^x & 0 \end{pmatrix},$$

$$A''' = \begin{pmatrix} 0 & 0 & 0 & 0 & \bar{\gamma}_h & -\bar{\beta}_h & -2\bar{\beta}_{z'} \\ 0 & 0 & 0 & -\bar{\gamma}_h & 0 & \bar{\alpha}_h & -2\bar{\alpha}_{z'} \\ 0 & 0 & \Lambda & \bar{\beta}_h & -\bar{\alpha}_h & 0 & -2\bar{\alpha}_{y'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & \bar{\beta}_{z'} & \bar{\alpha}_{z'} & \bar{\alpha}_{y'} & 0 \end{pmatrix}.$$

Let D_3 be a linear operator defined by the matrix A_3. By Proposition 1.2, D_3 is a derivation. Then $\nabla''' = \nabla'' - D_3$ is a local derivation.

Now we prove that the linear operator, defined by the matrix A''' is a local derivation.

Let a be an element in $\mathbb{M}_7$. Then we can write

$$a = a_1x + a_2y + a_3z + a_4x' + a_5y' + a_6z' + a_7h,$$

for some elements a_1, a_2, a_3, a_4, a_5, a_6, a_7 in $\mathbb{F}$. Throughout of the paper let $\bar{a} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T$.

If, for each element $a \in \mathbb{M}_7$, there exists a matrix B of the form in proposition 1.2 such that

$$B\bar{a} = A'''\bar{a},$$

then the linear operator, defined by the matrix A''' is a local derivation. In other words, if, for each element $a \in \mathbb{M}_7$, the system of linear equations

$$\begin{cases} a_1\alpha_x + a_2\beta_x + a_3\gamma_x + a_5\gamma_h - a_6\beta_h + 2a_7\beta_{z'} = a_5\bar{\gamma}_h - a_6\bar{\beta}_h - 2a_7\bar{\beta}_{z'}; \\ a_1\alpha_y + a_2\beta_y + a_3\gamma_y - a_4\gamma_h + a_6\alpha_h - 2a_7\alpha_{z'} = -a_4\bar{\gamma}_h + a_6\bar{\alpha}_h - 2a_7\bar{\alpha}_{z'}; \\ a_1\alpha_z + a_2\beta_z - a_3(\alpha_x + \beta_y) + a_4\beta_h - a_5\alpha_h + 2a_7\alpha_{y'} = a_3\Lambda + a_4\bar{\beta}_h - a_5\bar{\alpha}_h - 2a_7\bar{\alpha}_{y'}; \\ -a_2\alpha_{y'} - a_3\alpha_{z'} - a_4\alpha_x - a_5\alpha_y - a_6\alpha_z - 2a_7\alpha_h = 0; \\ a_1\alpha_{y'} - a_3\beta_{z'} - a_4\beta_x - a_5\beta_y - a_6\beta_z - 2a_7\beta_h = 0; \\ a_1\alpha_{z'} + a_2\beta_{z'} - a_4\gamma_x - a_5\gamma_y + a_6(\alpha_x + \beta_y) - 2a_7\gamma_h = -a_6\Lambda; \\ a_1\alpha_h + a_2\beta_h + a_3\gamma_h - a_4\beta_{z'} + a_5\alpha_{z'} - a_6\alpha_{y'} = a_4\bar{\beta}_{z'} + a_5\bar{\alpha}_{z'} + a_6\bar{\alpha}_{y'}. \end{cases} \tag{2.1}$$

has a solution with respect to the variables

$$\alpha_x^a,\ \beta_x^a,\ \gamma_x^a,\ \alpha_y^a,\ \beta_y^a,\ \gamma_y^a,\ \alpha_z^a,\ \beta_z^a,\ \alpha_{y'}^a,\ \alpha_{z'}^a,\ \beta_{z'}^a,\ \alpha_h^a,\ \beta_h^a,\ \gamma_h^a,$$

then the linear operator, defined by the matrix A''' is a local derivation. The main matrix of this system we can write as follows

$$\begin{pmatrix}
\alpha_x & \alpha_y & \alpha_z & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_x & \beta_y & \beta_z & \beta_{z'} & \beta_h & \gamma_x & \gamma_y & \gamma_h \\
a_1 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 2a_7 & -a_6 & a_3 & 0 & a_5 \\
0 & a_1 & 0 & 0 & -2a_7 & a_6 & 0 & a_2 & 0 & 0 & 0 & 0 & a_3 & -a_4 \\
-a_3 & 0 & a_1 & 2a_7 & 0 & -a_5 & 0 & -a_3 & a_2 & 0 & a_4 & 0 & 0 & 0 \\
-a_4 & -a_5 & -a_6 & -a_2 & -a_3 & -2a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & -a_4 & -a_5 & -a_6 & -a_3 & -2a_7 & 0 & 0 & 0 \\
a_6 & 0 & 0 & 0 & a_1 & 0 & 0 & a_6 & 0 & a_2 & 0 & -a_4 & -a_5 & -2a_7 \\
0 & 0 & 0 & -a_6 & a_5 & a_1 & 0 & 0 & 0 & -a_4 & a_2 & 0 & 0 & a_3
\end{pmatrix}$$

We will need the following matrix

$$\begin{pmatrix}
\Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\
0 & 0 & 0 & 0 & -2a_7 & -a_6 & a_5 \\
0 & 0 & -2a_7 & a_6 & 0 & 0 & -a_4 \\
a_3 & -2a_7 & 0 & -a_5 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_6 & a_5 & 0 & a_4 & 0 & 0
\end{pmatrix}$$

from the right part of this system of linear equations.

