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ON A STEVIĆ–SHARMA TYPE OPERATOR FROM $Q_K(p,q)$ SPACES TO BLOCH–TYPE SPACES

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Abstract. The aim of this paper is to investigate the boundedness and compactness of a Stević-Sharma type operator $T^n_{\psi_1,\psi_2,\varphi}$ from $Q_K(p,q)$ and $Q_{K,0}(p,q)$ spaces to Bloch-type spaces and little Bloch-type spaces.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$. Denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A positive continuous function ϕ on [0,1) is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0,1)$ such that (see [21])

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \ \lim_{r\to 1}\frac{\phi(r)}{(1-r)^s} = 0;$$
$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \ \lim_{r\to 1}\frac{\phi(r)}{(1-r)^t} = \infty.$$

Let $\mu : \mathbb{D} \to (0,+\infty)$ be a function that is normal and radial, i.e., $\mu(z) = \mu(|z|)$. An $f \in H(\mathbb{D})$ is said to belong to Bloch-type space, denoted by $\mathscr{B}_\mu$, if

$$\|f\|_{\mathscr{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

$\mathscr{B}_\mu$ is a Banach space under the above norm. When $\mu(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, the induced space becomes the α-Bloch space $\mathscr{B}^\alpha$. In particular, if $\alpha = 1$, then we get the

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classical Bloch space $\mathscr{B}$. The little Bloch-type space $\mathscr{B}_{\mu,0}$ consists of those functions f in $\mathscr{B}_\mu$ satisfying

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0,$$

and it can be shown that $\mathscr{B}_{\mu,0}$ is a closed subspace of $\mathscr{B}_\mu$. Some results on the Bloch-type spaces and operators on them can be found, for instance, in [1, 7, 8, 10, 11, 12, 15, 16, 19, 25, 29, 31, 33, 34, 35, 37, 38].

Let dA denote the normalized Lebesgue area measure in $\mathbb{D}$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function and $g(z,a)$ the Green function with logarithmic singularity at a, i.e., $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$. For $p > 0$, $q > -2$, $Q_K(p,q)$ space consists of those $f \in H(\mathbb{D})$ such that (see, for example, [17, 32])

$$\|f\|^p_{Q_K(p,q)} = |f(0)| + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) < \infty.$$

Under the norm $\|\cdot\|_{Q_K(p,q)}$, $Q_K(p,q)$ is a Banach space when $p \geqslant 1$. An $f \in H(\mathbb{D})$ is said to belong to $Q_{K,0}(p,q)$ space if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0.$$

Throughout the paper we assume that (see [32])

$$\int_0^1 (1-r^2)^q K(-\log r) r dr < \infty,$$

since otherwise $Q_K(p,q)$ consists only of constant functions. Recently, many researchers have studied various concrete operators from or to $Q_K(p,q)$ space. For instance, Kotilainen in [6] characterized the boundedness and compactness of composition operator between $\mathscr{B}^\alpha$ and $Q_K(p,q)$ spaces. Pan in [19] studied the boundedness and compactness of an integral-type operator from $Q_K(p,q)$ and $Q_{K,0}(p,q)$ spaces to $\mathscr{B}^\alpha$ and $\mathscr{B}^\alpha_0$. Some more related results can be found (see, e.g., [7, 8, 13, 20, 33, 34, 35] and the references therein).

Let $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, then φ and ψ induce a composition operator $C_\varphi f = f\circ\varphi$ and a multiplication operator $M_\psi f = \psi \cdot f$, respectively, where $f \in H(\mathbb{D})$. The product of these two operators is known as the weighted composition operator $W_{\psi,\varphi} f = \psi \cdot f\circ\varphi$ for $f \in H(\mathbb{D})$, which has been extensively studied. The differentiation operator D, which is defined by $(Df)(z) = f'(z)$, $f \in H(\mathbb{D})$, plays an important role in operator theory and dynamical system. The first papers on product-type operators including the differentiation operator dealt with the operators DC_φ and $C_\varphi D$ (see, for example, [5, 9, 12, 18, 23, 24, 26]). During recent years, there has been a great interest in studying the operator-theoretic properties of their product-type operators between various analytic function spaces (see, for instance, [1, 3, 4, 7, 10, 11, 12, 14, 15, 23, 25, 27, 28, 29, 34, 39]).

In [29], Stević et al. introduced the following operator:

$$(T^n_{\psi_1,\psi_2,\varphi}f)(z) = \psi_1(z)f^{(n)}(\varphi(z)) + \psi_2(z)f^{(n+1)}(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $n \in \mathbb{N}_0$, $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Note that when $n = 0$, the operator $T_{\psi_1,\psi_2,\varphi}$ is called the Stević-Sharma operator, which was introduced by Stević et al. in [27, 28] and has aroused great interest of experts recently (see, [3, 4, 14, 30, 36] and also related references therein). Moreover, we can get the general product-type operators by taking some specific choices of the involving symbols (see [29]). Quite recently, Abbasi and Zhu in [1] studied the boundedness, compactness and essential norm of $T^n_{\psi_1,\psi_2,\varphi}$ from the (little) Bloch space into Zygmund-type spaces. In [15], the boundedness and compactness of $T^n_{\psi_1,\psi_2,\varphi}$ from the logarithmic Bloch spaces to Zygmund-type spaces was investigated by Liu and Yu. In [39] Zhu et al. characterized the boundedness, compactness and essential norm of $T^n_{\psi_1,\psi_2,\varphi}$ on the Zygmund space.

