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A CLASS OF INTEGRO–MULTIPLICATION OPERATORS

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Abstract. This paper introduces a class of Integro-multiplication operators on Hilbert spaces of analytic functions with reproducing kernels of the form

$$K_\varphi(z,w) = \sum_{n=0}^{\infty} f(z)\overline{f(w)} \quad \text{with} \quad f(z) = (n+1)z^n + \varphi(z)z^{n+1},$$

where $\varphi \in H^\infty(\mathbb{D})$. Hyponormality and subnormality of the operators is explored in some special cases, particularly the case where $\varphi(z) = 1$. Additionally the idea of M-dominating matrices is introduced as a means of establishing the norms of these operators.

1. Introduction and preliminaries

The primary focus of this paper is the study of multiplication operators defined on some special types of reproducing kernel Hilbert spaces of analytic functions. The reproducing kernels are of the form

$$K_\varphi(z,w) = \sum_{n=0}^{\infty} f(z)\overline{f(w)} \quad \text{with} \quad f(z) = (n+1)z^n + \varphi(z)z^{n+1},$$

where $\varphi \in H^\infty(\mathbb{D})$. The appeal of these multiplication operators is that they can be realized as the perturbation by a nice integral operator of a multiplication operator on the Hardy space. This allows for an elegant Integro-Multiplicative realization of the functional calculus.

A special focus is placed on the space where $\phi(z) = 1$. In this case multiplication by z is a hyponormal operator which, while not subnormal, is so close to being subnormal that a subtle proof of non-subnormality is required that makes use of Lambert's requirement. The operator is shown to have unit norm, spectrum the closed unit disk, and an infinite rank self-commutator. In addition to the interesting behavior described above, the appeal of this particular operator, as well as the general class of operators, is that they all arise quite naturally on function spaces, they do not appear to have been looked at previously, and they have a rich functional calculus.

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The remainder of this section will be devoted to some general preliminaries regarding reproducing kernel Hilbert spaces. The next section will focus on the kernel K_φ where $\varphi \in H^\infty(\mathbb{D})$. Some specific properties of the associated reproducing kernel Hilbert space $H(K_\varphi)$ together with it's multiplication operators will be explored. Section 3 focuses on the case where $\varphi(z) = 1$, in which case $H(K_\varphi)$ is an analytic *tridiagonal* reproducing kernel Hilbert space. Section 4 establishes the hyponormality of the operator M_z of multiplication by z. Section 5 introduces M-dominating matrices and establishes a method of estimating the norm of such matrices. This is used to show that the operator M_z has norm equal to it's spectral radius. Section 6 is devoted to a subtle proof that the operator M_z is not subnormal. The final section addresses how our techniques would extend to a broader class of spaces as well as some open questions.

The function $K(z,w)$ is *positive definite* (denoted $K \succeq 0$) on the set $E \times E$ if for any finite collection $z_1, z_2, \cdots, z_n$ in E and any complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ the sum

$$\sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(z_i, z_j) \geqslant 0$$

with strict inequality unless all the α_i's are zero. It is well known that if $K \succeq 0$ on E, then the set of functions in z given by

$$\left\{ \sum_{j=1}^{n} \alpha_j K(z, w_j) : \alpha_1, \cdots, \alpha_n \in \mathbb{C}, w_1, \cdots, w_n \in E \right\}$$

has dense span in a Hilbert space $H(K)$ of functions on E with

$$\left\| \sum_{j=1}^{n} \alpha_j K(z, w_j) \right\|^2 = \sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(w_i, w_j).$$

A fundamental property of $H(K)$ is the *Reproducing Property* which states that for every w in E and f in $H(K)$, $f(w) = \langle f(z), K(z,w) \rangle$. Thus evaluation at w is a bounded linear functional for each w in E.

Conversely, it is well known that if F is a Hilbert space of functions defined on E such that evaluation at w is a bounded linear functional for each w in E, then there is a unique K defined on $E \times E$ such that $F = H(K)$. It follows from the reproducing property that $K(z,w) = \overline{K(w,z)}$. Hence if K is analytic in the first variable, then K is coanalytic in the second variable. In this case K is an *analytic* kernel. In later sections of this paper, E will always be the unit disk $\mathbb{D}$ and K will be an analytic kernel.

It is also well known, see N. Aronszajn [3], that if $\{f_n(z)\}$ is an orthonormal basis for a reproducing kernel Hilbert space of functions on E, then $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ for all z, w in E. Moreover if the largest common domain E' of the functions $\{f_n(z)\}$ is larger than E, then the largest natural domain of $H(K)$ is given by $Dom(K) = \{z \in E' : \sum_{n=0}^{\infty} |f_n(z)|^2 < \infty\}$. When K is analytic and E contains the open unit disk, $K(z,w) =$

$\sum_{i,j=0}^{\infty} a_{i,j} z^i \overline{w}^j$ has a Taylor series expansion about $(0,0)$ with coefficient matrix $A = [a_{i,j}]$. The matrix A is positive and if $A = BB^*$ is any factorization of A, then $H(K)$ is naturally isomorphic to the range space $R(B) = \{B\vec{x} : \vec{x} \in l^2_+\}$ via the map which identifies $B\vec{x}$ with the analytic function f whose Taylor coefficients are the components of $B\vec{x}$. Thus, when B has no kernel, the columns of B correspond to an orthonormal basis for $H(K)$. It should be noted that the matrices are not necessarily bounded, but this is not a problem for the general theory. The interested reader is referred to Adams, McGuire, and Paulsen [2] for more details.

An analytic kernel is *tridiagonal* if there exists an orthonormal basis of polynomials for $H(K)$ of the form $\{f_n(z) = (a_n + b_n z)z^n : n \geqslant 0\}$ and diagonal if $b_n = 0$ for all n. In this case the coefficient matrix A has bandwidth 3, hence the name tridiagonal, and A can be factored as LL^* where

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \vdots & b_2 & \ddots \end{pmatrix}.$$

The natural domain $Dom(K)$ of a tridiagonal kernel is either an open or closed disk about the origin together with at most one point not in the disk. This was shown in Adams, McGuire [1] in which the properties of M_z were considered. In particular, it was shown that if M_z, the operator of multiplication by z, is bounded, then $H(K)$ must contain the polynomials. For $a_n > 0$, this is equivalent to the sequence

$$\left\{1, \frac{b_n}{a_{n+1}}, \frac{b_n b_{n+1}}{a_{n+1} a_{n+2}}, \frac{b_n b_{n+1} b_{n+2}}{a_{n+1} a_{n+2} a_{n+3}}, \ldots\right\}$$

being absolutely square summable for each n.

2. The space $H(K_\varphi)$

For $\varphi = \sum_{n=0}^{\infty} \varphi_n z^n \in H^\infty$ and $n \geqslant 0$, let $f_n(z) = (n+1)z^n + \varphi(z) z^{n+1}$. With $K_\varphi(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$, the set of functions $\{f_n\}$ forms an orthonormal basis for $H(K_\varphi)$ and

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \varphi_0 & 2 & 0 & 0 & \ddots \\ \varphi_1 & \varphi_0 & 3 & 0 & \ddots \\ \varphi_2 & \varphi_1 & \varphi_0 & 4 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since φ is bounded, it is straightforward to determine that the natural domain $Dom(K_\varphi)$ is the unit disk. Let $H(\mathbb{D})$ denote the space of holomorphic functions on $\mathbb{D}$ and $H_0(\mathbb{D})$

the subset of $H(\mathbb{D})$ consisting of the functions which vanish at the origin. Consider the linear map $U : H_0(\mathbb{D}) \to H(\mathbb{D})$ defined by $U(f) = \varphi f + f'$. First, note that U is injective since $U(f) = \varphi f + f' = 0$ implies that $f' = -\varphi f$ with $f(0) = 0$. This equation has the unique solution $f = ce^{-\int_0^z \varphi(w)\, dw}$ where the constant c must satisfy $c = 0$. The map U is also onto, for if $g \in H(\mathbb{D})$, then we can uniquely solve the differential equation $g = U(f) = \varphi f + f'$ with $f(0) = 0$ to obtain $f(z) = \frac{1}{\psi(z)} \int_0^z \psi(w)g(w)\, dw$ where $\psi(z) = e^{\int_0^z \phi(w)\, dw}$. Thus U is invertible with inverse given by

$$(U^{-1}g)(z) = f(z) = \frac{1}{\psi(z)} \int_0^z \psi(w)g(w)\, dw.$$

If $\varphi(z) = 1$, then K_φ is a tridiagonal kernel. This special case will be thoroughly explored in the next section. If φ is a polynomial of degree m, then K_φ is a kernel of bandwidth $2m+3$.

THEOREM 1. *Let H^2 denote the usual Hardy space on the unit disk, H_0^2 the subspace spanned by $\{z^n : n \geqslant 1\}$, and $H(K_\varphi)$ the space with reproducing kernel $K_\varphi(z,w) = \sum_{n=0}^\infty f_n(z)\overline{f_n(w)}$ where $f_n(z) = (n+1)z^n + \varphi(z)z^{n+1}$.*

1. *$U : H_0^2 \to H(K_\varphi)$ given by $U(f) = \varphi f + f'$ is an isomorphism. In this case*
$$U^*g = U^{-1}g = \frac{1}{\psi(z)} \int_0^z \psi(w)g(w)dw$$
where $\psi(z) = \int_0^z \phi(w)\, dw$.
2. *$H(K_\varphi)$ consists of the functions $\{\frac{1}{\psi(z)} f'(z) : f \in H_0^2\}$;*
3. *$H(K_\varphi)$ contains the polynomials;*
4. *$\widehat{M}_z : H(K_\varphi) \to H(K_\varphi)$ is a bounded linear operator where $\widehat{M}_z(f) = zf$;*
5. *the spectrum of $\widehat{M}_z$ is the closed unit disk;*
6. *$\widehat{M}_z : H(K_\varphi) \to H(K_\varphi)$ is unitarily equivalent to $T : H_0^2 \to H_0^2$ where*
$$(Tf)(z) = zf(z) - \frac{1}{\psi(z)} \int_0^z \psi(w)f(w)\, dw.$$
7. *the multiplier algebra of $H(K_\varphi)$ is a subalgebra of H^∞ which includes all functions analytic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$.*
8. *If ϕ is a bounded multiplier of $H(K_\varphi)$ and $T_\phi : H_0^2 \to H_0^2$ by $T_\phi = U^*\widehat{M_\phi}U$ where $\widehat{M_\phi}$ is multiplication by ϕ on $H(K_\varphi)$, then*
$$(T_\phi h)(z) = \frac{1}{\psi(z)} \int_0^z \phi(w)\psi(w)(h(w) + h'(w))dw$$
or
$$(T_\phi h)(z) = \phi(z)h(z) - \frac{1}{\psi(z)} \int_0^z \psi(w)\phi'(w)h(w)dw.$$

Proof. Since $U(z^{n+1}) = f_n(z)$ for $n \geqslant 0$, U sends an orthonormal basis to an orthonormal basis. Hence the restriction of U to H_0^2 is an isomorphism onto $H(K_\varphi)$. Consequently $U^* = U^{-1}$ is given by $(U^*g)(z) = f(z) = \frac{1}{\psi(z)} \int_0^z \psi(w)g(w)\, dw$ as before. Also, H_0^2 is functionally equal to the set of functions $\{\frac{1}{\psi(z)} f(z) : f \in H_0^2\}$ as both $\frac{1}{\psi(z)}$ and $\psi(z)$ are bounded invertible functions on $\partial\mathbb{D}$. Since $U(\frac{1}{\psi(z)} f(z)) = \frac{\varphi(z)}{\psi(z)} f(z) - \frac{\varphi(z)}{\psi(z)} f(z) + \frac{1}{\psi(z)} f'(z) = \frac{1}{\psi(z)} f'(z)$, it follows that $H(K_\varphi)$ is a renorming of the functions $\{\frac{1}{\psi(z)} f'(z) : f \in H_0^2\}$.