We replace the 4-th row to the below of the matrices and vanish the (7,1)-th component of the first matrix:

$$\begin{pmatrix}
\alpha_x & \alpha_y & \alpha_z & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_x & \beta_y & \beta_z & \beta_{z'} & \beta_h & \gamma_x & \gamma_y & \gamma_h \\
a_1 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 2a_7 & -a_6 & a_3 & 0 & a_5 \\
0 & a_1 & 0 & 0 & -2a_7 & a_6 & 0 & a_2 & 0 & 0 & 0 & 0 & a_3 & -a_4 \\
-a_3 & 0 & a_1 & 2a_7 & 0 & -a_5 & 0 & -a_3 & a_2 & 0 & a_4 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & -a_4 & -a_5 & -a_6 & -a_3 & -2a_7 & 0 & 0 & 0 \\
a_6 & 0 & 0 & 0 & a_1 & 0 & 0 & a_6 & 0 & a_2 & 0 & -a_4 & -a_5 & -2a_7 \\
0 & 0 & 0 & -a_6 & a_5 & a_1 & 0 & 0 & 0 & -a_4 & a_2 & 0 & 0 & a_3 \\
0 & -a_5 & -a_6 & -a_2 & -a_3 & -2a_7 & \frac{a_4}{a_1}a_2 & 0 & 0 & \frac{a_4}{a_1}2a_7 & -\frac{a_4}{a_1}a_6 & \frac{a_4}{a_1}a_3 & 0 & \frac{a_4}{a_1}a_5
\end{pmatrix}$$

$$\begin{pmatrix}
\Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\
0 & 0 & 0 & 0 & -2a_7 & -a_6 & a_5 \\
0 & 0 & -2a_7 & a_6 & 0 & 0 & -a_4 \\
a_3 & -2a_7 & 0 & -a_5 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_6 & a_5 & 0 & a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\frac{a_4}{a_1}a_7 & -\frac{a_4}{a_1}a_6 & \frac{a_4}{a_1}a_5
\end{pmatrix}$$

and so on. Thus, the last 7-th row of the matrices vanishes and we have

$$\begin{pmatrix}
\alpha_x & \alpha_y & \alpha_z & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_x & \beta_y & \beta_z & \beta_{z'} & \beta_h & \gamma_x & \gamma_y & \gamma_h \\
a_1 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 2a_7 & -a_6 & a_3 & 0 & a_5 \\
0 & a_1 & 0 & 0 & -2a_7 & a_6 & 0 & a_2 & 0 & 0 & 0 & 0 & a_3 & -a_4 \\
-a_3 & 0 & a_1 & 2a_7 & 0 & -a_5 & 0 & -a_3 & a_2 & 0 & a_4 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & -a_4 & -a_5 & -a_6 & -a_3 & -2a_7 & 0 & 0 & 0 \\
a_6 & 0 & 0 & 0 & a_1 & 0 & 0 & a_6 & 0 & a_2 & 0 & -a_4 & -a_5 & -2a_7 \\
0 & 0 & 0 & -a_6 & a_5 & a_1 & 0 & 0 & 0 & -a_4 & a_2 & 0 & 0 & a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
\Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\
0 & 0 & 0 & 0 & -2a_7 & -a_6 & a_5 \\
0 & 0 & -2a_7 & a_6 & 0 & 0 & -a_4 \\
a_3 & -2a_7 & 0 & -a_5 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_6 & a_5 & 0 & a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

It is not hard to see that the appropriate system of linear equations has solution for any $a_2, a_3, a_4, a_5, a_6, a_7$ in $\mathbb{F}$.

Since x, y and z are symmetric with respect to the multiplication we have similarly calculations for $a_2 \neq 0$, $a_3 \neq 0$.

Thus, we may consider the case $a_1 = a_2 = a_3 = 0$. In this case we get

$$\begin{pmatrix}
\alpha_x & \alpha_y & \alpha_z & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_x & \beta_y & \beta_z & \beta_{z'} & \beta_h & \gamma_x & \gamma_y & \gamma_h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_7 & -a_6 & 0 & 0 & a_5 \\
0 & 0 & 0 & 0 & -2a_7 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_4 \\
0 & 0 & 0 & 2a_7 & 0 & -a_5 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\
-a_4 & -a_5 & -a_6 & 0 & 0 & -2a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_4 & -a_5 & -a_6 & 0 & -2a_7 & 0 & 0 & 0 \\
a_6 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & -a_4 & -a_5 & -2a_7 \\
0 & 0 & 0 & -a_6 & a_5 & 0 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
\Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\
0 & 0 & 0 & 0 & -2a_7 & -a_6 & a_5 \\
0 & 0 & -2a_7 & a_6 & 0 & 0 & -a_4 \\
0 & -2a_7 & 0 & -a_5 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_6 & a_5 & 0 & a_4 & 0 & 0
\end{pmatrix}$$