Inspired by the above results, the purpose of the paper is to study the boundedness and compactness of the Stević-Sharma type operator $T^n_{\psi_1,\psi_2,\varphi}$ from $Q_K(p,q)$ and $Q_{K,0}(p,q)$ spaces to Bloch-type spaces and little Bloch-type spaces.

Throughout the paper we use the letter C to denote a positive constant whose value may change at each occurrence. The notation abbreviation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y means that there is a positive constant C such that $X \leqslant CY$. Moreover, if both $X \lesssim Y$ and $Y \lesssim X$ hold, then one says that $X \approx Y$.

2. Auxiliary results

In this section, we state several auxiliary results which will be used in the proofs of the main results.

LEMMA 1. [22] *Let $f \in \mathscr{B}^\alpha$, $0 < \alpha < \infty$. Then*

$$|f(z)| \lesssim \begin{cases} \|f\|_{\mathscr{B}^\alpha}, & 0 < \alpha < 1, \\ \|f\|_{\mathscr{B}} \ln \frac{e}{1-|z|^2}, & \alpha = 1, \\ \frac{1}{(1-|z|^2)^{\alpha-1}} \|f\|_{\mathscr{B}^\alpha}, & \alpha > 1. \end{cases}$$

The following lemma is well-known (see [38]).

LEMMA 2. *Suppose $\alpha > 0$, $n \in \mathbb{N}$ and $f \in \mathscr{B}^\alpha$. Then*

$$\|f\|_{\mathscr{B}^\alpha} \approx |f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)|.$$

LEMMA 3. [32] *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. For $f \in Q_K(p,q)$, we have $f \in \mathscr{B}^{\frac{q+2}{p}}$ and*

$$\|f\|_{\mathscr{B}^{\frac{q+2}{p}}} \leqslant \|f\|_{Q_K(p,q)}.$$

The following lemma can be deduced by the standard arguments in [2, Proposition 3.11], consequently we omit the details.

LEMMA 4. *Let $p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Then the operator $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathscr{B}_\mu$ is compact if and only if $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q)$ (or $Q_{K,0}(p,q)$) $\to \mathscr{B}_\mu$ is bounded and for each sequence $\{f_k\}_{k\in\mathbb{N}}$ which is bounded in $Q_K(p,q)$ (or $Q_{K,0}(p,q)$) and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, we have $\|T^n_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_\mu}\to 0$ as $k\to\infty$.*

By the same method as [16, Lemma 1], we can get the lemma below.

LEMMA 5. *A closed set K in $\mathscr{B}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f'(z)|=0.$$

LEMMA 6. [37] *Fix $0<\alpha<1$ and let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathscr{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Then we have*

$$\lim_{k\to\infty}\sup_{z\in\mathbb{D}}|f_k(z)|=0.$$

3. Main results

In this section, our main results are stated and proved. We first give some characterizations of the boundedness and compactness of $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q)\to\mathscr{B}_\mu$ for $n\in\mathbb{N}$.

THEOREM 1. *Let $\psi_1,\psi_2\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}\left(\log\frac{1}{1-r}\right)^{\chi_{-1}(q)} r\,dr<\infty, \tag{1}$$

where $\chi_O(x)$ denotes the characteristic function of the set O. Then for each $n\in\mathbb{N}$, the following statements are equivalent.
(i) *The operator $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q)\to\mathscr{B}_\mu$ is bounded.*
(ii) *The operator $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q)\to\mathscr{B}_\mu$ is bounded.*
(iii)

$$M_1:=\sup_{z\in\mathbb{D}}\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}}<\infty, \tag{2}$$

$$M_2:=\sup_{z\in\mathbb{D}}\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}}<\infty, \tag{3}$$

$$M_3:=\sup_{z\in\mathbb{D}}\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}}<\infty. \tag{4}$$

Proof. (i)$\Rightarrow$(ii). It is obvious.

(ii)$\Rightarrow$(iii). Suppose that $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is bounded. Taking the functions $f_1(z) = \frac{z^n}{n!}$, $f_2(z) = \frac{z^{n+1}}{(n+1)!}$ and $f_3(z) = \frac{z^{n+2}}{(n+2)!}$, which are all in $Q_{K,0}(p,q)$, we get

$$L_1 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_1'(z)| \leqslant \left\| T^n_{\psi_1,\psi_2,\varphi} \frac{z^n}{n!} \right\|_{\mathscr{B}_\mu} < \infty, \tag{5}$$

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1'(z)\varphi(z) + \psi_1(z)\varphi'(z) + \psi_2'(z)| \leqslant \left\| T^n_{\psi_1,\psi_2,\varphi} \frac{z^{n+1}}{(n+1)!} \right\|_{\mathscr{B}_\mu} < \infty, \tag{6}$$

and

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu(z) \left| \psi_1'(z)\frac{\varphi(z)^2}{2} + (\psi_1(z)\varphi'(z) + \psi_2'(z))\varphi(z) + \psi_2(z)\varphi'(z) \right| \\ &\leqslant \left\| T^n_{\psi_1,\psi_2,\varphi} \frac{z^{n+2}}{(n+2)!} \right\|_{\mathscr{B}_\mu} < \infty, \end{aligned} \tag{7}$$

respectively. Employing (5), (6), the triangle inequality and the fact that $|\varphi(z)| < 1$, we obtain

$$L_2 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)| < \infty. \tag{8}$$