Let M_z denote multiplication by z on H_0^2 and $\widehat{M}_z$ multiplication by z on $H(K_\varphi)$. Applying integration by parts we see that

$$U^{-1}\Big(\frac{1}{\psi(z)} zf'(z)\Big) = \frac{1}{\psi(z)} \int_0^z \psi(w) \frac{1}{\psi(w)} wf'(w)dw = \frac{1}{\psi(z)} zf(z) - \frac{1}{\psi(z)} \int_0^z f(w)dw$$

is in H_0^2 whenever f is in H_0^2 since M_z is bounded on H_0^2 and the antiderivative of a function in H_0^2 is in H_0^2. Thus $H(K_\varphi)$ is invariant under multiplication by z and hence $\widehat{M}_z$ is bounded on $H(K_\varphi)$ by the Closed Graph Theorem. To see that the polynomials are in $H(K_\varphi)$, note that $U^{-1}(1) = \frac{1}{\psi(z)} \int_0^z \psi(w)\, dw$ is in H_0^2. Hence 1 is in $H(K_\varphi)$. Since M_z is bounded, $M_z^n(1) = z^n$ is in $H(K_\varphi)$ for all $n \geqslant 1$.

Since $U(M_z f) = \varphi(z)zf(z) + f(z) + zf'(z) = \widehat{M}_z U(f(z)) + f(z)$ and $\widehat{M}_z$ is bounded, f is in $H(K_\varphi)$ and we can write $f = UU^{-1}f$. Hence $U(M_z - U^{-1})f = \widehat{M}_z Uf$ which implies $\widehat{M}_z$ on $H(K_\varphi)$ is unitarily equivalent to $T = M_z - U^{-1}$ on H_0^2. Recall U^{-1} is defined on all holomorphic functions on $\mathbb{D}$ and hence is a well defined operator on H_0^2. Note that if f is in H_0^2, then

$$(Tf)(z) = zf(z) - \frac{1}{\psi(z)} \int_0^z \psi(w)f(w)\, dw.$$

It remains to show that the spectrum of $\widehat{M}_z$ is the closed unit disk and the multipliers of $H(K_\varphi)$ are in H^∞. A standard argument shows

$$\Big\langle (\widehat{M}_z - \lambda I)f(z), K_\varphi(z,\lambda) \Big\rangle = \langle (z-\lambda)f(z), K_\varphi(z,\lambda) \rangle = 0$$

for each $\lambda \in \mathbb{D}$ and $f \in H(K_\varphi)$; thus every point in the open unit disk is in the point spectrum of $\widehat{M}_z^*$. Hence the spectrum contains the closed unit disk.

Notice that if φ is analytic on an open set containing the closed unit disk, then φ is a uniform limit of polynomials on a neighborhood of the closed unit disk and so the continuity of the Reisz functional calculus implies that φ is a multiplier. Thus, in particular e^z, e^{-z}, and other such (entire) functions are multipliers of $H(K_\varphi)$. In particular, if $\lambda \notin \overline{\mathbb{D}}$, then $\frac{1}{z-\lambda}$ is a multiplier and hence $\sigma(\widehat{M}_z) \subset \overline{\mathbb{D}}$. The fact that the multiplier algebra is a subalgebra of H^∞ is a well known result regarding multipliers in functional Hilbert spaces going back to unpublished work of A. L. Shields, L. J. Wallen, and H.S. Shapiro in 1960 (see Lemma 2 and the subsequent remarks in Shields and Wallen [7]). The proof is short and is included for completeness. Simply note that if ϕ is a multiplier, then $\widehat{M}_\phi$ is bounded by the Closed Graph Theorem and hence

$$|\phi(w)K(w,w)| = \left|\left\langle \widehat{M}_\phi K(\cdot,w), K(\cdot,w)\right\rangle\right| \leqslant \|\widehat{M}_\phi\| \|K(\cdot,w)\|^2 = \|\widehat{M}_\phi\| K(w,w). \quad \square$$

PROPOSITION 1. *Let $U : H_0^2 \to H(K_\varphi)$ be given by $U(h) = \varphi h + h'$, then the following hold.*

1. *$U(\frac{1}{\psi(z)} f) = \frac{1}{\psi(z)} f'$, for every $f \in H_0^2$.*
2. *For every $f, g \in H_0^2$,*
$$\left\langle \frac{1}{\psi(z)} f', \frac{1}{\psi(z)} g' \right\rangle_{H(K_\varphi)} = \left\langle \frac{1}{\psi(z)} f, \frac{1}{\psi(z)} g \right\rangle_{H_0^2} = \int_{\partial\mathbb{D}} f\overline{g} \left|\frac{1}{\psi(z)}\right|^2 dm.$$
3. *For every $f \in H_0^2$,*
$$\left\| \frac{1}{\psi(z)} f' \right\|_{H(K_\varphi)}^2 = \left\| \frac{1}{\psi(z)} f \right\|_{H_0^2}^2 = \int_{\partial\mathbb{D}} |f|^2 \left|\frac{1}{\psi(z)}\right|^2 dm.$$
4. *$H^2 \subseteq H(K_\varphi)$.*

Proof. The first three items are all consequences of the proof of the above theorem. To see that $H^2 \subseteq H(K_\varphi)$, let $f \in H^2$. Then $h(z) = \frac{1}{\psi(z)} \int_0^z \psi(w) f(w) dw$ is in H_0^2 since multiplication by either $\frac{1}{\psi(z)}$ or $\psi(z)$ leaves H^2 invariant as both are in H^∞ and the antiderivative of an H^2 function is also in H^2 (consider its power series). Thus the function h defined above is in H_0^2. Now a simple computation shows that $f = \varphi h + h' = U(h) \in H(K_\varphi)$. $\square$

COROLLARY 1. *The operator $\widehat{M_{\frac{1}{\psi(z)}}}$ on $H(K_\varphi)$ is unitarily equivalent to the operator $T_{\frac{1}{\psi(z)}}$ on H_0^2 given by*

$$T_{\frac{1}{\psi(z)}}(h) = \frac{1}{\psi(z)} \int_0^z \varphi(w)h(w) + h'(w)dw = \frac{1}{\psi(z)} \int_0^z \varphi(w)h(w)dw + \frac{1}{\psi(z)} h$$

Proof. If $h \in H_0^2$, then

$$\begin{aligned} U^* \widehat{M_{\frac{1}{\psi(z)}}} U(h) &= U^* \widehat{M_{\frac{1}{\psi(z)}}} (\varphi h + h') = U^* \frac{1}{\psi(z)} (\varphi h + h') \\ &= \frac{1}{\psi(z)} \int_0^z \psi(w) \frac{1}{\psi(w)} (\varphi(w)h(w) + h'(w))dw \\ &= \frac{1}{\psi(z)} \int_0^z \varphi(w)h(w) + h'(w)dw. \quad \square \end{aligned}$$

PROPOSITION 2. *Assuming that ψ is one-to-one on the closed unit disk, then the operator $\widehat{M_z}$ on $H(K_\varphi)$ is subnormal if and only if $\widehat{M_{\frac{1}{\psi(z)}}}$ on $H(K_\varphi)$ is subnormal.*

Proof. If M_z is subnormal on $H(K_\varphi)$, then $M_{\frac{1}{\psi(z)}}$ is an analytic function of M_z and thus will also be subnormal. Conversely, assume that $M_{\frac{1}{\psi(z)}}$ is subnormal. Since $\frac{1}{\psi(z)}$ is analytic on the closed unit disk and one-to-one on the closed unit disk, it is a weak* generator for H^∞. Thus M_z is an analytic function of $M_{\frac{1}{\psi(z)}}$ and as such M_z must be subnormal. $\square$

Consider the space H^2_{ad} of all analytic functions on the unit disk whose antiderivatives belong to H^2. The norm on H^2_{ad} is $\|f\|^2_{H^2_{ad}} = \|\int_0^z f(w)dw\|^2_{H^2}$. So,

$$H^2_{ad} = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^2} < \infty \right\}.$$

The astute reader will note that the reproducing kernel and associated matrix L for the space H^2_{ad} are given by

$$K_{ad}(z,w) = \sum_{n=0}^{\infty} (n+1)^2 (\bar{w}z)^n \quad \text{and} \quad L_{ad} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \ddots \\ 0 & 0 & 3 & 0 & \ddots \\ 0 & 0 & 0 & 4 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

So L_{ad} consists of the diagonal entries of the matrix L for $H(K_\varphi)$. It is also worth noting that M_z on H^2_{ad} is the unilateral shift with matrix representation given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \ddots \\ 0 & \frac{2}{3} & 0 & 0 & \ddots \\ 0 & 0 & \frac{3}{4} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Among other things, the next proposition shows that $H^2_{ad} = P^2(\mu)$ for an appropriate measure μ on the open unit disk. The following propositions and results illustrate that various multiplication, integral operators, and combinations are subnormal on $P^2(\mu)$. Our goal in introducing this space is to make an additional connection to the space $H(K_\varphi)$ and establish the plausibility that some choices of ϕ can lead to $\widehat{M_z}$ being a subnormal operator. Even when this is not the case, it will be seen that $\widehat{M_z}$ is "close" to being a subnormal operator.

PROPOSITION 3. *If $h \in H(\mathbb{D})$, then*

$$\int_{\partial\mathbb{D}} \left| \int_0^z h(w)dw \right|^2 dm = \int_{\mathbb{D}} |h|^2 d\mu$$

where $d\mu = -4r\ln(r)dr\frac{d\theta}{2\pi} = -2\ln|z|\frac{dA}{\pi}$. Thus, the map $V : L^2_a(\mathbb{D},\mu) \to H^2_0$ given by $V(h) = \int_0^z h(w)dw$ is an onto isometry and $V^{-1}(f) = f'$.

Proof. First note that integration by parts shows that $\int_0^1 r^{2n}(-4r\ln(r))dr = \frac{1}{(n+1)^2}$ for all $n \geqslant 0$. Thus, $\|z^n\|^2_{L^2(\mu)} = \int |z^n|^2 d\mu = \frac{1}{(n+1)^2}$ for all $n \geqslant 0$. Also one easily checks that the functions $\{z^n\}_{n=0}^\infty$ are orthogonal in $L^2(\mu)$.

So, if $h(z) = \sum_{n=0}^\infty a_n z^n$ is in $H(\mathbb{D})$, then

$$F(z) := \int_0^z h(w)dw = \sum_{n=0}^\infty \frac{a_n}{n+1} z^{n+1}.$$

Using the orthogonality of the functions $\{z^n\}_{n=0}^\infty$ we get

$$\begin{aligned} \int_{\partial\mathbb{D}} \left| \int_0^z h(w)dw \right|^2 dm &= \|F\|^2_{H^2} = \sum_{n=0}^\infty \frac{|a_n|^2}{(n+1)^2} = \sum_{n=0}^\infty |a_n|^2 \|z^n\|^2_{L^2(\mu)} \\ &= \left\| \sum_{n=0}^\infty a_n z^n \right\|^2_{L^2(\mu)} = \int_{\mathbb{D}} |h|^2 d\mu. \quad \square \end{aligned}$$

COROLLARY 2. *(a) The operator of multiplication by a bounded analytic function $\phi(z)$ on H^2_{ad} is subnormal.*

(b) If $\phi \in H^\infty(\mathbb{D})$, then the operator $R_\phi : H^2_0 \to H^2_0$ given by

$$(R_\phi h)(z) = \int_0^z \phi(w)h'(w)dw$$

is subnormal.

(c) If $\phi \in H^\infty(\mathbb{D})$, then the operator, $Q_\phi : L^2_a(\mathbb{D},\mu) \to L^2_a(\mathbb{D},\mu)$ given by

$$(Q_\phi h)(z) = \phi(z)h(z) + \phi'(z)\int_0^z h(w)dw$$

is subnormal.