Now, suppose that $a_4 \neq 0$. Then, preforming some replacements, we get

$$
\begin{pmatrix}
\alpha_x & \gamma_h & \beta_x & \beta_{z'} & \gamma_x & \beta_h & \alpha_z & \beta_y & \beta_z & \alpha_{y'} & \alpha_h & \alpha_{z'} & \gamma_y & \alpha_y \\
0 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & -2a_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 2a_7 & -a_5 & 0 & 0 & 0 \\
-a_4 & 0 & 0 & 0 & 0 & 0 & -a_6 & 0 & 0 & 0 & -2a_7 & 0 & 0 & -a_5 \\
0 & 0 & -a_4 & 0 & 0 & -2a_7 & 0 & -a_5 & -a_6 & 0 & 0 & 0 & 0 & 0 \\
a_6 & -2a_7 & 0 & 0 & -a_4 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & -a_5 & 0 \\
0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0 & 0 & -a_6 & 0 & a_5 & 0 & 0 \\
0 & a_5 & 0 & 2a_7 & 0 & -a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\
0 & 0 & -2a_7 & a_6 & 0 & 0 & -a_4 \\
0 & -2a_7 & 0 & -a_5 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_6 & a_5 & 0 & a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & -2a_7 & -a_6 & a_5
\end{pmatrix}
$$

Now we replace some columns:

$$
\begin{pmatrix}
\gamma_h & \beta_h & \alpha_x & \beta_{z'} & \gamma_x & \beta_x & \alpha_z & \beta_y & \beta_z & \alpha_{y'} & \alpha_h & \alpha_{z'} & \gamma_y & \alpha_y \\
a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & -2a_7 & 0 & 0 \\
0 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_7 & -a_5 & 0 & 0 & 0 \\
0 & 0 & -a_4 & 0 & 0 & 0 & -a_6 & 0 & 0 & 0 & -2a_7 & 0 & 0 & -a_5 \\
0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0 & 0 & -a_6 & 0 & a_5 & 0 & 0 \\
-2a_7 & 0 & a_6 & 0 & -a_4 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & -a_5 & 0 \\
0 & -2a_7 & 0 & 0 & 0 & -a_4 & 0 & -a_5 & -a_6 & 0 & 0 & 0 & 0 & 0 \\
a_5 & -a_6 & 0 & 2a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\
0 & 0 & -2a_7 & a_6 & 0 & 0 & -a_4 \\
0 & -2a_7 & 0 & -a_5 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_6 & a_5 & 0 & a_4 & 0 & 0 \\
-a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2a_7 & -a_6 & a_5
\end{pmatrix}
$$

It is not difficult to see that the last 7-th row of the matrix can be vanished. It is not hard to see that the appropriate system of linear equations has solution for any a_5, a_6, a_7 in $\mathbb{F}$.

Since x', y' and z' are symmetric in the table of multiplication we have similarly calculations for $a_5 \neq 0$, $a_6 \neq 0$.

Thus, we may consider the case $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$. In this case we get

$$\begin{pmatrix} \alpha_x & \alpha_y & \alpha_z & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_x & \beta_y & \beta_z & \beta_{z'} & \beta_h & \gamma_x & \gamma_y & \gamma_h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2a_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \Lambda & \alpha_{y'} & \alpha_{z'} & \alpha_h & \beta_{z'} & \beta_h & \gamma_h \\ 0 & 0 & 0 & 0 & -2a_7 & 0 & 0 \\ 0 & 0 & -2a_7 & 0 & 0 & 0 & 0 \\ 0 & -2a_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

that is, we have

$$\begin{cases} 2a_7\beta_{z'} = -2a_7\bar{\beta}_{z'}; \\ -2a_7\alpha_{z'} = -2a_7\bar{\alpha}_{z'}; \\ 2a_7\alpha_{y'} = -2a_7\bar{\alpha}_{y'}; \\ -2a_7\alpha_h = 0; \\ -2a_7\beta_h = 0; \\ -2a_7\gamma_h = 0. \end{cases}$$

The last system of linear equation always has a solution. Hence, the system of linear equation (2.1) always has a solution. Therefore, the linear operator, defined by the matrix A''' is a local derivation.

Item (2) of the theorem follows by Proposition 1.2. This completes the proof. □

EXAMPLE 2.2. Let ∇ be a linear operator on $\mathbb{M}_7$ with the nonzero matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \bar{\gamma}_h & -\bar{\beta}_h & -2\bar{\beta}_{z'} \\ 0 & 0 & 0 & -\bar{\gamma}_h & 0 & \bar{\alpha}_h & -2\bar{\alpha}_{z'} \\ 0 & 0 & \Lambda & \bar{\beta}_h & -\bar{\alpha}_h & 0 & -2\bar{\alpha}_{y'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & \bar{\beta}_{z'} & \bar{\alpha}_{z'} & \bar{\alpha}_{y'} & 0 \end{pmatrix},$$

where $\bar{\alpha}_{y'}$, $\bar{\alpha}_{z'}$, $\bar{\alpha}_h$, $\bar{\beta}_{z'}$, $\bar{\beta}_h$, $\bar{\gamma}_h$, Λ are elements in the field $\mathbb{F}$. Then, by Theorem 2.1, ∇ is a local derivation which is not a derivation.