By using (5), (7) and (8), in the same manner, we have

$$L_3 := \sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)\varphi'(z)| < \infty. \tag{9}$$

For a fixed $w \in \mathbb{D}$, consider the function

$$\begin{aligned} f_{1,\varphi(w)}(z) = &\frac{\frac{q+2}{p}+n+2}{\frac{q+2}{p}+n} \prod_{j=0}^{n-1} \left(\frac{q+2}{p}+j+1\right)\left(\frac{q+2}{p}+j+2\right) \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} \\ &-2\frac{\frac{q+2}{p}+n+2}{\frac{q+2}{p}+n+1} \prod_{j=0}^{n-1} \left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+2\right) \frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} \\ &+\prod_{j=0}^{n-1} \left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+1\right) \frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}}. \end{aligned}$$

Using the condition (1), we see that $f_{1,\varphi(w)} \in Q_{K,0}(p,q)$ (see [6]). By a direct calculation, we obtain

$$f^{(n+1)}_{1,\varphi(w)}(\varphi(w)) = f^{(n+2)}_{1,\varphi(w)}(\varphi(w)) = 0,$$

and

$$f_{1,\varphi(w)}^{(n)}(\varphi(w))=\frac{2\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+1\right)\left(\frac{q+2}{p}+j+2\right)}{\left(\frac{q+2}{p}+n\right)\left(\frac{q+2}{p}+n+1\right)}\frac{\overline{\varphi(w)}^{n}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n-1}},$$

which along with the boundedness of $T^n_{\psi_1,\psi_2,\varphi}$ implies that

$$\begin{aligned}\infty &> \|T^n_{\psi_1,\psi_2,\varphi}f_{1,\varphi(w)}\|_{\mathscr{B}_\mu}\\ &\geqslant \mu(w)\left|(T^n_{\psi_1,\psi_2,\varphi}f_{1,\varphi(w)})'(w)\right| \gtrsim \frac{\mu(w)|\psi_1'(w)||\varphi(w)|^n}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n-1}}.\end{aligned} \tag{10}$$

From (5) and (10), we have

$$\begin{aligned}&\sup_{w\in\mathbb{D}}\frac{\mu(w)|\psi_1'(w)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n-1}}\\ \leqslant&\sup_{|\varphi(w)|\leqslant\frac12}\frac{\mu(w)|\psi_1'(w)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n-1}}+\sup_{\frac12<|\varphi(w)|<1}\frac{\mu(w)|\psi_1'(w)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n-1}}\\ \leqslant&\left(\frac43\right)^{\frac{q+2}{p}+n-1}\sup_{|\varphi(w)|\leqslant\frac12}\mu(w)|\psi_1'(w)|+2^n\sup_{\frac12<|\varphi(w)|<1}\frac{\mu(w)|\psi_1'(w)||\varphi(w)|^n}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n-1}}\\ <&\infty,\end{aligned}$$

where we used the fact $\frac{q+2}{p}+n-1>0$, which follows from the assumptions $q>-2$ and $n\in\mathbb{N}$. From this it follows that (2) holds.

For a fixed $w\in\mathbb{D}$, take the function

$$\begin{aligned}f_{2,\varphi(w)}(z)=&\frac{\frac{q+2}{p}+n+2}{\frac{q+2}{p}+n+1}\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j+1\right)\left(\frac{q+2}{p}+j+2\right)\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}}\\ &-\frac{\frac{2q+4}{p}+2n+3}{\frac{q+2}{p}+n+1}\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+2\right)\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}}\\ &+\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+1\right)\frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}}.\end{aligned}$$

Again, condition (1) says that $f_{2,\varphi(w)}\in Q_{K,0}(p,q)$. We also have

$$f_{2,\varphi(w)}^{(n)}(\varphi(w))=f_{2,\varphi(w)}^{(n+2)}(\varphi(w))=0,$$

and

$$f_{2,\varphi(w)}^{(n+1)}(\varphi(w))=\frac{-\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+1\right)\left(\frac{q+2}{p}+j+2\right)}{\frac{q+2}{p}+n+1}\frac{\overline{\varphi(w)}^{n+1}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n}},$$

which along with the boundedness of $T^n_{\psi_1,\psi_2,\varphi}$ implies that

$$\begin{aligned}\infty > \|T^n_{\psi_1,\psi_2,\varphi} f_{2,\varphi(w)}\|_{\mathscr{B}_\mu} &\geqslant \mu(w)\left|(T^n_{\psi_1,\psi_2,\varphi} f_{2,\varphi(w)})'(w)\right| \\ &\gtrsim \frac{\mu(w)|\psi_1(z)\varphi'(z)+\psi_2'(z)||\varphi(w)|^{n+1}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n}}.\end{aligned} \tag{11}$$

By using (8) and (11), we get

$$\begin{aligned}&\sup_{w\in\mathbb{D}} \frac{\mu(w)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n}} \\ \leqslant &\sup_{|\varphi(w)|\leqslant\frac{1}{2}} \frac{\mu(w)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n}} + \sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n}} \\ \leqslant &\left(\frac{4}{3}\right)^{\frac{q+2}{p}+n} \sup_{|\varphi(w)|\leqslant\frac{1}{2}} \mu(w)|\psi_1(z)\varphi'(z)+\psi_2'(z)| \\ &+2^{n+1} \sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|\psi_1(z)\varphi'(z)+\psi_2'(z)||\varphi(w)|^{n+1}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n}} \\ < &\infty.\end{aligned}$$

Thus (3) holds.