Proof. (a) Notice that the identity map $H^2_{ad} \to L^2_a(\mathbb{D},\mu)$ is an onto isometry. Since M_ϕ on $L^2_a(\mathbb{D},\mu)$ is subnormal, then M_ϕ on H^2_{ad} is also subnormal, since the identity map intertwines the two operators.

Now let $V : L^2_a(\mathbb{D},\mu) \to H^2_0$ be the operator given in Proposition 3, that is, $V(h) = \int_0^z h(w)dw$. For (b) notice that $R_\phi = VM_\phi V^{-1}$ holds where M_ϕ acts on $L^2_a(\mathbb{D},\mu)$ and for (c), notice that $Q = V^{-1}M_z V$ holds where M_z acts on H^2_0. $\square$

PROPOSITION 4. *If $f \in H(K_\varphi)$, then*

$$\|f\|^2_{H(K_\varphi)} = \int_{\partial\mathbb{D}} \left| \int_0^z \psi(w) f(w) \right|^2 \cdot \left| \frac{1}{\psi(z)} \right|^2 dm(z) .$$

Proof. Let $f \in H(K_\varphi)$ and let $h = U^{-1}(f)$ where U is the unitary $U : H_0^2 \to H(K_\varphi)$ given by $U(h) = \varphi h + h'$. Recall that U^{-1} is given by

$$U^{-1}(f) = \frac{1}{\psi(z)} \int_0^z \psi(w) f(w) dw ,$$

so

$$\begin{aligned} \|f\|^2_{H(K_\varphi)} = \|U^{-1}(f)\|^2_{H_0^2} &= \left\| \frac{1}{\psi(z)} \int_0^z \psi(w) f(w) dw \right\|^2_{H_0^2} \\ &= \int_{\partial\mathbb{D}} \left| \int_0^z \psi(w) f(w) dw \right|^2 \cdot \left| \frac{1}{\psi(z)} \right|^2 dm(z). \quad \square \end{aligned}$$

PROPOSITION 5. *For $n \geqslant 0$, let $f_n(z) = (n+1)z^n + \phi(z)z^{n+1}$, let $K_\varphi(z,w) = \sum_{n=0}^\infty f_n(z)\overline{f_n(w)}$ and let $d\mu = -4r\ln(r)dr\frac{d\theta}{2\pi}$. Then the following hold:*

1. *As sets of functions, $H(K_\varphi) = L_a^2(\mathbb{D},\mu)$.*
2. *The identity map from $L_a^2(\mathbb{D},\mu)$ to $H(K_\varphi)$ is a bounded invertible linear operator.*
3. *M_z on $H(K_\varphi)$ is similar to the subnormal operator M_z on $L_a^2(\mathbb{D},\mu)$. Since $\{f_n\}_{n=0}^\infty$ is an orthonormal basis for $H(K_\varphi)$, then U is a unitary map from $L_a^2(\mathbb{D},\mu)$ onto $H(K_\varphi)$.*
4. *If $\widehat{M_z}$ is multiplication by z on $H(K_\varphi)$, then $\widehat{M_z}$ is unitarily equivalent to the operator T on $L_a^2(\mathbb{D},\mu)$ given by*

   $$(Tf)(z) = zf(z) + \frac{\varphi(z)}{\psi(z)} \int_0^z \psi(w) \left(\int_0^w f(t)dt \right) dw$$

   via the unitary map $\tilde{U} : L_a^2(\mathbb{D},\mu) \to H(K_\varphi)$ given by

   $$(Uf)(z) = f(z) + \varphi(z) \int_0^z f(w)dw.$$

Proof. Define $\tilde{U} : Hol(\mathbb{D}) \to Hol(\mathbb{D})$ by $(\tilde{U}f)(z) = f(z) + \varphi(z)\int_0^z f(w)dw$. Notice that $\{(n+1)z^n\}_{n=0}^\infty$ is an orthonormal basis for $L_a^2(\mathbb{D},\mu)$ and, for all $n \geqslant 0$,

$$\tilde{U}((n+1)z^n) = (n+1)z^n + \varphi(z)z^{n+1} = f_n(z).$$

Thus $\tilde{U}: L^2_a(\mathbb{D},\mu) \to H(K_\varphi)$ is an onto unitary map, since it maps an orthonormal basis for one space onto an orthonormal basis for the other space. So,

$$\tilde{U}(L^2_a(\mathbb{D},\mu)) = H(K_\varphi). \tag{i}$$

We will now show that $\tilde{U}(L^2_a(\mathbb{D},\mu)) = L^2_a(\mathbb{D},\mu)$ and thus we will have that $L^2_a(\mathbb{D},\mu) = H(K_\varphi)$ as spaces of functions.

One can check that $\tilde{U}: Hol(\mathbb{D}) \to Hol(\mathbb{D})$ is a bijection, in fact its inverse is given by

$$(\tilde{U}^{-1}g)(z) = g(z) - \frac{\varphi(z)}{\psi(z)} \int_0^z \psi(w)g(w)dw.$$

To see this write $f(z) = (\tilde{U}^{-1}g)(z)$ and note $(\tilde{U}f)(z) = f(z) + \varphi(z)\int_0^z f(w)\,dw = g(z)$. Letting $h(z) = \int_0^z f(w)\,dw$ gives $h'(z) = f(z)$ and hence $h'(z) + \varphi(z)h(z) = g(z)$. Solving the differential equation we obtain $h(z) = \frac{1}{\psi(z)}\int_0^z \psi(w)g(w)\,dw$. Differentiating with the product rule and recalling that $(\frac{1}{\psi(z)})' = -\frac{\varphi(z)}{\psi(z)}$, we obtain the desired formula

$$(\tilde{U}^{-1}g)(z) = f(z) = g(z) - \frac{\varphi(z)}{\psi(z)} \int_0^z \psi(z)g(w)\,dw.$$

Notice that if $f(z) = \sum_{n=0}^\infty a_n z^n$, then

$$\|f\|^2_{L^2_a(\mathbb{D},\mu)} = \sum_{n=0}^\infty \frac{|a_n|^2}{(n+1)^2}. \tag{ii}$$

Thus $f \in L^2_a(\mathbb{D},d\mu)$ if and only if $\sum_{n=0}^\infty \frac{|a_n|^2}{(n+1)^2} < \infty$. Equation (ii) and the fact that φ, ψ, and $\frac{1}{\psi}$ are H^∞ functions can then be used to show that if $g \in L^2_a(\mathbb{D},\mu)$, then $U(g) \in L^2_a(\mathbb{D},\mu)$ and $U^{-1}(g) \in L^2_a(\mathbb{D},\mu)$. Hence $\tilde{U}(L^2_a(\mathbb{D},\mu)) \subseteq L^2_a(\mathbb{D},\mu)$ and $U^{-1}(L^2_a(\mathbb{D},\mu)) \subseteq L^2_a(\mathbb{D},\mu)$, which implies that $U(L^2_a(\mathbb{D},\mu)) = L^2_a(\mathbb{D},\mu)$. This together with (i) gives that $L^2_a(\mathbb{D},\mu) = H(K_\varphi)$. Thus (1) holds. Item (2) follows from the closed graph theorem, and (3) follows from (2).

To verify (4) requires computing

$$\begin{aligned}
&\tilde{U}^{-1}\widehat{M_z}\tilde{U}f(z)\\
&= zf(z) + z\varphi(z)\int_0^z f(w)dw - \frac{\varphi(z)}{\psi(z)}\int_0^z \psi(w)\left[wf(w) + w\varphi(w)\int_0^w f(t)dt\right]dw\\
&= zf(z) + z\varphi(z)\int_0^z f(w)dw - \frac{\varphi(z)}{\psi(z)}\int_0^z \psi(w)wf(w)dw\\
&\quad - \frac{\varphi(z)}{\psi(z)}\int_0^z \psi(w)w\varphi(w)\left(\int_0^w f(t)dt\right)dw.
\end{aligned}$$

The integral

$$\int_0^z \psi(w)w\varphi(w)\left(\int_0^w f(t)dt\right)dw$$

is equal to

$$z\psi(z)\int_0^z f(t)dt - \int_0^z w\psi(w)f(w)dw - \int_0^z \psi(w)\left(\int_0^w f(t)dt\right)dw.$$

One can see this by either interchanging the order of integration or using integration by parts with $u = w(\int_0^w f(t)dt)$ and $dv = \psi(w)\varphi(w)dw$. Replacing this integral in the above and cancelling like terms results in

$$\tilde{U}^{-1}\widehat{M_z}\tilde{U} = zf(z) + \frac{\varphi(z)}{\psi(z)}\int_0^z \psi(w)\left(\int_0^w f(t)dt\right)dw. \quad \square$$

3. The space $H(K_\varphi)$ with $\varphi(z) = 1$

We now consider the case when $\varphi(z) = 1$. In this case, $\psi(z) = e^z$, and, for $n \geqslant 0$, $f_n(z) = (n+1)z^n + z^{n+1}$. It is clear from the introductory section that $\{f_n\}$ forms an orthonormal basis for the tridiagonal reproducing kernel Hilbert space $H(K)$ with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & \ddots \\ 0 & 1 & 3 & 0 & \ddots \\ 0 & 0 & 1 & 4 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is also easy to determine that the natural domain $Dom(K)$ is the unit disk and that K has the closed form expression of

$$K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} = \frac{1+\bar{w}z}{(1-\bar{w}z)^3} + (\bar{w}+z)\frac{1}{(1-\bar{w}z)^2} + \frac{\bar{w}z}{1-\bar{w}z}.$$

We now interpret Theorem 1 and our previous results in this special case.

First note the linear map $U : H_0(\mathbb{D}) \to H(\mathbb{D})$ defined by $U(f) = f + f'$ has inverse given by

$$(U^{-1}g)(z) = f(z) = e^{-z}\int_0^z e^w g(w)\,dw.$$

THEOREM 2. *With H^2 denote the usual Hardy space on the unit disk, H_0^2 the subspace spanned by $\{\underline{z^n : n \geqslant 1}\}$, and $H(K)$ the space with reproducing kernel $K(z,w) = K_w(z) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ where $f_n(z) = (n+1)z^n + z^{n+1}$, we have the following results:.*

1. *$U : H_0^2 \to H(K)$ given by $U(f) = f + f'$ is an isomorphism. In this case*

$$U^*g = U^{-1}g = e^{-z}\int_0^z e^w g(w)dw;$$

2. *$H(K)$ consists of the functions $\{e^{-z}f'(z) : f \in H_0^2\}$;*
3. *$H(K)$ contains the polynomials;*
4. *$\widehat{M}_z : H(K) \to H(K)$ is a bounded operator where $\widehat{M}_z(f) = zf$;*
5. *the spectrum of $\widehat{M}_z$ is the closed unit disk;*
6. *$\widehat{M}_z : H(K) \to H(K)$ is unitarily equivalent to $T : H_0^2 \to H_0^2$ where*

$$(Tf)(z) = zf(z) - e^{-z}\int_0^z e^w f(w)\,dw.$$

7. *the multiplier algebra of $H(K)$ is a subalgebra of H^∞ which includes all functions analytic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$.*
8. *If ϕ is a bounded multiplier of $H(K)$ and $T_\phi : H_0^2 \to H_0^2$ by $T_\phi = U^*\widehat{M_\phi}U$ where $\widehat{M_\phi}$ is multiplication by ϕ on $H(K)$, then*

$$(T_\phi h)(z) = e^{-z}\int_0^z \phi(w)e^w(h(w)+h'(w))dw$$

or

$$(T_\phi h)(z) = \varphi(z)h(z) - e^{-z}\int_0^z \phi'(w)e^w h(w)dw.$$

Proof. Interpret Theorem 1 with $\varphi(z) = 1$. □

COROLLARY 3. *With $U : H_0^2 \to H(K)$ given by $U(h) = h + h'$, then the following hold:*