For example, the linear operator with the nonzero matrix

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

is a local derivation which is not a derivation.

3. 2-Local derivations of $\mathbb{M}_7$

Let $\mathbb{M}$ be an algebra. A (not necessary linear) map $\Delta\colon \mathbb{M}\to\mathbb{M}$ is called a 2-local derivation if for any two elements x, $y\in\mathbb{M}$ there exists a derivation $D_{x,y}\colon \mathbb{M}\to\mathbb{M}$ such that $\Delta(x)=D_{x,y}(x)$, $\Delta(y)=D_{x,y}(y)$. The following theorem was proved by Sh.Ayupov, A.Elduque and K.Kudaybergenov in [4]. Here we give a direct proof of this theorem.

THEOREM 3.1. *Each 2-local derivation of $\mathbb{M}_7$ is a derivation.*

Proof. Let $\triangle$ be an arbitrary 2-local derivation of $\mathbb{M}_7$. By definition, for every a, $b\in\mathbb{M}_7$ there exists a derivation $D_{a,b}$ of $\mathbb{M}_7$ such that

$$\triangle(a)=D_{a,b}(a),\quad \triangle(b)=D_{a,b}(b).$$

By Proposition 1.2, the derivation $D_{a,b}$ has the following matrix form:

$$A^{a,b}=\begin{pmatrix} \alpha_x^{ab} & \beta_x^{ab} & \gamma_x^{ab} & 0 & \gamma_h^{ab} & -\beta_h^{ab} & 2\beta_{z'}^{ab}\\ \alpha_y^{ab} & \beta_y^{ab} & \gamma_y^{ab} & -\gamma_h^{ab} & 0 & \alpha_h^{ab} & -2\alpha_{z'}^{ab}\\ \alpha_z^{ab} & \beta_z^{ab} & -\alpha_x^{ab}-\beta_y^{ab} & \beta_h^{ab} & -\alpha_h^{ab} & 0 & 2\alpha_{y'}^{ab}\\ 0 & -\alpha_{y'}^{ab} & -\alpha_{z'}^{ab} & -\alpha_x^{ab} & -\alpha_y^{ab} & -\alpha_z^{ab} & -2\alpha_h^{ab}\\ \alpha_{y'}^{ab} & 0 & -\beta_{z'}^{ab} & -\beta_x^{ab} & -\beta_y^{ab} & -\beta_z^{ab} & -2\beta_h^{ab}\\ \alpha_{z'}^{ab} & \beta_{z'}^{ab} & 0 & -\gamma_x^{ab} & -\gamma_y^{ab} & \alpha_x^{ab}+\beta_y^{ab} & -2\gamma_h^{ab}\\ \alpha_h^{ab} & \beta_h^{ab} & \gamma_h^{ab} & -\beta_{z'}^{ab} & \alpha_{z'}^{ab} & -\alpha_{y'}^{ab} & 0 \end{pmatrix}.$$

Let $a=\lambda_x x+\lambda_y y+\lambda_z z+\lambda_{x'}x'+\lambda_{y'}y'+\lambda_{z'}z'+\lambda_h h$ be an arbitrary element from $\mathbb{M}_7$. For every $v\in\mathbb{M}_7$ there exists a derivation $D_{v,a}$ such that

$$\triangle(v)=D_{v,a}(v),\quad \triangle(a)=D_{v,a}(a).$$

Then from

$$D_{h,v}(h)=D_{h,a}(h),\ v\in\mathbb{M}_7$$

it follows that

$$\beta_{z'}^{hv}x-\alpha_{z'}^{hv}y+\alpha_{y'}^{hv}z-\alpha_h^{hv}x'-\beta_h^{hv}y'-\gamma_h^{hv}z'$$
$$=\beta_{z'}^{ha}x-\alpha_{z'}^{ha}y+\alpha_{y'}^{ha}z-\alpha_h^{ha}x'-\beta_h^{ha}y'-\gamma_h^{ha}z'.$$

Hence,

$$\beta_{z'}^{hv} = \beta_{z'}^{ha}, \quad \alpha_{z'}^{hv} = \alpha_{z'}^{ha}, \quad \alpha_{y'}^{hv} = \alpha_{y'}^{ha},$$
$$\alpha_h^{hv} = \alpha_h^{ha}, \quad \beta_h^{hv} = \beta_h^{ha}, \quad \gamma_h^{hv} = \gamma_h^{ha}.$$