For a fixed $w\in\mathbb{D}$, set

$$\begin{aligned}f_{3,\varphi(w)}(z) = &\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j+1\right)\left(\frac{q+2}{p}+j+2\right)\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} \\ &-2\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+2\right)\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} \\ &+\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+1\right)\frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}}.\end{aligned}$$

Then we have $f_{3,\varphi(w)}\in Q_{K,0}(p,q)$ in light of condition (1) and, moreover,

$$f^{(n)}_{3,\varphi(w)}(\varphi(w)) = f^{(n+1)}_{3,\varphi(w)}(\varphi(w)) = 0,$$

and

$$\begin{aligned}&f^{(n+2)}_{3,\varphi(w)}(\varphi(w)) \\ =&2\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)\left(\frac{q+2}{p}+j+1\right)\left(\frac{q+2}{p}+j+2\right)\frac{\overline{\varphi(w)}^{n+2}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n+1}},\end{aligned}$$

which along with the boundedness of $T^n_{\psi_1,\psi_2,\varphi}$ implies that

$$\begin{aligned}\infty &> \|T^n_{\psi_1,\psi_2,\varphi} f_{3,\varphi(w)}\|_{\mathscr{B}_\mu} \\ &\geqslant \mu(w)\big|(T^n_{\psi_1,\psi_2,\varphi} f_{3,\varphi(w)})'(w)\big| \gtrsim \frac{\mu(w)|\psi_2(z)\varphi'(z)||\varphi(w)|^{n+2}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n+1}}.\end{aligned} \tag{12}$$

From (9) and (12), we obtain

$$\begin{aligned}&\sup_{w\in\mathbb{D}} \frac{\mu(w)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n+1}} \\ &\leqslant \sup_{|\varphi(w)|\leqslant\frac{1}{2}} \frac{\mu(w)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n+1}} + \sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n+1}} \\ &\leqslant \Big(\frac{4}{3}\Big)^{\frac{q+2}{p}+n+1} \sup_{|\varphi(w)|\leqslant\frac{1}{2}} \mu(w)|\psi_2(z)\varphi'(z)| + 2^{n+2} \sup_{\frac{1}{2}<|\varphi(w)|<1} \frac{\mu(w)|\psi_2(z)\varphi'(z)||\varphi(w)|^{n+2}}{(1-|\varphi(w)|^2)^{\frac{q+2}{p}+n+1}} \\ &<\infty.\end{aligned}$$

From this it follows that (4) holds.

(iii)$\Rightarrow$(i). Suppose that conditions (2)–(4) hold. By using Lemmas 2 and 3, for each $f \in Q_K(p,q)$, we have

$$\begin{aligned}&\mu(z)|(T^n_{\psi_1,\psi_2,\varphi} f)'(z)| \\ &\leqslant \mu(z)|\psi_1'(z)||f^{(n)}(\varphi(z))| + \mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)||f^{(n+1)}(\varphi(z))| \\ &\quad + \mu(z)|\psi_2(z)\varphi'(z)||f^{(n+2)}(\varphi(z))| \\ &\lesssim \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}}\|f\|_{\mathscr{B}^{\frac{q+2}{p}}} + \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}}\|f\|_{\mathscr{B}^{\frac{q+2}{p}}} \\ &\quad + \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}}\|f\|_{\mathscr{B}^{\frac{q+2}{p}}} \\ &\leqslant (M_1+M_2+M_3)\|f\|_{Q_K(p,q)}.\end{aligned} \tag{13}$$

Furthermore,

$$\begin{aligned}|(T^n_{\psi_1,\psi_2,\varphi} f)(0)| &\leqslant |\psi_1(0)f^{(n)}(\varphi(0))| + |\psi_2(0)f^{(n+1)}(\varphi(0))| \\ &\lesssim \Bigg[\frac{|\psi_1(0)|}{(1-|\varphi(0)|^2)^{\frac{q+2}{p}+n-1}} + \frac{|\psi_2(0)|}{(1-|\varphi(0)|^2)^{\frac{q+2}{p}+n}}\Bigg]\|f\|_{Q_K(p,q)}.\end{aligned} \tag{14}$$

In view of (13) and (14), we conclude that $T^n_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_\mu$ is bounded. □

THEOREM 2. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (1) holds. Then for each $n \in \mathbb{N}$, the following statements are equivalent.*

(i) *The operator* $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_\mu$ *is compact.*
(ii) *The operator* $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_\mu$ *is compact.*
(iii) *The operator* $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_\mu$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}} = 0, \tag{15}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}} = 0, \tag{16}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}} = 0. \tag{17}$$

Proof. (i)$\Rightarrow$(ii). It is clear.

