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ON n–LIE DERIVATIONS OF TRIANGULAR ALGEBRAS

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Abstract. This article is about the study of n-Lie derivations on triangular algebras under certain restrictions and it is shown that every n-Lie derivation apparently split into an extremal n-derivation and an n-linear central map on triangular algebras. In addition, we implement this result to the classical examples of triangular algebras.

1. Historical development

Inspecting the assorted properties of derivation defined through the well known Leibniz rule under the influence of various algebraic structures is an immense topic of study among the algebraists since past few decades. A number of articles has been published related to n-derivations on different rings and algebras (see [10, 7, 14, 2, 6, 9, 8, 15, 16, 22, 4, 1, 11, 12, 21, 5, 3] and references therein). In the year 1993, Brešar et al. [10] proved that "every biderivation over a noncommutative prime ring can be described as inner biderivation". Also, in [7] Brešar investigated biderivations on semiprime rings. The readers are encouraged to read the survey paper [9, Section 3] where applications of biderivations to other fields are also described. Benkovič in [6] defined the concept of an extremal biderivation and proved that "under certain conditions a biderivation of a triangular algebra is a sum of an extremal and an inner biderivation." Ghosseiri [15] showed that "every biderivation of upper triangular matrix rings is decomposed into the sum of three biderivations D, ψ and Δ, where $D(E_{11}, E_{11}) = 0$, ψ is an extremal biderivation and Δ is a special kind of biderivation." Moreover, they proved that " every biderivation of upper triangular matrices over a noncommutative prime ring is inner which extended some results due to Benkovič [6]." Wang et al. [23] proved that "every n-derivation $(n \geqslant 3)$ is an extremal n-derivation for a certain class of triangular algebras." Besides associative algebras or rings, numerous authors studied biderivation and related maps on various types of Lie algebras for example see [11, 12, 21, 17] and references therein.

The study of Lie derivations and related maps are also active area of research in last few decades (see [13, 9] and references therein). Cheung [13] initiated the study of maps on triangular algebras and proved that commuting maps and Lie derivations has proper form on triangular algebras. In [18] Liang et al. introduced the notion of Lie biderivation on triangular algebras and showed that "under certain mild assumptions,

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Lie biderivation is the sum of an inner biderivation, an extremal biderivation and a some central bilinear mapping."

In the light of above mentioned literature, in this article we evaluate the n-Lie derivations on triangular algebras and prove that every n-Lie derivation can be written as $\phi = \zeta + \varphi$, where ζ is an extremal n-derivation such that $\zeta(x_1, x_2, \cdots, x_n) = [x_1, [x_2, \cdots, [x_n, \phi(e, e, \cdots, e)] \cdots]]$ and φ is an n-linear central map.

2. Basic definitions & preliminaries

Throughout this paper, R will always denote a commutative ring with unity. Let $\mathscr{A}$ be an algebra over a commutative ring R with unity. For any $x, y \in \mathscr{A}$, $[x, y] = xy - yx$ denotes the Lie product (or commutator) and $\mathfrak{Z}(\mathscr{A})$ denote the center of $\mathscr{A}$. An R-linear map $d : \mathscr{A} \to \mathscr{A}$ is said to be a derivation (resp. Lie derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d([x, y]) = [d(x), y] + [x, d(y)]$) for all $x, y \in \mathscr{A}$. If derivation d takes form $d(x) = [x, a]$ for some fixed $a \in \mathscr{A}$, then d is called an inner derivation on $\mathscr{A}$.

A bilinear map $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is said to be a biderivation (resp. bi-Lie derivation or some author mention it as Lie biderivation [18]), if it is a derivation (resp. Lie derivation) with respect to both components, i.e.,

$$\phi(xy, z) = \phi(x, z)y + x\phi(y, z) \text{ and } \phi(x, yz) = \phi(x, y)z + y\phi(x, z)$$
$$(\text{resp. } \phi([x, y], z) = [\phi(x, z), y] + [x, \phi(y, z)] \text{ and } \phi(x, [y, z]) = [\phi(x, y), z] + [y, \phi(x, z)])$$

for all $x, y, z \in \mathscr{A}$. An n-linear map $\phi : \mathscr{A} \times \mathscr{A} \times \cdots \times \mathscr{A} \to \mathscr{A}$ is said to be a n-derivation (resp. n-Lie derivation), if it is a derivation (resp. Lie derivation) with respect to all components. A n-derivation is called a permuting n-derivation if $\phi(x_1, x_2, \cdots, x_n) = \phi(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)})$ for all $x_1, x_2, \cdots, x_n \in \mathscr{A}$ and $\sigma \in S_n$, where S_n denotes the symmetric group of degree n. A permuting 2-derivation is said to be a symmetric biderivation.

If $\mathscr{A}$ is a noncommutative algebra, then the map $\phi(x, y) = \lambda[x, y]$ for all $x, y \in \mathscr{A}$, where $\lambda \in \mathfrak{Z}(\mathscr{A})$ is called an inner biderivation. A permuting n-derivation $\zeta : \mathscr{A} \times \mathscr{A} \times \cdots \times \mathscr{A} \to \mathscr{A}$ is said to be an extremal n-derivation if it is of the form $\zeta(x_1, x_2, \cdots, x_n) = [x_1, [x_2, \cdots, [x_n, a] \cdots]]$ for all $x_1, x_2, \cdots, x_n \in \mathscr{A}$, where $a \in \mathscr{A}$ and $a \notin \mathfrak{Z}(\mathscr{A})$ such that $[[\mathscr{A}, \mathscr{A}], a] = 0$. An extremal 2-derivation is said to be an extremal biderivation.