1. *For every $f,g \in H_0^2$,*

$$\langle e^{-z}f', e^{-z}g'\rangle_{H(K)} = \langle e^{-z}f, e^{-z}g\rangle_{H_0^2} = \int_{\partial\mathbb{D}} f\overline{g}|e^{-z}|^2 dm.$$

2. *For every $f \in H_0^2$,*

$$\|e^{-z}f'\|_{H(K)}^2 = \|e^{-z}f\|_{H_0^2}^2 = \int_{\partial\mathbb{D}} |f|^2|e^{-z}|^2 dm$$

3. *$H^2 \subset H(K)$*

The above formula for T is amenable to calculating $T(z^n)$ for $n \geqslant 1$ and hence a matrix representation for T on H_0^2. A short calculation shows that for $n \geqslant 1$,

$$\begin{aligned} T(z^n) &= z^{n+1} + (-1)^n n!\left[e^{-z} + \sum_{k=0}^{n}(-1)^{k+1}\frac{z^k}{k!}\right] \\ &= z^{n+1} + (-1)^n n!\sum_{k=n+1}^{\infty}(-1)^k\frac{z^k}{k!} = (1-\frac{1}{n+1})z^{n+1} + (-1)^n n!\sum_{k=n+2}^{\infty}(-1)^k\frac{z^k}{k!}. \end{aligned}$$

Thus, for $n \geqslant 1$,

$$||T(z^n)||^2 = \left(1 - \frac{1}{n+1}\right)^2 + \sum_{k=n+2}^{\infty} \left(\frac{n!}{k!}\right)^2.$$

The matrix form for T relative to the basis $\{z^n\}_{n=1}^{\infty}$ or $\widehat{M}_z$ relative to the basis $\{f_n(z)\}_{n=1}^{\infty}$ is now seen to be

$$\widehat{M_z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ \frac{1!}{3!} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ -\frac{1!}{4!} & \frac{2!}{4!} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ \frac{1!}{5!} & -\frac{2!}{5!} & \frac{3!}{5!} & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & \ddots \\ -\frac{1!}{6!} & \frac{2!}{6!} & -\frac{3!}{6!} & \frac{4!}{6!} & \frac{5}{6} & 0 & 0 & 0 & 0 & \ddots \\ \frac{1!}{7!} & -\frac{2!}{7!} & \frac{3!}{7!} & -\frac{4!}{7!} & \frac{5!}{7!} & \frac{6}{7} & 0 & 0 & 0 & \ddots \\ -\frac{1!}{8!} & \frac{2!}{8!} & -\frac{3!}{8!} & \frac{4!}{8!} & -\frac{5!}{8!} & \frac{6!}{8!} & \frac{7}{8} & 0 & 0 & \ddots \\ \frac{1!}{9!} & -\frac{2!}{9!} & \frac{3!}{9!} & -\frac{4!}{9!} & \frac{5!}{9!} & -\frac{6!}{9!} & \frac{7!}{9!} & \frac{8}{9} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Note that the subdiagonal entries of this matrix form of $\widehat{M}_z$ are the shift elements of the subnormal unilateral shift of multiplication by z on the space $L^2(\mathbb{D},\mu)$ where $d\mu = -4r\ln(r)dr\frac{d\theta}{2\pi}$.

Also, we can see that for $n \geqslant 0$,

$$U^*(z^n) = e^{-z}\int_0^z e^w w^n \, dw$$

which gives

$$U^*(z^n) = (-1)^{n+1} n! \left[e^{-z} - \sum_{k=0}^{n} (-1)^k \frac{z^k}{k!}\right] = n! \sum_{k=n+1}^{\infty} (-1)^{n+k+1} \frac{z^k}{k!}.$$

Thus

$$||U^*(z^n)||^2 = n!\big(e^{-z} - P_n(z)\big) = \sum_{k=n+1}^{\infty} \left(\frac{n!}{k!}\right)^2$$

where $P_n(z)$ is the nth Taylor polynomial of e^{-z} expanded about the origin. It is now easy to calculate the Grammian matrix as, for $n \geqslant m$,

$$\begin{aligned} \langle z^n, z^m\rangle_{H(K)} &= \langle U^*(z^n), U^*(z^m)\rangle_{H_0^2} \\ &= \left\langle n! \sum_{k=n+1}^{\infty} (-1)^{n+k+1} \frac{z^k}{k!}\,,\; m! \sum_{k=m+1}^{\infty} (-1)^{m+k+1} \frac{z^k}{k!} \right\rangle_{H_0^2} \\ &= n!m! \sum_{k=n+1}^{\infty} \frac{(-1)^{n+m}}{(k!)^2}. \end{aligned}$$

From this formula, we can easily express

$$\langle z^n, z^m\rangle = (-1)^{n-m}\frac{m!}{n!}\langle z^n, z^n\rangle$$

in terms of the diagonal elements $\langle z^n, z^n\rangle = \sum_{k=n+1}^{\infty}\frac{(n!)^2}{(k!)^2}$ as indicated in the following matrix form of the Grammian:

$$G = \begin{pmatrix} \langle 1,1\rangle & \frac{-0!}{1!}\langle z,z\rangle & \frac{0!}{2!}\langle z^2,z^2\rangle & \frac{-0!}{3!}\langle z^3,z^3\rangle & \cdots \\ \frac{-0!}{1!}\langle z,z\rangle & \langle z,z\rangle & \frac{-1!}{2!}\langle z^2,z^2\rangle & \frac{1!}{3!}\langle z^3,z^3\rangle & \ddots \\ \frac{0!}{2!}\langle z^2,z^2\rangle & \frac{-1!}{2!}\langle z^2,z^2\rangle & \langle z^2,z^2\rangle & \frac{-2!}{3!}\langle z^3,z^3\rangle & \ddots \\ \frac{-0!}{3!}\langle z^3,z^3\rangle & \frac{1!}{3!}\langle z^3,z^3\rangle & \frac{-2!}{3!}\langle z^3,z^3\rangle & \langle z^3,z^3\rangle & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4. $\widehat{M}_z$ is hyponormal

First we compute the matrix $\left(m_{j,k}\right)_{j,k=0}^{\infty}$ of the self-commutator

$$[\widehat{M}_z^*, \widehat{M}_z] = \widehat{M}_z^*\widehat{M}_z - \widehat{M}_z\widehat{M}_z^*.$$

The diagonal entries $m_{n,n}$ will be denoted by d_n. By virtue of being self adjoint, we only compute the entries $m_{j,k}$ where $j \leqslant k$.

To begin, let B denote the matrix

$$B = \begin{pmatrix} \frac{1!}{2!} & 0 & 0 & 0 & \cdots \\ -\frac{1!}{3!} & \frac{2!}{3!} & 0 & 0 & \ddots \\ \frac{1!}{4!} & -\frac{2!}{4!} & \frac{3!}{4!} & 0 & \ddots \\ -\frac{1!}{5!} & \frac{2!}{5!} & -\frac{3!}{5!} & \frac{4!}{5!} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

and let S denote the matrix of the usual unilateral shift. Then $\widehat{M}_z = S(I-B)$. Showing $\widehat{M}_z$ is hyponormal is equivalent to showing that $Q(\vec{x}) = ||\widehat{M}_z\vec{x}||^2 - ||\widehat{M}_z^*\vec{x}||^2$ is positive definite. Notice that

$$Q(\vec{x}) = ||\vec{x}||^2 - ||S^*\vec{x}||^2 + ||B\vec{x}||^2 - ||B^*S^*\vec{x}||^2 - 2\langle B\vec{x},\vec{x}\rangle + 2\langle B^*S^*\vec{x}, S^*\vec{x}\rangle$$

Since all entries of the matrix B are real valued, we can limit consideration to vectors $\vec{x}$ where all entries are real valued as well. With that assumption,

$$B\vec{x} = \begin{pmatrix} +\frac{1!}{2!}x_0 \\ -\frac{1!}{3!}x_0 + \frac{2!}{3!}x_1 \\ +\frac{1!}{4!}x_0 - \frac{2!}{4!}x_1 + \frac{3!}{4!}x_2 \\ -\frac{1!}{5!}x_0 + \frac{2!}{5!}x_1 - \frac{3!}{5!}x_2 + \frac{4!}{5!}x_3 \\ \vdots \end{pmatrix}$$

and

$$B^*S^*\vec{x} = \begin{pmatrix} \frac{1!}{2!}x_1 - \frac{1!}{3!}x_2 + \frac{1!}{4!}x_3 - \frac{1!}{5!}x_4 + \cdots \\ \frac{2!}{3!}x_2 - \frac{2!}{4!}x_3 + \frac{2!}{5!}x_4 - \cdots \\ \frac{3!}{4!}x_3 - \frac{3!}{5!}x_4 + \cdots \\ \frac{4!}{5!}x_4 - \cdots \\ \vdots \end{pmatrix}.$$

Thus,

$$\langle B\vec{x}, B\vec{x}\rangle = \sum_{j=0}^{\infty}\sum_{s=j}^{\infty}\left(\frac{(j+1)!}{(s+2)!}\right)^2 x_j^2 + 2\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\sum_{s=k}^{\infty}(-1)^{j+k}\frac{(j+1)!(k+1)!}{(s+2)!^2}x_jx_k \,,$$

$$\langle B^*S^*\vec{x}, B^*S^*\vec{x}\rangle = \sum_{j=1}^{\infty}\sum_{s=1}^{j}\left(\frac{s!}{(j+1)!}\right)^2 x_j^2 + 2\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\sum_{s=1}^{j}(-1)^{j+k}\frac{(s!)^2}{(j+1)!(k+1)!}x_jx_k \,,$$

$$\langle B\vec{x}, \vec{x}\rangle = \sum_{j=0}^{\infty}\frac{(j+1)!}{(j+2)!}x_j^2 + \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}(-1)^{j+k}\frac{(j+1)!}{(k+2)!}x_jx_k \,,$$

and

$$\langle B^*S^*\vec{x}, S^*\vec{x}\rangle = \sum_{j=1}^{\infty}\frac{j!}{(j+1)!}x_j^2 + \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}(-1)^{j+k}\frac{j!}{(k+1)!}x_jx_k.$$

Hence $Q(\vec{x}) = \sum_{j=0}^{\infty} d_j x_j^2 + 2\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} m_{j,k}x_jx_k$ where

$$d_0 = \sum_{s=2}^{\infty}\frac{1}{s!^2} \quad \text{and} \quad m_{0,k} = (-1)^k\left(-\frac{1}{(k+2)!} + \sum_{s=k}^{\infty}\frac{(k+1)!}{(s+2)!^2}\right)$$

for $k > 0$.

If $j > 0$ and $k > j$, then

$$d_k = \frac{2}{k+1} - \frac{2}{k+2} + \sum_{s=k}^{\infty}\left(\frac{(k+1)!}{(s+2)!}\right)^2 - \sum_{s=1}^{k}\left(\frac{s!}{(k+1)!}\right)^2$$

and

$$m_{j,k} = (-1)^{j+k}\left[\frac{j!}{(k+1)!} - \frac{(j+1)!}{(k+2)!} + \sum_{s=k}^{\infty}\frac{(j+1)!(k+1)!}{((s+2)!)^2} - \sum_{s=1}^{j}\frac{s!^2}{(j+1)!(k+1)!}\right].$$

Let $p(x,h)$ denote the rising Pochhammer symbol if h is positive and the falling Pochhammer symbol if h is negative. Thus $p(x,h) = x(x+1)\cdots(x+h-1)$ if h is positive and $p(x,h) = x(x-1)\cdots(x+h+1)$ if h is negative. Making use of this symbol allows for a more convenient rewriting of d_k and $m_{j,k}$ which we summarize below in the following proposition.