(ii)$\Rightarrow$(iii). Suppose that $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is compact, and consequently bounded. Then by Theorem 1 the boundedness of $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_\mu$ follows. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ satisfying $|\varphi(z_k)| \to 1$ as $k\to\infty$. Set

$$f_{l,k}(z) = f_{l,\varphi(z_k)}(z), \quad l = 1,2,3,$$

where $f_{l,\varphi(z_k)}$ is defined in the proof of Theorem 1. Moreover, we have $\{f_{l,k}\}_{k\in\mathbb{N},l=1,2,3}$ are norm bounded sequences in $Q_{K,0}(p,q)$, and it is easily seen that $f_{l,k}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By Lemma 4, it yields

$$\lim_{k\to\infty} \|T^n_{\psi_1,\psi_2,\varphi} f_{l,k}\|_{\mathscr{B}_\mu} = 0, \quad l=1,2,3. \tag{18}$$

On the other hand, from (10)–(12) it follows that

$$\frac{\mu(z_k)|\psi_1'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{q+2}{p}+n-1}} \lesssim \|T^n_{\psi_1,\psi_2,\varphi} f_{1,k}\|_{\mathscr{B}_\mu}, \tag{19}$$

$$\frac{\mu(z_k)|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{\frac{q+2}{p}+n}} \lesssim \|T^n_{\psi_1,\psi_2,\varphi} f_{2,k}\|_{\mathscr{B}_\mu}, \tag{20}$$

$$\frac{\mu(z_k)|\psi_2(z_k)\varphi'(z_k)||\varphi(z_k)|^{n+2}}{(1-|\varphi(z_k)|^2)^{\frac{q+2}{p}+n+1}} \lesssim \|T^n_{\psi_1,\psi_2,\varphi} f_{3,k}\|_{\mathscr{B}_\mu}. \tag{21}$$

Letting $k\to\infty$ in (19)–(21) and employing (18), we can see that (15)–(17) hold.

(iii)$\Rightarrow$(i). Suppose that $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_\mu$ is bounded and that (15)–(17) hold. Then by Theorem 1 we have $L_1, L_2, L_3 < \infty$, where L_1, L_2, L_3 are defined in (5), (8) and (9), respectively. Moreover, for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}} < \varepsilon, \tag{22}$$

$$\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}} < \varepsilon, \tag{23}$$

$$\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}} < \varepsilon, \tag{24}$$

whenever $\delta < |\varphi(z)| < 1$.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $Q_K(p,q)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{Q_K(p,q)} \lesssim 1$ and $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$ as $k\to\infty$. Applying Lemma 2 we have

$$\begin{aligned}
&\|T^n_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_\mu}\\
=&|(T^n_{\psi_1,\psi_2,\varphi}f_k)(0)| + \sup_{z\in\mathbb{D}}\mu(z)|(T^n_{\psi_1,\psi_2,\varphi}f_k)'(z)|\\
\leqslant&|\psi_1(0)f_k^{(n)}(\varphi(0))| + |\psi_2(0)f_k^{(n+1)}(\varphi(0))|\\
&+\sup_{|\varphi(z)|\leqslant\delta}\mu(z)|\psi_1'(z)||f_k^{(n)}(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1}\mu(z)|\psi_1'(z)||f_k^{(n)}(\varphi(z))|\\
&+\sup_{|\varphi(z)|\leqslant\delta}\mu(z)|\psi_1(z)\varphi'(z)+\psi_2(z)||f_k^{(n+1)}(\varphi(z))|\\
&+\sup_{\delta<|\varphi(z)|<1}\mu(z)|\psi_1(z)\varphi'(z)+\psi_2(z)||f_k^{(n+1)}(\varphi(z))|\\
&+\sup_{|\varphi(z)|\leqslant\delta}\mu(z)|\psi_2(z)\varphi'(z)||f_k^{(n+2)}(\varphi(z))|\\
&+\sup_{\delta<|\varphi(z)|<1}\mu(z)|\psi_2(z)\varphi'(z)||f_k^{(n+2)}(\varphi(z))|\\
\lesssim&|\psi_1(0)||f_k^{(n)}(\varphi(0))| + |\psi_2(0)||f_k^{(n+1)}(\varphi(0))|\\
&+L_1\sup_{|\varphi(z)|\leqslant\delta}|f_k^{(n)}(\varphi(z))| + L_2\sup_{|\varphi(z)|\leqslant\delta}|f_k^{(n+1)}(\varphi(z))| + L_3\sup_{|\varphi(z)|\leqslant\delta}|f_k^{(n+2)}(\varphi(z))|\\
&+\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}}\\
&+\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}} + \sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}}\\
\leqslant&|\psi_1(0)||f_k^{(n)}(\varphi(0))| + |\psi_2(0)||f_k^{(n+1)}(\varphi(0))|\\
&+L_1\sup_{|w|\leqslant\delta}|f_k^{(n)}(w)| + L_2\sup_{|w|\leqslant\delta}|f_k^{(n+1)}(w)| + L_3\sup_{|w|\leqslant\delta}|f_k^{(n+2)}(w)| + 3\varepsilon.
\end{aligned}\tag{25}$$

Since $f_k\to 0$ uniformly on compact subset of $\mathbb{D}$ as $k\to\infty$, we conclude that $f_k^{(n)}$, $f_k^{(n+1)}$ and $f_k^{(n+2)}$ also do by Cauchy's estimate. In particular, $\{\varphi(0)\}$ and $\{w : |w|\leqslant\delta\}$ are compact subsets of $\mathbb{D}$, hence letting $k\to\infty$ in (25) yields

$$\limsup_{k\to\infty}\|T^n_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_\mu} \leqslant 3\varepsilon.$$

By the arbitrariness of ε it follows that $\lim_{n\to\infty}\|T^n_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_\mu} = 0$, from which by Lemma 4 we deduce that $T^n_{\psi_1,\psi_2,\varphi} : Q_K(p,q)\to\mathscr{B}_\mu$ is compact. □

We next turn to investigating the case that target space is $\mathscr{B}_{\mu,0}$. For the boundedness, the following theorem includes the case $n=0$.