Then we can write

$$A^{h,a} = \begin{pmatrix} \alpha_x^{ha} & \beta_x^{ha} & \gamma_x^{ha} & 0 & \gamma_h^{ha} & -\beta_h^{ha} & 2\beta_{z'}^{hv} \\ \alpha_y^{ha} & \beta_y^{ha} & \gamma_y^{ha} & -\gamma_h^{ha} & 0 & \alpha_h^{ha} & -2\alpha_{z'}^{hv} \\ \alpha_z^{ha} & \beta_z^{ha} & -\alpha_x^{ha}-\beta_y^{ha} & \beta_h^{ha} & -\alpha_h^{ha} & 0 & 2\alpha_{y'}^{hv} \\ 0 & -\alpha_{y'}^{ha} & -\alpha_{z'}^{ha} & -\alpha_x^{ha} & -\alpha_y^{ha} & -\alpha_z^{ha} & -2\alpha_h^{hv} \\ \alpha_{y'}^{ha} & 0 & -\beta_{z'}^{ha} & -\beta_x^{ha} & -\beta_y^{ha} & -\beta_z^{ha} & -2\beta_h^{hv} \\ \alpha_{z'}^{ha} & \beta_{z'}^{ha} & 0 & -\gamma_x^{ha} & -\gamma_y^{ha} & \alpha_x^{ha}+\beta_y^{ha} & -2\gamma_h^{hv} \\ \alpha_h^{hv} & \beta_h^{hv} & \gamma_h^{hv} & -\beta_{z'}^{hv} & \alpha_{z'}^{hv} & -\alpha_{y'}^{hv} & 0 \end{pmatrix}.$$

Hence,

$$\triangle(a) = D_{h,a}(a) = \mu_x^{ha}x + \mu_y^{ha}y + \mu_z^{ha}z + \mu_{x'}^{ha}x' + \mu_{y'}^{ha}y'\mu_{z'}^{ha}z'$$
$$+(\alpha_h^{hv}\lambda_x + \beta_h^{hv}\lambda_y + \gamma_h^{hv}\lambda_z - \beta_{z'}^{hv}\lambda_{x'} + \alpha_{z'}^{hv}\lambda_{y'} - \alpha_{y'}^{hv}\lambda_{z'})h,$$

for some elements μ_x^{ha}, μ_y^{ha}, μ_z^{ha}, $\mu_{x'}^{ha}$, $\mu_{y'}^{ha}$, $\mu_{z'}^{ha} \in \mathbb{C}$. Similarly, from

$$D_{x,v}(x) = D_{x,a}(x), \ v \in \mathbb{M}_7$$

it follows that

$$\alpha_x^{xv} = \alpha_x^{xa}, \quad \alpha_y^{xv} = \alpha_y^{xa}, \quad \alpha_z^{xv} = \alpha_z^{xa},$$
$$\alpha_{y'}^{xv} = \alpha_{y'}^{xa}, \quad \alpha_{z'}^{xv} = \alpha_{z'}^{xa}, \quad \alpha_h^{xv} = \alpha_h^{xa}.$$

Then we can write

$$\triangle(a) = D_{xa}(a) = \mu_x^{xa}x + \mu_y^{xa}y + \mu_z^{xa}z$$
$$+(-\alpha_{y'}^{xv}\lambda_y - \alpha_{z'}^{xv}\lambda_z - \alpha_x^{xv}\lambda_{x'} - \alpha_y^{xv}\lambda_{y'} - \alpha_z^{xv}\lambda_{z'} - 2\alpha_h^{xv}\lambda_h)x'$$
$$+\mu_{y'}^{xa}y' + \mu_{z'}^{xa}z' + \mu_h^{xa}h$$

for some elements μ_x^{xa}, μ_y^{xa}, μ_z^{xa}, $\mu_{y'}^{xa}$, $\mu_{z'}^{xa}$, $\mu_h^{xa} \in \mathbb{C}$.

From

$$D_{y,v}(y) = D_{y,a}(y), \ v \in \mathbb{M}_7$$

it follows that

$$\beta_x^{yv} = \beta_x^{ya}, \quad \beta_y^{yv} = \beta_y^{ya}, \quad \beta_z^{yv} = \beta_z^{ya},$$
$$\alpha_{y'}^{yv} = \alpha_{y'}^{ya}, \quad \beta_{z'}^{yv} = \beta_{z'}^{ya}, \quad \beta_h^{yv} = \beta_h^{ya}.$$

Then we can write

$$\triangle(a) = D_{ya}(a) = \mu_x^{ya}x + \mu_y^{ya}y + \mu_z^{ya}z + \mu_{x'}^{ya}x'$$
$$+(\alpha_{y'}^{yv}\lambda_x - \beta_{z'}^{yv}\lambda_z - \beta_x^{yv}\lambda_{x'} - \beta_y^{yv}\lambda_{y'} - \beta_z^{yv}\lambda_{z'} - 2\beta_h^{yv}\lambda_h)y'$$
$$+\mu_{z'}^{ya}z' + \mu_h^{ya}h$$

for some elements μ_x^{ya}, μ_y^{ya}, μ_z^{ya}, $\mu_{x'}^{ya}$, $\mu_{z'}^{ya}$, $\mu_h^{ya} \in \mathbb{C}$.