PROPOSITION 6. *With*

$$p_k = \sum_{s=1}^{\infty} \left(\frac{1}{p(k+2,s)} \right)^2, \quad q_j = \sum_{t=1}^{j} \left(\frac{1}{p(j+1,-t)} \right)^2,$$

and

$$b_{j,k} = \frac{k-j+1}{(j+1)(k+2)} + p_k - q_j,$$

the diagonal entries d_k *of* $[\widehat{M}_z^*, \widehat{M}_z]$ *can be expressed as*

$$d_k = \frac{1}{(k+1)(k+2)} + b_{k,k}$$

and, for $0 < j < k$, *the lower triangular entries* $m_{j,k}$ *are given by*

$$m_{j,k} = (-1)^h \frac{1}{p(k+1,-h)} b_{j,k} \text{ where } h = k-j.$$

Proof. Simply compare to the formulas in the proposition to the formulas immediately before the proposition. $\square$

The following proposition provides the framework for the proof that $\widehat{M}_z$ is hyponormal.

PROPOSITION 7. *Let* $A = (a_{j,k})_{j,k=0}^{\infty}$ *be a self adjoint matrix with positive diagonal entries* $d_k = a_{k,k}$ *and let* $A_N = (a_{j,k})_{0 \leqslant j,k \leqslant N}$ *be the* $(N+1) \times (N+1)$ *cutdown of* A. *Let* $\vec{a}_n$ *denote the* n *by* 1 *column vector with first entry* $a_{0,n}$ *and last entry* $a_{n-1,n}$. *If, for some* $N \in \mathbb{Z}^+$,

1. $\inf_{||\vec{f}||=1} \left\langle A_N \vec{f}, \vec{f} \right\rangle \geqslant d_{N+1}$, *and*
2. $d_n - ||\vec{a}_n|| \geqslant d_{n+1}$ *for all* $n \geqslant N+1$,

then $A > 0$.

Proof. It suffices to show that $\inf\limits_{||\vec{f}||=1} \left\langle A_N \vec{f}, \vec{f} \right\rangle \geqslant d_{n+1}$ for all $n \geqslant N$. This will be established by induction. The base case is the hypothesis, so assume that $\lambda_{n-1} = \inf\limits_{||\vec{f}||=1} \left\langle A_{n-1} \vec{f}, \vec{f} \right\rangle \geqslant d_n$. Let $\vec{f} \in \mathbb{C}^n$ and let $\alpha \in \mathbb{C}$ with $||\vec{f}||^2 + |\alpha|^2 = 1$. Then

$$\begin{aligned}
\left\langle A_n \begin{pmatrix} \vec{f} \\ \alpha \end{pmatrix}, \begin{pmatrix} \vec{f} \\ \alpha \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} A_{n-1} & \vec{a}_n \\ \vec{a}_n^* & d_n \end{pmatrix} \begin{pmatrix} \vec{f} \\ \alpha \end{pmatrix}, \begin{pmatrix} \vec{f} \\ \alpha \end{pmatrix} \right\rangle \\
&= \left\langle A_{n-1} \vec{f}, \vec{f} \right\rangle + d_n |\alpha|^2 + 2Re \left(\alpha \left\langle \vec{a}_n, \vec{f} \right\rangle \right) \\
&\geqslant \lambda_{n-1} ||\vec{f}||^2 + d_n |\alpha|^2 - 2|\alpha| \, ||\vec{f}|| \, ||\vec{a}_n|| \\
&\geqslant \min(\lambda_{n-1}, d_n) - ||\vec{a}_n|| \\
&= d_n - ||\vec{a}_n|| \\
&\geqslant d_{n+1}. \quad \square
\end{aligned}$$

We will argue that $\widehat{M}_z$ is positive definite by showing that the proposition above applies to the matrix $(m_{j,k})$ where $m_{n,n} = d_n$. Due to the complexity of the formulas, brute force will be applied to bounding $||\vec{a}_n||$ for small n. For larger values of n, the derived bounds will suffice. For some arguments, we appeal to simple computations on ugly rational functions which can easily be verified using a symbolic algebra system such as Mathematica on a computer. Perhaps a more elegant approach suffices, but we have not found it. We will first prove a small lemma, followed by the main result of this section.

LEMMA 1. *If $j,k \geqslant 2$ then*

$$\frac{1}{(k+2)^2}+\frac{1}{(k+2)^2(k+3)^2} < p_k < \frac{1}{(k+2)^2-1}$$
$$\frac{1}{(j+1)^2-1} < q_j < \frac{1}{(j+1)^2}+\frac{1}{(j+1)j^3}$$

Proof. The inequalities for p_k follow easily from the series definition. The left hand inequality for q_j follows from the fact that $\sum_{s=j}^{\infty}\left(\frac{1}{j+1}\right)^2 \leqslant \frac{1}{(j+1)!}$ as long as $j \geqslant 2$. The right hand inequality for q_j follows by induction on j. The base case follows from $q_1 = \frac{1}{4} < \frac{3}{4}$ To accomplish the induction step, note that q_j satisfies the order preserving recursion: $q_{j+1} = \frac{1+q_j}{(j+2)^2}$. Applying the recursion to both sides of the induction hypothesis completes the lemma. □

THEOREM 3. *The operator $\widehat{M}_z$ on $H(K)$ is hyponormal where*

$$K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \quad , \ f_n(z) = ((n+1)+z)^n.$$

Proof. By Proposition 7, the theorem requires that $d_n - d_{n+1}$ exceeds

$$||\vec{a}_n|| = \left(\sum_{j=0}^{n-1} |m_{j,n}|^2\right)^{1/2}.$$

Notice, by Proposition 6,

$$\begin{aligned}
d_n - d_{n+1} &= \frac{2}{(n+1)(n+2)} - \frac{2}{(n+2)(n+3)} + (p_n - p_{n+1}) - (q_n - q_{n+1}) \\
&> \frac{4}{(n+1)(n+2)(n+3)} + \left(\frac{1}{(n+2)^2} + \frac{1}{(n+2)^2(n+3)^2} - \frac{1}{(n+3)^2-1}\right) \\
&\quad - \left(\frac{1}{(n+1)^2} + \frac{1}{(n+1)n^3} - \frac{1}{(n+2)^2-1}\right) \\
&= \frac{4-r(n)}{(n+1)(n+2)(n+3)}
\end{aligned}$$

where $r(n)$ is a rational function in n which decreases for positive n. Specifically

$$r(n) = \frac{144+420n+484n^2+359n^3+186n^4+53n^5+6n^6}{n^3(n+1)(n+2)(n+3)(n+4)}$$

and $r(n)$ decreases for $n \geqslant 1$ as the derivative of $r = r(n)$ is negative for $n \geqslant 1$. Also $r(n) \to 0$ as $n \to \infty$.

By Proposition 6, $|m_{j,k}| = \dfrac{1}{p(k+1,-h)}|b_{j,k}|$. First we show that $b_{j,k} > 0$ for $1 \leqslant j < k$ and use this positivity to bound $b_{j,k}$ from above. First note that $p_k, q_j > 0$ for all $j,k > 0$ Since $k > j$,

$$\begin{aligned} b_{j,k} &= \frac{1}{(j+1)} - \frac{1}{k+2} + p_k - q_j \\ &\geqslant \frac{1}{(j+1)} - \frac{1}{j+3} + 0 - \frac{1}{(j+1)^2} - \frac{1}{(j+1)j^3} \\ &> \frac{1}{(j+1)} - \frac{1}{j+3} - \frac{1}{(j+1)^2} - \frac{1}{(j+1)j^3}. \end{aligned}$$

Evaluating this last expression at $j = 3+t$ results in a ratio of polynomials in t, all of whose coefficients are positive. Using $q_1 = 1/4$ and $q_2 = 1/9 + 1/36$ and the lower bound for p_k, it is a simple matter to verify the inequality for $j = 1,2$ as well.

Having established the positivity of $b_{j,k}$, notice that, for $1 \leqslant j < k$,

$$\begin{aligned} b_{j,k} &< \frac{1}{(j+1)} - \frac{1}{k+2} + \frac{1}{(k+2)^2-1} - \frac{1}{(j+1)^2-1} \\ &< \frac{1}{j+1} - \frac{1}{k+2} + \frac{1}{(k+2)^2} - \frac{1}{(j+1)^2} \\ &= \frac{j}{(j+1)^2} - \frac{k+1}{(k+2)^2}. \end{aligned}$$

For $j = n-1$ and $k = n$, this leads to

$$|m_{n-1,n}| < \frac{2n^2-4}{n^2(n+1)(n+2)^2}.$$

For $j = n-2 = k$, this leads to

$$|m_{n-2,n}| < \frac{3(n^2-n-3)}{(n-1)^2 n(n+1)(n+2)^2}.$$

For $j < n-2$, the cruder approximation $b_{j,n} < \frac{1}{j+1}$ leads to $|m_{j,n}| < \frac{j!}{(n+1)!}$. Additionally, $|m_{0,n}| < \frac{1}{(n+2)!}$.

Thus,

$$\begin{aligned}
||\vec{a}_n||^2 &= |m_{0,n}|^2 + \sum_{j=1}^{n-3} |m_{j,n}|^2 + |m_{n-2,n}|^2 + |m_{n-1,n}|^2 \\
&< \left(\frac{1}{(n+2)!}\right)^2 + \sum_{j=1}^{n-3} \left(\frac{j!}{(n+1)!}\right)^2 + \left(\frac{3(n^2-n-3)}{(n-1)^2 n(n+1)(n+2)^2}\right)^2 \\
&\quad + \left(\frac{2n^2-4}{n^2(n+1)(n+2)^2}\right)^2.
\end{aligned}$$

Note that if $n > 3$,

$$\begin{aligned}
\sum_{j=1}^{n-3} \left(\frac{j!}{(n+1)!}\right)^2 &= \frac{q_{n-3}}{(n+1)^2 n^2 (n-1)^2} \\
&< \frac{\frac{1}{(n-2)^2} + \frac{1}{(n-2)(n-3)^3}}{(n+1)^2 n^2 (n-1)^2} \\
&= \frac{n^3 - 9n^2 + 28n - 29}{(n+1)^2 n^2 (n-1)^2 (n-2)^2 (n-3)^3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
||\vec{a}_n||^2 &< \left(\frac{1}{(n+2)!}\right)^2 + \frac{n^3 - 9n^2 + 28n - 29}{(n+1)^2 n^2 (n-1)^2 (n-2)^2 (n-3)^3} \\
&\quad + \left(\frac{3(n^2-n-3)}{(n-1)^2 n(n+1)(n+2)^2}\right)^2 + \left(\frac{2n^2-4}{n^2(n+1)(n+2)^2}\right)^2 \\
&= \left(\frac{1}{(n+2)!}\right)^2 + \frac{r_1(n)}{(n+1)^2(n+2)^2(n+3)^2}
\end{aligned}$$

where $r_1(n)$ is the obvious rational function of n obtained on factoring $(n+1)^2(n+2)^2(n+3)^2$ out of the denominator in the last three terms. Note that a quick inspection of these last three terms and their relationship to $r_1(n)$ shows that $r_1(n) \to 4$ as $n \to \infty$. Additionally, a tedious computation (or use of a computer algebra system) will show that $r_1'(n+4) = -\frac{\phi_1(n)}{\phi_2(n)}$ where ϕ_1 and ϕ_2 are polynomials in n with all positive coefficients. Hence $r_1(n)$ is decreasing for $n \geqslant 4$, a fact that also can be verified by plotting the function.