THEOREM 3. *Let $\psi_1,\psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Then for each $n \in \mathbb{N}_0$, $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ is bounded if and only if $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is bounded and*

$$\lim_{|z|\to 1} \mu(z)|\psi_1'(z)| = 0, \tag{26}$$

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)| = 0, \tag{27}$$

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)\varphi'(z)| = 0. \tag{28}$$

Proof. Assume that $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ is bounded, then it is evident that $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is bounded and for every $f \in Q_{K,0}(p,q)$, we have $T^n_{\psi_1,\psi_2,\varphi} f \in \mathscr{B}_{\mu,0}$. Taking $f_1(z) = \frac{z^n}{n!} \in Q_{K,0}(p,q)$ yields

$$\lim_{|z|\to 1} \mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi}\frac{z^n}{n!}\right)'(z)\right| = \lim_{|z|\to 1} \mu(z)|\psi_1'(z)| = 0.$$

That is, (26) holds. Instead of using the functions $f_2(z) = \frac{z^{n+1}}{(n+1)!}$ and $f_3(z) = \frac{z^{n+2}}{(n+2)!} \in Q_{K,0}(p,q)$, we obtain

$$\lim_{|z|\to 1} \mu(z)|\psi_1'(z)\varphi(z)+\psi_1(z)\varphi'(z)+\psi_2'(z)| = 0,$$

$$\lim_{|z|\to 1} \mu(z)\left|\psi_1'(z)\frac{\varphi(z)^2}{2} + (\psi_1(z)\varphi'(z)+\psi_2'(z))\varphi(z)+\psi_2(z)\varphi'(z)\right| = 0,$$

respectively. By using (26), the triangle inequality and the fact that $|\varphi(z)|<1$, we deduce that (27) and (28) hold.

Conversely, suppose that $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is bounded and (26)–(28) hold. Then for each polynomial $p(z)$, we have

$$\begin{aligned}
&\mu(z)|(T^n_{\psi_1,\psi_2,\varphi}p)'(z)| \\
\leqslant&\mu(z)|\psi_1'(z)||p^{(n)}(\varphi(z))| + \mu(z)|\psi_1(z)\varphi'(z)+\psi_2(z)||p^{(n+1)}(\varphi(z))| \\
&+\mu(z)|\psi_2(z)\varphi'(z)||p^{(n+2)}(\varphi(z))| \\
\lesssim&\mu(z)|\psi_1'(z)| + \mu(z)|\psi_1(z)\varphi'(z)+\psi_2(z)| + \mu(z)|\psi_2(z)\varphi'(z)|.
\end{aligned}$$

Letting $|z|\to 1$ in the above inequality and employing (26)–(28) gives

$$\lim_{|z|\to 1} \mu(z)|(T^n_{\psi_1,\psi_2,\varphi}p)'(z)| = 0,$$

which says that $T^n_{\psi_1,\psi_2,\varphi}p \in \mathscr{B}_{\mu,0}$. Since the set of all polynomials is dense in $Q_{K,0}(p,q)$ (see [6]), and hence for each $f \in Q_{K,0}(p,q)$, there is a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\|p_k - f\|_{Q_K(p,q)} = 0$, which along with the boundedness of $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ implies that

$$\|T^n_{\psi_1,\psi_2,\varphi}p_k - T^n_{\psi_1,\psi_2,\varphi}f\|_{\mathscr{B}_\mu} \leqslant \|T^n_{\psi_1,\psi_2,\varphi}\|_{Q_{K,0}(p,q)\to\mathscr{B}_\mu} \cdot \|p_k - f\|_{Q_K(p,q)} \to 0,$$

as $k\to\infty$. Since $\mathscr{B}_{\mu,0}$ is a closed subspace of $\mathscr{B}_\mu$, we have $T^n_{\psi_1,\psi_2,\varphi}f \in \mathscr{B}_{\mu,0}$, and consequently $T^n_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ is bounded. $\square$

THEOREM 4. *Let* $\psi_1,\psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p>0$, $q>-2$ *and* K *be a nonnegative nondecreasing function on* $[0,\infty)$ *such that* (1) *holds. Then for each* $n \in \mathbb{N}$, *the following statements are equivalent.*
(i) *The operator* $T^n_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_{\mu,0}$ *is compact.*
(ii) *The operator* $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ *is compact.*
(iii)

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}} = 0, \tag{29}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}} = 0, \tag{30}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}} = 0. \tag{31}$$

Proof. (i)$\Rightarrow$(ii). It is obvious.

(ii)$\Rightarrow$(iii). Suppose that $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ is compact, and the compactness of $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_{\mu}$ follows. From Theorem 2, for any $\varepsilon>0$, there exists $\delta \in (0,1)$ such that (22)–(24) hold whenever $\delta < |\varphi(z)| < 1$. Moreover, the compactness of $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ implies that $T^n_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ is bounded. Then (26)–(28) follow from Theorem 3, and for any $\varepsilon>0$, there exists $\eta \in (0,1)$ such that

$$\mu(z)|\psi_1'(z)| \leqslant \varepsilon(1-\delta^2)^{\frac{q+2}{p}+n-1}, \tag{32}$$

$$\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)| \leqslant \varepsilon(1-\delta^2)^{\frac{q+2}{p}+n}, \tag{33}$$

$$\mu(z)|\psi_2(z)\varphi'(z)| \leqslant \varepsilon(1-\delta^2)^{\frac{q+2}{p}+n+1}, \tag{34}$$

whenever $\eta < |z| < 1$. From (22), when $\eta<|z|<1$ and $\delta<|\varphi(z)|<1$, we have

$$\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}} < \varepsilon. \tag{35}$$

On the other hand, when $\eta<|z|<1$ and $|\varphi(z)| \leqslant \delta$, using (32) yields

$$\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}} \leqslant \frac{\mu(z)|\psi_1'(z)|}{(1-\delta^2)^{\frac{q+2}{p}+n-1}} < \varepsilon. \tag{36}$$

From (35) and (36) we deduce that (29) holds. Employing (23) and (33), (24) and (34), with the similar arguments, we can get (30) and (31).