From

$$D_{z,v}(z) = D_{z,a}(z),\ v \in \mathbb{M}_7$$

it follows that

$$\gamma_x^{zv} = \gamma_x^{za}, \quad \gamma_y^{zv} = \gamma_y^{za}, \quad \alpha_x^{zv} + \beta_y^{zv} = \alpha_x^{za} + \beta_y^{za},$$
$$\alpha_{z'}^{zv} = \alpha_{z'}^{za}, \quad \beta_{z'}^{zv} = \beta_{z'}^{za}, \quad \gamma_h^{zv} = \gamma_h^{za}.$$

Then we can write

$$\triangle(a) = D_{za}(a) = \mu_x^{za} x + \mu_y^{za} y + \mu_z^{za} z + \mu_{x'}^{za} x' + \mu_{y'}^{za} y'$$
$$+(\alpha_{z'}^{zv}\lambda_x + \beta_{z'}^{zv}\lambda_y - \gamma_x^{zv}\lambda_{x'} - \gamma_y^{zv}\lambda_{y'} + (\alpha_x^{zv} + \beta_y^{zv})\lambda_{z'} - 2\gamma_h^{zv}\lambda_h)z'$$
$$+\mu_h^{za} h$$

for some elements μ_x^{za}, μ_y^{za}, μ_z^{za}, $\mu_{x'}^{za}$, $\mu_{y'}^{za}$, $\mu_h^{za} \in \mathbb{C}$.

From

$$D_{x',v}(x') = D_{x',a}(x'),\ v \in \mathbb{M}_7$$

it follows that

$$\gamma_h^{x'v} = \gamma_h^{x'a}, \quad \beta_h^{x'v} = \beta_h^{x'a}, \quad \alpha_x^{x'v} = \alpha_x^{x'a},$$
$$\beta_x^{x'v} = \beta_x^{x'a}, \quad \gamma_x^{x'v} = \gamma_x^{x'a}, \quad \beta_{z'}^{x'v} = \beta_{z'}^{x'a}.$$

Then we can write

$$\triangle(a) = D_{x'a}(a) = (\alpha_x^{x'v}\lambda_x + \beta_x^{x'v}\lambda_y - \gamma_x^{x'v}\lambda_z + \gamma_h^{x'v}\lambda_{y'} - \beta_h^{x'v}\lambda_{z'} + 2\beta_{z'}^{x'v}\lambda_h)x$$
$$+\mu_y^{x'a} y + \mu_z^{x'a} z + +\mu_{x'}^{x'a} x' + \mu_{y'}^{x'a} y' + \mu_{z'}^{x'a} z' + \mu_h^{x'a} h$$

for some elements $\mu_y^{x'a}$, $\mu_z^{x'a}$, $\mu_{x'}^{x'a}$, $\mu_{y'}^{x'a}$, $\mu_{z'}^{x'a}$, $\mu_h^{x'a} \in \mathbb{C}$.

From

$$D_{y',v}(y') = D_{y',a}(y'),\ v \in \mathbb{M}_7$$

it follows that

$$\gamma_h^{y'v} = \gamma_h^{y'a}, \quad \alpha_h^{y'v} = \alpha_h^{y'a}, \quad \alpha_y^{y'v} = \alpha_y^{y'a},$$
$$\beta_y^{y'v} = \beta_y^{y'a}, \quad \gamma_y^{y'v} = \gamma_y^{y'a}, \quad \alpha_{z'}^{y'v} = \alpha_{z'}^{y'a}.$$

Then we can write

$$\triangle(a) = D_{y'a}(a) = \mu_x^{y'a} x + (\alpha_y^{y'v}\lambda_x + \beta_y^{y'v}\lambda_y + \gamma_y^{y'v}\lambda_z - \gamma_h^{y'v}\lambda_{x'} + \alpha_h^{y'v}\lambda_{z'} - 2\alpha_{z'}^{y'v}\lambda_h)y$$
$$+\mu_z^{y'a} z + +\mu_{x'}^{y'a} x' + \mu_{y'}^{y'a} y' + \mu_{z'}^{y'a} z' + \mu_h^{y'a} h$$

for some elements $\mu_x^{y'a}$, $\mu_z^{y'a}$, $\mu_{x'}^{y'a}$, $\mu_{y'}^{y'a}$, $\mu_{z'}^{y'a}$, $\mu_h^{y'a} \in \mathbb{C}$.

From

$$D_{z',v}(z') = D_{z',a}(z'),\ v \in \mathbb{M}_7$$

it follows that

$$\beta_h^{z'v} = \beta_h^{z'a}, \quad \alpha_h^{z'v} = \alpha_h^{z'a}, \quad \alpha_z^{z'v} = \alpha_z^{z'a},$$
$$\beta_z^{z'v} = \beta_z^{z'a}, \quad \alpha_x^{z'v} + \beta_y^{z'v} = \alpha_x^{z'a} + \beta_y^{z'a}, \quad \alpha_{y'}^{z'v} = \alpha_{y'}^{z'a}.$$