We claim that, for all $n \geqslant 3$,

$$||\vec{a}_n||^2 < (d_n - d_{n+1})^2.$$

For $3 \leqslant n \leqslant 4$, direct computation using $p_n > 1/(n+2)^2$ and the exact values of q_j verifies the inequality. In light of the upper estimate for $||\vec{a}_n||$ and the lower estimate for $d_n - d_{n+1}$, it suffices to show that

$$\left(\frac{1}{(n+2)!}\right)^2 + \frac{r_1(n)}{(n+1)^2(n+2)^2(n+3)^2} < \left(\frac{4-r(n)}{(n+1)(n+2)(n+3)}\right)^2$$

for all $n \geqslant 5$. As $\lim_{n\to\infty} r_1(n) = 4$ and $(4-r(n))^2$ increases to 16 as $n \to \infty$, it is apparent that $||\vec{a}_n|| < d_n - d_{n+1}$ for large n. Since $r_1(n)$ is decreasing for $n \geqslant 4$ and it is easily seen that $\frac{(n+1)(n+2)(n+3)}{(n+2)!}$ is decreasing with n, to finish the proof of the claim, it is enough to verify

$$\left(\frac{(n+1)^2(n+2)^2(n+3)^2}{(n+2)!}\right)^2 + r_1(n) < (4-r(n))^2$$

for $n = 5$ which is easily done. To finish the proof that T_z is hyponormal, it suffices to prove that

$$\inf_{||\vec{f}||=1} \langle A_2\vec{f},\vec{f}\,\rangle > d_3$$

Approximating A_2 and d_3 to 6 decimal places gives

$$\inf_{||\vec{f}||=1} \langle A_2\vec{f},\vec{f}\,\rangle \approx .086 > .070 \approx d_3$$

where we use the fact that for a positive n by n matrix A, $\inf_{||\vec{f}||=1} \langle A\vec{f},\vec{f}\rangle$ equals the smallest eigenvalue. $\square$

5. A norm estimate for M-dominated matrices and $||\widehat{M}_z|| = 1$

In general it is very difficult to calculate the norm of a matrix or operator. In this section we develop an effective, but somewhat cumbersome, method for estimating the norm of an operator. As will be seen, it is particularly effective for lower triangular matrices alike in matrix form to our operator $\widehat{M}_z$. We begin with some terms and notation followed by a simple 3×3 lower triangular matrix example. Following that we state and prove a norm-estimation theorem which we then use to calculate the norm of $\widehat{M}_z$.

DEFINITION 1. If $\vec{x} = (x_1, x_2, \cdots, x_N)$ is a in vector $\mathbb{C}^n$ or l^2_+, then the support of $\vec{x}$ is defined by $supp(\vec{x}) = \{k : x_k \neq 0\}$.

DEFINITION 2. Let $T = [t_{n,k}]$ and $D = [d_{n,k}]$ be $N \times N$ matrices such that for each n and k, either $d_{n,k} = t_{n,k}$ or $d_{n,k} = 0$. Let $\vec{t}_n$, $\vec{d}_n$ be the nth row-vectors of T, D respectively and set $\vec{v}_n = \vec{t}_n - \vec{d}_n$. If $\vec{x} = (x_1, x_2, \cdots, x_N) \in \mathbb{C}^n$, then express $\vec{x}$ as $\vec{x} = \vec{x}_{n,0} + \vec{x}_{n,1} + \vec{x}_{n,2}$ where, for $j = 0, 1, 2$,

- the coordinates of $\vec{x}_{n,j}$ are either the same as the coordinates of $\vec{x}$ or are zero,
- $supp(\vec{d}_n) = supp(\vec{x}_{n,2})$ and
- $supp(\vec{v}_n) = supp(\vec{x}_{n,1})$.

Note that $k \in supp(\vec{x}_{n,0})$ if and only if $t_{n,k} = 0$.

For $M > 0$, the matrix D is said to be M-dominating for T if there exist non-negative sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ satisfying:

1. for each row $\vec{t}_n$ of T_N,

$$|\vec{t}_n \cdot \vec{x}|^2 \leqslant \varepsilon_n \|\vec{x}_{n,1}\|^2 + (M^2 - \delta_n)\|\vec{x}_{n,2}\|^2,$$

and

2. for each i between 1 and N,

$$\sum_{n \in \mathscr{A}_i} \varepsilon_n < \sum_{n \in \mathscr{B}_i} \delta_n$$

where

$$\mathscr{A}_i = \{n : i \in supp(\vec{v}_n)\} \quad \text{and} \quad \mathscr{B}_i = \{n : i \in supp(\vec{d}_n)\}.$$

If T and D are bounded infinite matrices, then D is M-dominating for T if, for each $N \in \mathbb{N}$, the $N \times N$ cut down D_n of D is M-dominating for the $N \times N$ cut down T_n of T.

REMARK 1. It is worth noting that if we let $A_n \geqslant \|\vec{v}_n\|^2$, $B_n \geqslant \|\vec{v}_n\|\|\vec{d}_n\|$, and $C_n \geqslant \|\vec{d}_n\|^2$, then condition 1 of Definition 2 can be established by finding non-negative sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ for which the quadratic inequality

$$(A_n - \varepsilon)\|\vec{x}_{n,1}\|^2 + 2B_n\|\vec{x}_{n,1}\|\|\vec{x}_{n,2}\| + (C_n - M^2 + \delta_n)\|\vec{x}_{n,2}\|^2 \leqslant 0$$

holds since

$$\begin{aligned}|\vec{t}_n \cdot \vec{x}|^2 &\leqslant \|\vec{v}_n\|^2\|\vec{x}_{n,1}\|^2 + 2\|\vec{v}_n\|\|\vec{d}_n\|\|\vec{x}_{n,1}\|\|\vec{x}_{n,2}\| + \|\vec{d}_n\|^2\|\vec{x}_{n,2}\|^2 \\ &\leqslant A_n\|\vec{x}_{n,1}\|^2 + 2B_n\|\vec{x}_{n,1}\|\|\vec{x}_{n,2}\| + C_n\|\vec{x}_{n,2}\|^2.\end{aligned}$$

The quadratic inequality can be expressed in dot product form as

$$\left\langle \begin{bmatrix} (A_n - \varepsilon_n) & B_n \\ B_n & (C_n - M^2 + \delta_n) \end{bmatrix} \begin{bmatrix} \|\vec{x}_{n,1}\| \\ \|\vec{x}_{n,2}\| \end{bmatrix}, \begin{bmatrix} \|\vec{x}_{n,1}\| \\ \|\vec{x}_{n,2}\| \end{bmatrix} \right\rangle \leqslant 0.$$

The two conditions $(A_n - \varepsilon_n) < 0$ and

$$det\left(\begin{bmatrix} (A_n - \varepsilon_n) & B_n \\ B_n & (C_n - M^2 + \delta_n) \end{bmatrix} \right) \geqslant 0$$

are sufficient for the quadratic inequality to be non-positive.

EXAMPLE 1. Let $T = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, so $\vec{t}_1 = (2,0,0)$, $\vec{t}_2 = (1,2,0)$, $\vec{t}_3 = (0,1,2)$.

We will show that T is 3-dominated by $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Note that $\vec{d}_1 = (2,0,0)$, $\vec{d}_2 = (0,2,0)$, $\vec{d}_3 = (0,0,2)$ while $\vec{v}_1 = (0,0,0)$, $\vec{v}_2 = (1,0,0)$, $\vec{v}_3 = (0,1,0)$.

If $\vec{x} = (x_1, x_2, x_3)$, then

$$\begin{array}{lll} \vec{x}_{10} = (0, x_2, x_3) & \vec{x}_{11} = (0,0,0) & \vec{x}_{12} = (x_1, 0, 0) \\ \vec{x}_{20} = (0,0,x_3) & \vec{x}_{21} = (x_1,0,0) & \vec{x}_{22} = (0,x_2,0) \\ \vec{x}_{30} = (x_1,0,0) & \vec{x}_{31} = (0,x_2,0) & \vec{x}_{32} = (0,0,x_3) \end{array}$$

while $\mathscr{A}_1 = \{2\}$, $\mathscr{B}_1 = \{1\}$, $\mathscr{A}_2 = \{3\}$, $\mathscr{B}_2 = \{2\}$, $\mathscr{A}_3 = \emptyset$, and $\mathscr{B}_3 = \{3\}$. Letting $A_1 = ||\vec{v}_1||^2$, $B_1 = ||\vec{v}_1||||\vec{d}_n|| = 0$, $C_1 = ||\vec{d}_n||^2 =^2 = 4$, and $0 < \varepsilon_1$, we see that $A_1 - \varepsilon_1 = -\varepsilon_1 < 0$ and

$$\begin{aligned} det\left(\begin{bmatrix} (A_1 - \varepsilon_1) & B_1 \\ B_1 & (C_1 - M^2 + \delta_1) \end{bmatrix}\right) &= det\left(\begin{bmatrix} -\varepsilon_1 & 0 \\ 0 & (4 - M^2 + \delta_1) \end{bmatrix}\right) \\ &= -\varepsilon_1(4 - M^2 + \delta_1) \geqslant 0 \end{aligned}$$

provided that $M^2 \geqslant 4 + \delta_1$. Proceeding, we take $A_2 = 1$, $C_2 = 4$, $B_1 = 2$, and the conditions imply $(1-\varepsilon_2) < 0$ and $(1-\varepsilon_2)(4 - M^2 + \delta_2) \geqslant 4$. Similarly, with $A_3 = 1$, $C_3 = 4$, and $B_3 = 2$, we have $(1-\varepsilon_3) < 0$ and $(1-\varepsilon_3)(4 - M^2 + \delta_3) \geqslant 4$. Since $\sum_{n \in \mathscr{A}_i} \varepsilon_n < \sum_{n \in \mathscr{B}_i} \delta_n$, we see $1 < \varepsilon_2 < \delta_1$, $1 < \varepsilon_3 < \delta_2$, while δ_3 can be any positive number. Letting $\varepsilon_2 = \varepsilon_3 = 1 + \varepsilon$ where $\varepsilon > 0$ and $\delta_1 = \delta_2 = 1 + \delta$ where $\delta > 0$, we are led to $M^2 \geqslant \frac{4}{\varepsilon} + \varepsilon + 5 + \delta$ which is minimized by letting $\delta \to 0$ and $\varepsilon = 2$. This results in $M^2 = 9$ or $M = 3$. The next result shows that this provides an estimate of $||T|| \leqslant 3$. The actual value can be numerically computed to be about 2.76.

THEOREM 4. *Let H be a separable complex Hilbert space with orthonormal basis $\mathscr{B} = \{x_1, x_2, x_3, \cdots\}$, assume that $T = D + V$, where T, D, and V are continuous linear operators on H, and for each $n = 1, 2, 3, \ldots$, the row vectors $\vec{d}_n$ and $\vec{v}_n$ of D and V respectively have disjoint support. If, for each $N \in \mathbb{N}$, T_N is M-dominated by D_N, then*

1. *with $\vec{t}_n$ denoting the nth row of T_N, for each $\vec{x} \in \mathbb{C}^N$,*

$$||T_N\vec{x}||^2 = \sum_{n=1}^{N} |\vec{t}_n \cdot \vec{x}|^2 \leqslant M^2 ||\vec{x}||^2;$$

2. $\|T\| \leqslant M$.