(iii)$\Rightarrow$(i). Assume that (29)–(31) hold. Let $f \in Q_K(p,q)$, analysis similar to (13) in the proof of Theorem 1 shows that

$$\mu(z)|(T^n_{\psi_1,\psi_2,\varphi}f)'(z)| \lesssim \Bigg(\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n-1}} + \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n}} + \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+n+1}} \Bigg) \|f\|_{Q_K(p,q)}.$$

Taking the supremum in the above inequality over all $f \in Q_K(p,q)$ such that $\|f\|_{Q_K(p,q)} \leqslant 1$ and letting $|z| \to 1$, we have

$$\lim_{|z|\to 1} \sup_{\|f\|_{Q_K(p,q)}\leqslant 1} \mu(z)|(T^n_{\psi_1,\psi_2,\varphi}f)'(z)| = 0.$$

Therefore, the operator $T^n_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_{\mu,0}$ is compact by Lemma 5. $\square$

We are now in a position to consider the case $n = 0$. For this purpose, we need to break the problem into two different cases: $q+2 \geqslant p$ and $q+2 < p$.

THEOREM 5. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ such that $q+2 \geqslant p$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (1) holds. Then the following statements are equivalent.*
(i) *The operator $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_\mu$ is bounded.*
(ii) *The operator $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is bounded.*
(iii)

$$N_1 := \begin{cases} \sup\limits_{z\in\mathbb{D}} \mu(z)|\psi_1'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty, & q+2 = p, \\ \sup\limits_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1}} < \infty, & q+2 > p, \end{cases}$$

$$N_2 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty,$$

$$N_3 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} < \infty.$$

Proof. It is immediate that (i)$\Rightarrow$(ii) holds. For the implication (ii)$\Rightarrow$(iii), suppose that $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is bounded. By using the functions

$$g_{1,\varphi(w)}(z) = \frac{\frac{q+2}{p}+2}{\frac{q+2}{p}} \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} - 2\frac{\frac{q+2}{p}+2}{\frac{q+2}{p}+1} \frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} + \frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}},$$

$$g_{2,\varphi(w)}(z) = \frac{\frac{q+2}{p}+2}{\frac{q+2}{p}+1} \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} - \frac{\frac{2q+4}{p}+3}{\frac{q+2}{p}+1} \frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} + \frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}},$$

$$g_{3,\varphi(w)}(z) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}}} - 2\frac{(1-|\varphi(w)|^2)^2}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+1}} + \frac{(1-|\varphi(w)|^2)^3}{(1-\overline{\varphi(w)}z)^{\frac{q+2}{p}+2}}.$$

where $w \in \mathbb{D}$, with the similar arguments in the proof of Theorem 1, we get $N_1 < \infty$ when $q+2 > p$ and $N_2, N_3 < \infty$. For the case $q+2 = p$, consider the function

$$g_{\varphi(w)}(z) = \ln \frac{e}{1-\overline{\varphi(w)}z},$$

where $w \in \mathbb{D}$. Then $g_{\varphi(w)} \in Q_{K,0}(p,q)$ (see [6]) and it is easy to calculate that

$$g_{\varphi(w)}(\varphi(w)) = \ln \frac{e}{1-|\varphi(w)|^2}, \quad g'_{\varphi(w)}(\varphi(w)) = \frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2},$$

$$g''_{\varphi(w)}(\varphi(w)) = \frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^2},$$

which along with the boundedness of $T_{\psi_1,\psi_2,\varphi}$ and the triangle inequality implies that

$$\begin{aligned} \infty >& \|T_{\psi_1,\psi_2,\varphi} g_{\varphi(w)}\|_{\mathscr{B}_\mu} \\ \geqslant& \mu(w)\left|(T_{\psi_1,\psi_2,\varphi} g_{\varphi(w)})'(w)\right| \\ \geqslant& \mu(w)|\psi_1'(w)| \ln \frac{e}{1-|\varphi(w)|^2} \\ &- \frac{\mu(w)|\psi_1(w)\varphi'(w)+\psi_2'(w)||\varphi(w)|}{1-|\varphi(w)|^2} - \frac{\mu(w)|\psi_2(w)\varphi'(w)||\varphi(w)|^2}{(1-|\varphi(w)|^2)^2}. \end{aligned}$$

From $N_2, N_3 < \infty$ and the fact that $|\varphi(w)| < 1$ it follows that $N_1 < \infty$.