Then we can write

$$\triangle(a) = D_{z'a}(a) = \mu_x^{z'a}x + \mu_y^{z'a}y$$
$$+(\alpha_z^{z'v}\lambda_x + \beta_z^{z'v}\lambda_y - (\alpha_x^{z'v} + \beta_y^{z'v})\lambda_z - \beta_h^{z'v}\lambda_{x'} - \alpha_h^{z'v}\lambda_{y'} + 2\alpha_{y'}^{z'v}\lambda_h)z$$
$$+\mu_{x'}^{z'a}x' + \mu_{y'}^{z'a}y' + \mu_{z'}^{z'a}z' + \mu_h^{z'a}h$$

for some elements $\mu_x^{z'a}$, $\mu_y^{z'a}$, $\mu_{x'}^{z'a}$, $\mu_{y'}^{z'a}$, $\mu_{z'}^{z'a}$, $\mu_h^{z'a} \in \mathbb{C}$. Hence,

$$\triangle(a) = D_{h,a}(a) = D_{x,a}(a) = D_{y,a}(a) = D_{z,a}(a) = D_{x'a}(a) = D_{y',a}(a) = D_{z',a}(a)$$
$$= (\alpha_x^{x'v_1}\lambda_x + \beta_x^{x'v_1}\lambda_y - \gamma_x^{x'v_1}\lambda_z + \gamma_h^{x'v_1}\lambda_{y'} - \beta_h^{x'v_1}\lambda_{z'} + 2\beta_{z'}^{x'v_1}\lambda_h)x$$
$$+(\alpha_y^{y'v_2}\lambda_x + \beta_y^{y'v_2}\lambda_y + \gamma_y^{y'v_2}\lambda_z - \gamma_h^{y'v_2}\lambda_{x'} + \alpha_h^{y'v_2}\lambda_{z'} - 2\alpha_{z'}^{y'v_2}\lambda_h)y$$
$$+(\alpha_z^{z'v_3}\lambda_x + \beta_z^{z'v_3}\lambda_y - (\alpha_x^{z'v_3} + \beta_y^{z'v_3})\lambda_z - \beta_h^{z'v_3}\lambda_{x'} - \alpha_h^{z'v_3}\lambda_{y'} + 2\alpha_{y'}^{z'v_3}\lambda_h)z$$
$$+(-\alpha_{y'}^{xv_4}\lambda_y - \alpha_{z'}^{xv_4}\lambda_z - \alpha_x^{xv_4}\lambda_{x'} - \alpha_y^{xv_4}\lambda_{y'} - \alpha_z^{xv_4}\lambda_{z'} - 2\alpha_h^{xv_4}\lambda_h)x'$$
$$+(\alpha_{y'}^{yv_5}\lambda_x - \beta_{z'}^{yv_5}\lambda_z - \beta_x^{yv_5}\lambda_{x'} - \beta_y^{yv_5}\lambda_{y'} - \beta_z^{yv_5}\lambda_{z'} - 2\beta_h^{yv_5}\lambda_h)y'$$
$$+(\alpha_{z'}^{zv_6}\lambda_x + \beta_{z'}^{zv_6}\lambda_y - \gamma_x^{zv_6}\lambda_{x'} - \gamma_y^{zv_6}\lambda_{y'} + (\alpha_x^{zv_6} + \beta_y^{zv})\lambda_{z'} - 2\gamma_h^{zv_6}\lambda_h)z'$$
$$+(\alpha_h^{hv_7}\lambda_x + \beta_h^{hv_7}\lambda_y + \gamma_h^{hv_7}\lambda_z - \beta_{z'}^{hv_7}\lambda_{x'} + \alpha_{z'}^{hv_7}\lambda_{y'} - \alpha_{y'}^{hv_7}\lambda_{z'})h \tag{1.1}$$

for any v_1, v_2, v_3, v_4, v_5, v_6, $v_7 \in \mathbb{M}_7$. Note that the components in this last sum do not depend on the element a. Therefore the map $\triangle$ is linear and it is a local derivation.

Now, by Theorem 2.1, the linear map $\triangle$ has the following matrix

$$A = \begin{pmatrix} \alpha_x & \beta_x & \gamma_x & 0 & \bar{\gamma}_h & -\bar{\beta}_h & 2\bar{\beta}_{z'} \\ \alpha_y & \beta_y & \gamma_y & -\bar{\gamma}_h & 0 & \bar{\alpha}_h & -2\bar{\alpha}_{z'} \\ \alpha_z & \beta_z & -\Lambda & \bar{\beta}_h & -\bar{\alpha}_h & 0 & 2\bar{\alpha}_{y'} \\ 0 & -\alpha_{y'} & -\alpha_{z'} & -\alpha_x & -\alpha_y & -\alpha_z & -2\alpha_h \\ \alpha_{y'} & 0 & -\beta_{z'} & -\beta_x & -\beta_y & -\beta_z & -2\beta_h \\ \alpha_{z'} & \beta_{z'} & 0 & -\gamma_x & -\gamma_y & \Lambda & -2\gamma_h \\ \alpha_h & \beta_h & \gamma_h & -\bar{\beta}_{z'} & \bar{\alpha}_{z'} & -\bar{\alpha}_{y'} & 0 \end{pmatrix}.$$