Proof. Let P_N denote the projection onto $\mathscr{B}_n = \{x_1, x_2, \cdots, x_N\}$. Since $T_N = P_N T P_N$ converges in the strong operator topology to T, (2) will hold if $\|T_N\| \leqslant M$ for all $N \in \mathbb{N}$. To that end we observe that, for each $\vec{x} \in \mathbb{C}^N$,

$$||T_N\vec{x}||^2 = \sum_{n=1}^{N} |\vec{t}_n . \vec{x}|^2$$

By assumption D_N M-dominates T_N, which implies the existence of non-negative sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ such that:

1. for each row $\vec{t}_n$ of T_N,

$$|\vec{t}_n \cdot \vec{x}|^2 \leqslant 0 \cdot \|\vec{x}_{n,0}\| + \|\vec{v}_n\|^2 \|\vec{x}_{n,1}\|^2 + 2\|\vec{v}_n\| \|\vec{d}_n\| \|\vec{x}_{n,1}\| \|\vec{x}_{n,2}\| + \|\vec{d}_n\|^2 \|\vec{x}_{n,2}\|^2$$
$$\leqslant \varepsilon_n \|\vec{x}_{n,1}\|^2 + (M^2 - \delta_n)\|\vec{x}_{n,2}\|^2,$$

and

2. for each i between 1 and N,

$$\sum_{n \in \mathscr{A}_i} \varepsilon_n < \sum_{n \in \mathscr{B}_i} \delta_n$$

where
$$\mathscr{A}_i = \{n : i \in supp(\vec{v}_n)\} \quad \text{and} \quad \mathscr{B}_i = \{n : i \in supp(\vec{d}_n)\}.$$

Hence,

$$\|T_N \vec{x}\|^2 = \sum_{n=1}^{N} |\vec{t}_n . \vec{x}|^2 \leqslant \sum_{n=1}^{N} \left(\varepsilon_n \|\vec{x}_{n,1}\|^2 + (M^2 - \delta_n)\|\vec{x}_{n,2}\|^2 \right)$$
$$= \sum_{n=1}^{N} \left(\sum_{n \in \mathscr{A}_i} \varepsilon_n x_i^2 + \sum_{n \in \mathscr{B}_i} (M^2 - \delta_n) x_i^2 \right).$$

Applying (2), we obtain

$$\|T_N \vec{x}\|^2 \leqslant \sum_{n=1}^{N} \left(\sum_{i \in \mathscr{A}_i \cup \mathscr{B}_i} M^2 x_i^2 \right) \leqslant M^2 \|\vec{x}\|^2.$$

Hence $\|T_N\| \leqslant M$. □

PROPOSITION 8. *If $\widehat{M}_z$ denotes multiplication by z on $H(K)$ where $K(z,w) = K_w(z) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ with $f_n(z) = (n+1)z^n + z^{n+1}$, then $||\widehat{M}_z|| = 1$.*

Proof. Previously it was shown that the spectrum of $\widehat{M}_z$ was the closed unit disk, so $||\widehat{M}_z|| \geqslant 1$. Additionally a matrix form for $\widehat{M}_z$ was determined in section 2. For convenience we make all of the entries of the matrix form of $\widehat{M}_z$ positive and use Theorem 4 to show that this matrix has norm at most 1, thereby establishing the result. Note that making the entries all positive will not change our norm estimate since the choices of A_n, B_n, and C_n in the M-dominating scheme only rely on the norms of the rows of D and V.

Let $T = [t_{n,k}]$ be the lower triangular matrix with diagonal entries $t_{n,n} = \frac{n}{n+1}$ and $t_{n,m} = \frac{m!}{(n+1)!}$ for $n \geqslant 2$ and $m = 1, 2, \ldots, n-1$. Let D be the diagonal matrix with diagonal entries $\{\frac{n}{n+1}\}$, and for $n = 1, \ldots, N$, let $\vec{d}_n = \frac{n}{n+1}\vec{e}_n$ where $\vec{e}_n$ is the canonical basis row-vector. Let $\vec{v}_1 = \vec{0}$ and $\vec{v}_n = \sum_{k=1}^{n-1} \frac{k!}{(n+1)!}\vec{e}_k$ for $n \geqslant 2$. Hence $\vec{x}_{n,2} = x_n \vec{e}_n$, $\vec{x}_{n,1} = \sum_{k=1}^{n-1} x_k \vec{e}_k$, and $\vec{x}_{n,0} = \sum_{k=n+1}^{N} x_k \vec{e}_k$. We shall show below that T is M-dominated by D with $M=1$. Since $\sigma(T)=\overline{\mathbb{D}}$, invoking the previous theorem will then show that T is a contraction.

When $n=1$, $|\vec{t}_1\cdot\vec{x}|^2=|\vec{d}_1\cdot\vec{x}|^2=\frac{1}{4}$. Letting $A_1=0$, $C_1=\frac{1}{4}$, $B_1=0$, $M=1$, $\varepsilon_1=0$, and $\delta_1=\frac{3}{4}$, we have $A_1-\varepsilon_1=0\leqslant 0$, $C_1-M^2+\delta_1=\frac{1}{2^2}-1+\frac{3}{4}=0\leqslant 0$, and the matrix below is negative semidefinite as it is the zero matrix

$$\begin{bmatrix} A_1-\varepsilon_1 & B_1 \\ B_1 & (C_1-M^2+\delta_1)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

When $n=2$, $\vec{d}_2=\frac{2}{3}\vec{e}_2$ and $\vec{v}_2=\frac{1}{6}\vec{e}_1$. Letting $A_2=\frac{1}{6^2}$, $C_2=\frac{4}{9}$, $B_2=\frac{1}{9}$, $M=1$, $\varepsilon_2=\frac{1}{2^2}$, and $\delta_2=\frac{1}{2}$, we have $A_2-\varepsilon_2=-\frac{2}{9}\leqslant 0$ and

$$\begin{vmatrix} A_2-\varepsilon_2 & B_2 \\ B_2 & (C_2-M^2+\delta_2)\end{vmatrix}=\begin{vmatrix} -\frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9}-1+\frac{1}{2}\end{vmatrix}=0\geqslant 0.$$

Note that for $n\geqslant 3$,

$$\|\vec{v}_n\|^2=\sum_{k=1}^{n-1}\frac{k!^2}{(n+1)!^2}\leqslant\left(1-\frac{1}{(n-1)(n-2)}\right)\frac{(n-1)!^2}{(n+1)!^2}\leqslant\frac{1}{n^4}.$$

With $A_n=\frac{1}{n^4}$, $C_n=\frac{n^2}{(n+1)^2}$, $B_n=\frac{1}{n(n+1)}$, $M=1$, $\varepsilon=\frac{1}{n^2}$, and $\delta_n=\frac{1}{n}$, it is routine to check that for $n\geqslant 3$, $A_n-\varepsilon=\frac{1}{n^4}-\frac{1}{n^2}<0$, and

$$\begin{aligned}\begin{vmatrix} (A_n-\varepsilon_n) & B_n \\ B_n & (C_n-M^2+\delta_n)\end{vmatrix}&=\begin{vmatrix} (\frac{1}{n^4}-\frac{1}{n^2}) & \frac{1}{n(n+1)} \\ \frac{1}{n(n+1)} & (\frac{n^2}{(n+1)^2}-1+\frac{1}{n})\end{vmatrix}\\ &=\frac{n^4-2n^3-2n^2+n+1}{n^5(n+1)^2}\geqslant 0.\end{aligned}$$

It remains to show that for each i between 1 and N,

$$\sum_{n\in\mathscr{A}_i}\varepsilon_n<\sum_{n\in\mathscr{B}_i}\delta_n$$

where

$$\mathscr{A}_i=\{n: i\in supp(\vec{v}_n)\}\quad\text{and}\quad\mathscr{B}_i=\{n: i\in supp(\vec{d}_n)\}.$$

First note that $\mathscr{B}_i=\{n: n=i\}$ is a singleton set and that $\mathscr{A}_i=\{n: i+1\leqslant n\leqslant N-1\}$. When $i=1$,

$$\sum_{n\in\mathscr{A}_i}\varepsilon_n<\sum_{k=2}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}-1<\sum_{n\in\mathscr{B}_i}\delta_n=\delta_1=\frac{3}{4}.$$

For $i\geqslant 2$,

$$\sum_{n\in\mathscr{A}_i}\varepsilon_n<\sum_{k=i+1}^{\infty}\frac{1}{k^2}\leqslant\int_i^{\infty}\frac{1}{x^2}=\frac{1}{i}=\sum_{n\in\mathscr{B}_i}\delta_n=\delta_i=\frac{1}{i}.$$

Thus T is 1-dominated by D and $\|T\|\leqslant 1$. Thus $\|\widehat{M}_z\|\leqslant 1$. Since the spectral radius of $\widehat{M}_z$ is 1, $\|\widehat{M}_z\|=1$. $\square$

6. Subnormality and $\widehat{M}_z$

We now return to the question of the subnormality of $\widehat{M}_z$. Thus far it has been established that $\widehat{M}_z$ is a hyponormal operator whose norm is equal to its spectral radius. These are both necessary conditions for $\widehat{M}_z$ to be a subnormal operator. Additionally, it has been shown that $\widehat{M}_z$ is similar to the subnormal operator M_z on $L^2_a(\mathbb{D},\mu)$ and that for all multipliers ϕ of $H(K)$, $\widehat{M}_\phi$ is a perturbation by an integral operator of the Toeplitz operator M_ϕ on $L^2_a(\mathbb{D},\mu)$. In fact the operator $\widehat{M}_z$ is sufficiently "close" to being a subnormal operator that it passes most of the simpler tests for subnormality. In this section we show that $\widehat{M}_z$ is not a subnormal operator by making use of a modification of an old result of Alan Lambert [5].

We begin by looking at the formula

$$\sum_{k=n+1}^{\infty}\left(\frac{n!}{k!}\right)^2=\frac{1}{(n+1)^2}+\frac{1}{(n+1)^2(n+2)^2}+\frac{1}{(n+1)^2(n+2)^2(n+3)^2}+\cdots$$

for the diagonal elements of the Grammian computed at the end of Section 3, we can see that the function

$$f(x)=\frac{1}{(x+1)^2}+\frac{1}{(x+1)^2(x+2)^2}+\frac{1}{(x+1)^2(x+2)^2(x+3)^2}+\cdots$$

is a limit of the sequence of completely monotone functions

$$f_n(x)=\sum_{k=1}^{n}\frac{1}{(x+1)^2\cdots(x+k)^2}.$$

Since the set of completely monotone functions is closed with respect to pointwise convergence (see page 5 of Schilling, Song, and Vondraček), f is a completely monotone function. The function f is also known as the hypergeometric function $\frac{{}_1F_2(1;x+2,x+2;1)}{(x+1)^2}$ and it is easily seen to satisfy the relation

$$f(n)=\frac{1}{(n+1)^2}\Big[1+f(n+1)\Big].$$

Since f is completely monotonic, it is the Laplace transform of a *unique* positive measure μ on $[0,\infty)$ (see Widder [9]). That unique measure is determined in the next proposition.

PROPOSITION 9. *The function*

$$f(n)=\frac{1}{(n+1)^2}+\frac{1}{(n+1)^2(n+2)^2}+\frac{1}{(n+1)^2(n+2)^2(n+3)^2}+\cdots$$

is the Laplace transform of the measure $h(t)dt$ where

$$h(t)=e^{-t}\left(\frac{I_0(2)K_0(2e^{\frac{-t}{2}})-K_0(2)I_0(2e^{\frac{-t}{2}})}{I_1(2)K_0(2)+I_0(2)K_0(2)}\right).$$

where K_0 denotes the BesselK function and I_0 the Bessel0 function.

Proof. Recall the Laplace transform of a function g is defined by $\mathscr{L}\big(g(t)\big)(n) = \int_0^\infty e^{-nt}g(t)\,dt$. Assume that $f(n) = \mathscr{L}\big(h(t)\big)$ for some unknown $h \geqslant 0$ and note that elementary properties of the Laplace transform imply $\mathscr{L}\big(e^{-t}h(t)\big) = f(n+1)$ as well as $\mathscr{L}\big(te^{-t}\big) = \frac{1}{(n+1)^2}$.