(iii)$\Rightarrow$(i). Assume that $N_1, N_2, N_3 < \infty$. Using Lemmas 1, 2 and 3, similar to the proof of Theorem 1, for each $f \in Q_K(p,q)$, we have

$$\mu(z)|(T^n_{\psi_1,\psi_2,\varphi} f)'(z)| \leqslant (N_1+N_2+N_3)\|f\|_{Q_K(p,q)}.$$

Thus the implication follows. □

THEOREM 6. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ such that $q+2 \geqslant p$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (1) holds. Then the following statements are equivalent.*

(i) *The operator $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_\mu$ is compact.*

(ii) *The operator $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is compact.*

(iii) *The operator* $T_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_\mu$ *is bounded and*

$$R_1 := \begin{cases} \lim\limits_{|\varphi(z)|\to 1} \mu(z)|\psi_1'(z)|\ln\frac{e}{1-|\varphi(z)|^2} = 0, & q+2 = p, \\ \lim\limits_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1}} = 0, & q+2 > p, \end{cases}$$

$$R_2 := \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0,$$

$$R_3 := \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} = 0.$$

Proof. It is obvious that (i)$\Rightarrow$(ii) holds. To verify (ii)$\Rightarrow$(iii), suppose that $T_{\psi_1,\psi_2,\varphi}: Q_{K,0}(p,q) \to \mathscr{B}_\mu$ is compact, and consequently bounded. From Theorem 5 it follows that $T_{\psi_1,\psi_2,\varphi}: Q_K(p,q) \to \mathscr{B}_\mu$ is bounded. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$g_{l,k}(z) = g_{l,\varphi(z_k)}(z), \quad l = 1,2,3,$$

where $g_{l,\varphi(z_k)}$ is defined in the proof of Theorem 5. Analysis similar to that in the proof of Theorem 2 shows that $R_1 = 0$ for $q+2 > p$ and $R_2 = R_3 = 0$. For the case $q+2 = p$, take the function

$$g_k(z) = \left(\ln\frac{e}{1-\overline{\varphi(z_k)}z}\right)^2\left(\ln\frac{e}{1-|\varphi(z_k)|^2}\right)^{-1},$$

then $\{g_k\}_{k\in\mathbb{N}}$ is a bounded sequences in $Q_{K,0}(p,q)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By Lemma 4, it yields

$$\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi}g_k\|_{\mathscr{B}_\mu} = 0. \tag{37}$$

Then we have

$$\begin{aligned} &\|T_{\psi_1,\psi_2,\varphi}g_k\|_{\mathscr{B}_\mu} \\ \geqslant&\mu(z_k)\left|(T_{\psi_1,\psi_2,\varphi}g_k)'(z_k)\right| \\ \geqslant&\mu(z_k)|\psi_1'(z_k)|\ln\frac{e}{1-|\varphi(z_k)|^2} - \frac{2\mu(z_k)|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2} \\ &-\left(\frac{2}{\ln\frac{e}{1-|\varphi(z_k)|^2}}+2\right)\frac{\mu(z_k)|\psi_2(z_k)\varphi'(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^2}. \end{aligned} \tag{38}$$

Letting $k \to \infty$ in (38) and employing (37), the fact that $R_2 = R_3 = 0$, we get $R_1 = 0$.

The proof of implication (iii)$\Rightarrow$(i) runs as that of Theorem 2 by using Lemma 1 and we omit the details. □

Similar to the proof of Theorem 4, we have the following theorem.

THEOREM 7. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ *such that* $q+2 \geqslant p$ *and* K *be a nonnegative nondecreasing function on* $[0,\infty)$ *such that* (1) *holds. Then the following statements are equivalent.*
(i) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_{\mu,0}$ *is compact.*
(ii) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_{\mu,0}$ *is compact.*
(iii)

$$\begin{cases} \lim\limits_{|z|\to 1} \mu(z)|\psi_1'(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0, & q+2 = p, \\ \lim\limits_{|z|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1}} = 0, & q+2 > p, \end{cases}$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0,$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} = 0.$$

Finally, when $q+2 < p$, we give the following results, whose proofs run essentially as before and we omit the details.

THEOREM 8. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ *such that* $q+2 < p$ *and* K *be a nonnegative nondecreasing function on* $[0,\infty)$ *such that* (1) *holds. Then the following statements are equivalent.*
(i) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_{\mu}$ *is bounded.*
(ii) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_{\mu}$ *is bounded.*
(iii) $\psi_1 \in \mathscr{B}_{\mu}$ *and*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty,$$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} < \infty.$$

By using Lemma 6, the characterizations of compactness follows.

THEOREM 9. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $p > 0$, $q > -2$ *such that* $q+2 < p$ *and* K *be a nonnegative nondecreasing function on* $[0,\infty)$ *such that* (1) *holds. Then the following statements are equivalent.*
(i) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_{\mu}$ *is compact.*
(ii) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_{K,0}(p,q) \to \mathscr{B}_{\mu}$ *is compact.*
(iii) *The operator* $T_{\psi_1,\psi_2,\varphi} : Q_K(p,q) \to \mathscr{B}_{\mu}$ *is bounded and* $\psi_1 \in \mathscr{B}_{\mu}$,

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} = 0.$$

THEOREM 10. *Let $\psi_1,\psi_2\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $p>0$, $q>-2$ such that $q+2<p$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (1) holds. Then the following statements are equivalent.*

(i) *The operator $T_{\psi_1,\psi_2,\varphi}:Q_K(p,q)\to\mathscr{B}_{\mu,0}$ is compact.*

(ii) *The operator $T_{\psi_1,\psi_2,\varphi}:Q_{K,0}(p,q)\to\mathscr{B}_{\mu,0}$ is compact.*

(iii) $\psi_1\in\mathscr{B}_{\mu,0}$ *and*

$$\lim_{|z|\to 1}\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}=0,$$
$$\lim_{|z|\to 1}\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}}=0.$$

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