From

$$A^{z,x'}\bar{z} = A\bar{z}, \quad A^{z,x'}\bar{x'} = A\bar{x'}.$$

it follows that

$$\gamma_h = \gamma_h^{z,x'} = \bar{\gamma}_h,$$

i.e.,

$$\gamma_h = \bar{\gamma}_h.$$

Similarly, from

$$A^{y,x'}\,\overline{y} = A\,\overline{y}, \quad A^{y,x'}\,\overline{x'} = A\overline{x'}.$$

it follows that

$$\beta_h = \beta_h^{y,x'} = \bar{\beta}_h,$$

i.e.,

$$\beta_h = \bar{\beta}_h.$$

and so on. Thus, we get

$$\bar{\alpha}_h = \alpha_h, \quad \bar{\alpha}_{y'} = \alpha_{y'}, \quad \bar{\alpha}_{z'} = \alpha_{z'},$$

$$\bar{\beta}_{z'} = \beta_{z'}, \quad \bar{\beta}_h = \beta_h, \quad \bar{\gamma}_h = \gamma_h.$$

Hence, the linear map $\triangle$ has the following matrix

$$A = \begin{pmatrix} \alpha_x & \beta_x & \gamma_x & 0 & \gamma_h & -\beta_h & 2\beta_{z'} \\ \alpha_y & \beta_y & \gamma_y & -\gamma_h & 0 & \alpha_h & -2\alpha_{z'} \\ \alpha_z & \beta_z & -\Lambda & \beta_h & -\alpha_h & 0 & 2\alpha_{y'} \\ 0 & -\alpha_{y'} & -\alpha_{z'} & -\alpha_x & -\alpha_y & -\alpha_z & -2\alpha_h \\ \alpha_{y'} & 0 & -\beta_{z'} & -\beta_x & -\beta_y & -\beta_z & -2\beta_h \\ \alpha_{z'} & \beta_{z'} & 0 & -\gamma_x & -\gamma_y & \Lambda & -2\gamma_h \\ \alpha_h & \beta_h & \gamma_h & -\beta_{z'} & \alpha_{z'} & -\alpha_{y'} & 0 \end{pmatrix}.$$

Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Lambda' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\Lambda' = \Lambda - (\alpha_x + \beta_y)$. Let D_1 be the linear operator defined by the matrix $A - A_1$. By Proposition 1.2, D_1 is a derivation. From this it follows that $\triangle' = \triangle - D_1$ is a new local derivation with the matrix A_1.

We take

$$\Delta(z) = A_1\,\overline{z} = A^{z,x}\,\overline{z} = A^{z,y}\,\overline{z},$$

$$\Delta(x) = A_1\,\overline{x} = A^{z,x}\,\overline{x}, \quad \Delta(y) = A_1\,\overline{y} = A^{z,y}\,\overline{y}.$$

From this it follows that

$$\Lambda = \alpha_x^{x,z} + \beta_y^{x,z} = \alpha_x^{y,z} + \beta_y^{y,z}$$

and

$$\alpha_x^{x,z} = 0, \beta_y^{y,z} = 0.$$

Hence,

$$\Lambda = \beta_y^{x,z} = \alpha_x^{y,z}. \tag{1}$$

Now, take

$$\Delta(x+y) = \Delta(x) + \Delta(y).$$

Then

$$\alpha_y^{x+y,x} + \beta_y^{x+y,x} = \alpha_y^{x,z} + \beta_y^{x,z}, \quad \alpha_y^{x,z} = \alpha_y^{x+y,x} = 0, \quad \beta_y^{x+y,x} = \beta_y^{x,z}.$$

$$\alpha_y^{x+y,x} + \beta_y^{x+y,x} = \alpha_y^{x,y} + \beta_y^{x,y}, \quad \alpha_y^{x,y} = \alpha_y^{x+y,x} = 0, \quad \beta_y^{x+y,x} = \beta_y^{x,y} = 0.$$

Hence,

$$\beta_y^{x,z} = \beta_y^{x+y,x} = \beta_y^{x,y} = 0.$$

By (1) we get

$$\Lambda = 0.$$

Therefore, $\Lambda = \alpha_x + \beta_y$ and $\triangle$ is a derivation. This completes the proof. □

4. Local and 2-local derivations of binary Lie algebras

The variety of binary Lie algebras was introduced in [21] and it is defined by the following property: *each 2-generated subalgebra of a binary Lie algebra is a Lie algebra*. It is known that the variety of Malcev and the variety of anticommutative $\mathfrak{CD}$-algebras are proper subvarieties of the variety of binary Lie algebras. On the other hand, it was proved that each complex finite-dimensional semisimple binary Lie algebra is Malcev [12] and each complex finite-dimensional semisimple Malcev algebra is a direct sum of some simple Lie algebras and some copies of $\mathbb{M}_7$ [19]. It was proved that every 2-local derivation of a complex finite-dimensional Lie algebra or $\mathbb{M}_7$ is a derivation (see, [10] and Theorem 2.1). Hence, we have the following result.

COROLLARY 4.1. *Let Ξ be a 2-local derivation of a complex finite-dimensional semisimple binary Lie algebra. Then Ξ is a derivation.*

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