Since $f(n) = \frac{1}{(n+1)^2}\big[1+f(n+1)\big]$,

$$\mathscr{L}^{-1}\big(f(n)\big) = \mathscr{L}^{-1}\left(\frac{1}{(n+1)^2}\right) + \mathscr{L}^{-1}\left(\frac{1}{(n+1)^2}f(n+1)\right).$$

Since the transform of the convolution of two functions is the product of the transforms, $h(t) = te^{-t} + \phi(t)$ where

$$\phi(t) = (te^{-t}) * (e^{-t}h(t)) = \int_0^t (t-\tau)e^{-(t-\tau)}e^{-\tau}h(\tau)\,d\tau = e^{-t}\int_0^t (t-\tau)h(\tau)\,d\tau.$$

Hence $h(t)$ is equal to

$$te^{-t} + te^{-t}\int_0^t h(\tau)\,d\tau - e^{-t}\int_0^t \tau h(\tau)\,d\tau = te^{-t}\left(1+\int_0^t h(\tau)\,d\tau\right) - e^{-t}\int_0^t \tau h(\tau)\,d\tau.$$

Multiplying by e^t yields

$$e^t h(t) = t\left(1+\int_0^t h(\tau)\,d\tau\right) - \int_0^t \tau h(\tau)\,d\tau.$$

Evaluating at $t=0$ shows that $h(0)=0$. Differentiating with respect to t yields

$$e^t\left(h'(t)+h(t)\right) = \left(1+\int_0^t h(\tau)\,d\tau\right) + t\,h(t) - t\,h(t) = 1 + \int_0^t h(\tau)\,d\tau.$$

Evaluating again at $t=0$ shows that $h'(0)=1$. A final differentiation yields

$$e^t\left(h''(t)+2h'(t)+h(t)\right) = h(t)$$

from which we obtain the following second order linear equation with initial conditions

$$h''(t)+2h'(t)+(1-e^{-t})h(t) = 0 \qquad h(0)=0, \quad h'(0)=1.$$

This equation can now be solved by the series method or by a computer algebra solution such as *Mathematica* or Maple to obtain the solution

$$h(t) = e^{-t}\left(\frac{I_0(2)K_0(2e^{\frac{-t}{2}}) - K_0(2)I_0(2e^{\frac{-t}{2}})}{I_1(2)K_0(2)+I_0(2)K_0(2)}\right). \quad \square$$

Note that it is now a routine change of variables to draw the measure $h(t)dt$ on $[0,\infty)$ back to the measure $q(t)dt$ on $[0,1]$ resulting in

$$q(t) = 2t\left(\frac{I_0(2)K_0(2\sqrt{t^2}) - K_0(2)I_0(2\sqrt{t^2})}{I_1(2)K_0(2)+I_0(2)K_0(2)}\right) \text{ and } f(n) = \langle z^n, z^n\rangle = \int_0^1 t^{2n}q(t)dt.$$

The next proposition is a slight reformulation of a characterization of subnormality due to Alan Lambert [5].

PROPOSITION 10. *Let S be an operator on a Hilbert space $\mathscr{H}$ with $ker(S) = \{0\}$ and $||S|| = 1$. The operator S is subnormal if and only if $\{||S^n f||^2||\}_{n=0}^{\infty}$ is a completely monotone sequence for each $f \in \mathscr{H}$.*

Proof. Lambert's [5] result asserts that S is a subnormal operator if and only if $\left\{\frac{||S^{n+1}f||}{||S^n f||}\right\}_{n=0}^{\infty}$ is the weight sequence of a subnormal weighted shift for each nonzero $f \in \mathscr{H}$. Note that if $\{f_n(z) = \sqrt{a_n}z^n\}$ is an orthonormal basis for a reproducing kernel Hilbert space with kernel $K(z,w) = \sum_{n=0}^{\infty} a_n(\bar{w}z)^n$, then the operator M_z of multiplication by z is a weighted shift with weight sequence $\left\{\sqrt{\frac{a_n}{a_{n+1}}}\right\}$. Moreover, Shield's [6] shows that all weighted shifts arise in this fashion. It is well known that a weighted shift is subnormal if and only if $\{\frac{a_0}{a_n}\}$ is a moment sequence (see Conway [4],page 57) if and only if $\{\frac{a_0}{a_n}\}$ is a completely monotonic sequence (see Widder [9]). Setting $a_n = \frac{1}{||S^n f||^2}$ we see that Lambert's condition is equivalent to the assertion that $\{\frac{||S^n f||^2||}{||f||^2}\}_{n=0}^{\infty}$ is a completely monotonic sequence for each nonzero $f \in \mathscr{H}$. Since $\{\frac{||S^n f||^2||}{||f||^2}\}_{n=0}^{\infty}$ is a completely monotonic sequence if and only if $\{||S^n f||^2||\}_{n=0}^{\infty}$ is a completely monotone sequence the result is established. $\square$

PROPOSITION 11. *The operator $\widehat{M}_z$ of multiplication by z on $H(K)$ is not a subnormal operator when $f_n(z) = ((n+1)+z)z^n$ and $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$.*

Proof. It has been established that the spectrum of $\widehat{M}_z$ is the closed unit disk and the norm is equal to its spectral radius. Also, it is clear that the kernel of $\widehat{M}_z$ is trivial. Hence the conditions of Proposition 10 result hold and we need only produce a function f_0 such that $\{||\widehat{M}_z^n f_0||^2||\}_{n=0}^{\infty}$ is not a completely monotonic sequence. To that end consider $f_0(z) = \frac{1}{10} + z$ and note that

$$\begin{aligned}
g(n) = ||\widehat{M}_z^n f_0||^2 &= \left|\left|\frac{1}{10}z^n + z^{n+1}\right|\right|^2 \\
&= \frac{1}{100}\langle z^n, z^n\rangle + \frac{2}{10}\langle z^n, z^{n+1}\rangle + \langle z^{n+1}, z^{n+1}\rangle \\
&= \frac{1}{100}f(n) + \left[1 - \frac{1}{5}\frac{1}{n+1}\right]f(n+1)
\end{aligned}$$

since $\langle z^n, z^{n+1}\rangle = \langle z^{n+1}, z^n\rangle = \frac{-1}{n+1}\langle z^{n+1}, z^{n+1}\rangle$. Here $f(n) = \langle z^n, z^n\rangle$ is the completely monotone sequence from Proposition 9 above. If $g(n)$ is completely monotonic, then it is the Laplace transform of a unique positive measure on $[0,\infty)$. Hence the inverse transform of $g(n)$ must be a positive function on $[0,\infty)$. Note that by Proposition 9 and elementary properties of the Lapace transform we see

$$\begin{aligned}
\mathscr{L}^{-1}(g) &= \mathscr{L}^{-1}\left(\frac{1}{100}f(n) + \left[1 - \frac{1}{5}\frac{1}{n+1}\right]f(n+1)\right) \\
&= \frac{1}{100}h(t) + e^{-t}h(t) - \frac{1}{5}\left(e^{-t}\right) * \left(e^{-t}h(t)\right)
\end{aligned}$$

where $*$ denotes convolution. Since

$$\left(e^{-t}\right) * \left(e^{-t}h(t)\right) = \int_0^t e^{-(t-\tau)} e^{-\tau} h(\tau)\, d\tau = e^{-t} \int_0^t h(\tau)\, d\tau,$$

we have

$$\hat{g}(t) = L^{-1}(g) = \left(\frac{1}{100} + e^{-t}\right) h(t) - \frac{1}{5} e^{-t} \int_0^t h(\tau)\, d\tau.$$

Since e^t is always positive, we can consider instead the positivity of

$$e^t \hat{g}(t) = \left(\frac{1}{100} e^t + 1\right) h(t) - \frac{1}{5} \int_0^t h(\tau)\, d\tau.$$

A numerical computation shows that $e^{6.2}\hat{g}(6.2) \approx -.114402$ with all digits significant, hence g is not completely monotone and $\widehat{M}_z$ is not subnormal. $\square$

7. Open questions and concluding remarks

A distinctive aspect of the operators looked at in this paper is that they are perturbations of multiplication operators by integral operators. While this presents the usual computational challenges of anti-differentiation, it also opens a new door to examples of near subnormal operators with a rich functional calculus. Many concepts of near subnormal operators have been introduced over the years such as polynomially hyponormal, n-hyponormal, weakly subnormal, ... It would be interesting to see how our operator $\widehat{M}_z$ relates to these other concepts of nearly subnormal.

In general it is difficult to compute and work with the matrix form of $\widehat{M}_z$ on $H(K_\varphi)$ unless $\varphi(z)$ has a particularly simple form such as $\varphi(z) = 1$. A natural question is what occurs if $\varphi(z) = a$ for $|a| < 1$. In this case, for $n \geqslant 1$,

$$T(z^n) = \left(1 - \frac{1}{n+1}\right) z^{n+1} + (-1)^n \frac{n!}{a^{n+1}} \sum_{k=n+2}^{\infty} (-1)^k a^k \frac{z^k}{k!}$$

and

$$||T(z^n)||^2 = \left(1 - \frac{1}{n+1}\right)^2 + \sum_{k=n+2}^{\infty} \left(\frac{n!}{k!}\right)^2 |a|^{2(k-n-1)}.$$

The matrix form for T relative to the basis $\{z^n\}_{n=1}^{\infty}$ or $\widehat{M}_z$ relative to the basis $\{f_n(z)\}_{n=1}^{\infty}$

is given by

$$\widehat{M_z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ a\frac{1!}{3!} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ -a^2\frac{1!}{4!} & a\frac{2!}{4!} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ a^3\frac{1!}{5!} & -a^2\frac{2!}{5!} & a\frac{3!}{5!} & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & \ddots \\ -a^4\frac{1!}{6!} & a^3\frac{2!}{6!} & -a^2\frac{3!}{6!} & a\frac{4!}{6!} & \frac{5}{6} & 0 & 0 & 0 & 0 & \ddots \\ a^5\frac{1!}{7!} & -a^4\frac{2!}{7!} & a^3\frac{3!}{7!} & -a^2\frac{4!}{7!} & a\frac{5!}{7!} & \frac{6}{7} & 0 & 0 & 0 & \ddots \\ -a^6\frac{1!}{8!} & a^5\frac{2!}{8!} & -a^4\frac{3!}{8!} & a^3\frac{4!}{8!} & -a^2\frac{5!}{8!} & a\frac{6!}{8!} & \frac{7}{8} & 0 & 0 & \ddots \\ a^7\frac{1!}{9!} & -a^6\frac{2!}{9!} & a^5\frac{3!}{9!} & -a^4\frac{4!}{9!} & a^3\frac{5!}{9!} & -\frac{a^2 6!}{9!} & a\frac{7!}{9!} & \frac{8}{9} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

As in the case of $\varphi(z) = 1$, the subdiagonal entries of this matrix form of $\widehat{M_z}$ are the shift elements of the subnormal unilateral shift of multiplication by z on the space $L^2(\mathbb{D},\mu)$ where $d\mu = -4r\ln(r)dr\frac{d\theta}{2\pi}$. The effect of the constant a is that the perturbation from this unilateral shift is even smaller than before. This has no effect on the hyponormality of the operator. Although it is unlikely that the operator is subnormal for any value a, the inherent difficulty in checking monotonicity as well as inverse Laplace transforms leaves an open question as to whether there exists a value of a for which $\widehat{M_z}$ is subnormal.

In the cases $\varphi(z) = \frac{1}{2-z}$ and $\varphi(z) = \frac{2z}{z^2+\alpha}$ it is fairly routine to determine the form of $T(z^n)$ and hence a matrix representation of $\widehat{M_z}$. While an interesting exercise, neither of these cases result in $\widehat{M_z}$ being even a hyponormal operator.

The spaces $H(K_\varphi)$ and operators introduced in this paper are but a first step in the direction of Integro-Multiplication operators on analytic reproducing kernel Hilbert spaces. One can generalize the mapping $U(f) = \varphi f + f'$ from $H_0^2 \to H(K)$ in many ways. For example $U(f) = \varphi_1 f + \varphi_2 f' + f''$, $U(f) = f + f' + f''$, or many other combinations of derivatives. The spaces are naturally associated with differential and integral equations and the properties of the multiplication operators are wide open for exploration.

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