Let A and B be unital algebras over R and let M be an (A, B)-bimodule which is faithful as a left A-module and also as a right B-module. The R-algebra

$$\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B}) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B} \right\}$$

under the usual matrix operations is called triangular algebra. The center of $\mathfrak{T}$ is

$$\mathrm{Z}(\mathfrak{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb \ \forall\, m \in \mathrm{M} \right\}.$$

Define two natural projections $\pi_{\mathrm{A}} : \mathfrak{T} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathfrak{T} \to \mathrm{B}$ by

$$\pi_{\mathrm{A}}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = a \text{ and } \pi_{\mathrm{B}}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = b.$$

Moreover, $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) \subseteq \mathrm{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) \subseteq \mathrm{Z}(\mathrm{B})$ and there exists a unique algebraic isomorphism $\tau : \pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) \to \pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})), m \in \mathrm{M}$.

Let 1_{A} (resp. 1_{B}) be the identity of the algebra A (resp. B) and let I be the unity of triangular algebra $\mathfrak{T}$. Throughout, this paper we shall use the following notions: $e = \begin{bmatrix} 1_{\mathrm{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $f = I - e = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{B}} \end{bmatrix}$ are the idempotent elements of $\mathfrak{T}$ and $\mathrm{A} = e\mathfrak{T}e$, $\mathrm{M} = e\mathfrak{T}f$, $\mathrm{B} = f\mathfrak{T}f$. Thus, $\mathfrak{T} = e\mathfrak{T}f + e\mathfrak{T}f + f\mathfrak{T}f = \mathrm{A} + \mathrm{M} + \mathrm{B}$. Also, $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T}))$ and $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T}))$ are isomorphic to $e\mathrm{Z}(\mathfrak{T})e$ and $f\mathrm{Z}(\mathfrak{T})f$ respectively. Then there is an algebra isomorphisms $\tau : e\mathrm{Z}(\mathfrak{T})e \to f\mathrm{Z}(\mathfrak{T})f$ such that $am = m\tau(a)$ for all $m \in e\mathfrak{T}f$.

An (A,B)-bimodule homomorphism $\mathfrak{f} : \mathrm{M} \to \mathrm{M}$ is of the standard form if there exist $a_0 \in \mathfrak{Z}(\mathrm{A})$, $b_0 \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{f}(m) = a_0 m + m b_0$ for all $m \in \mathrm{M}$.

Now we should mention some important results which are subsequently used in this article.

LEMMA 2.1. [6, Corollary 3.4] *Suppose that every derivation of the triangular algebra $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ is inner. Then every bimodule homomorphism $\mathfrak{f} : \mathrm{M} \to \mathrm{M}$ is of the standard form.*

LEMMA 2.2. *Let $\phi : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ be a bi-Lie derivation on $\mathfrak{T}$. Then ϕ satisfies*

$$[\phi(x,y),[u,v]] + [[x,v],\phi(u,y)] = [[x,y],\phi(u,v)] + [\phi(x,v),[u,y]] \textit{ for all } x,y,u,v \in \mathfrak{T}.$$

Proof. For any $x,y,u,v \in \mathfrak{T}$, we have

$$\begin{aligned} \phi([x,u],[y,v]) &= [\phi(x,[y,v]),u] + [x,\phi(u,[y,v])] \\ &= [[\phi(x,y),v],u] + [[y,\phi(x,v)],u] + [x,[\phi(u,y),v]] + [x,[y,\phi(u,v)]]. \end{aligned}$$

On the other way,

$$\begin{aligned} \phi([x,u],[y,v]) &= [\phi([x,u],y),v] + [y,\phi([x,u],v)] \\ &= [[\phi(x,y),u],v] + [[x,\phi(u,y)],v] + [y,[\phi(x,v),u]] + [y,[x,\phi(u,v)]]. \end{aligned}$$

On comparing above two expressions, it is easy to obtain that

$$[\phi(x,y),[u,v]] - [[x,y],\phi(u,v)] = [\phi(x,v),[u,y]] - [[x,v],\phi(u,y)]$$

for all $x,y,u,v \in \mathfrak{T}$. $\square$

3. Key content

In this part, we establish the main result of the article by induction for $n \geqslant 3$ and one can look into [13, 18] for $n=1,2$ respectively. Primarily, we establish the result for $n=3$ and then we apply induction for $n>3$ as follows:

THEOREM 3.1. *Let $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ be a triangular algebra. If the following conditions hold:*

1. $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{B})$;
2. *at least one of the algebras* A *and* B *is noncommutative;*
3. *every bimodule homomorphism* $\mathfrak{f} : \mathrm{M} \to \mathrm{M}$ *is of the standard form;*
4. *If* $\alpha a = 0$, $\alpha \in \mathfrak{Z}(\mathfrak{T})$, $0 \neq a \in \mathfrak{T}$, *then* $\alpha = 0$.

Then every 3-Lie derivation $\phi : \mathfrak{T}\times\mathfrak{T}\times\mathfrak{T} \to \mathfrak{T}$ can be written as $\phi = \zeta + \varphi$, where ζ is an extremal 3-derivation such that $\zeta(x,y,z) = [x,[y,[z,\phi(e,e,e)]]]$, where $\phi(e,e,e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$ and φ is a 3-linear central map on $\mathfrak{T}$.

Proof. Let us fix $z \in \mathfrak{T}$ and define a map $\phi_z : \mathfrak{T}\times\mathfrak{T} \to \mathfrak{T}$ by

$$\phi_z(x,y) = \phi(x,y,z) \text{ for all } x,y,z \in \mathfrak{T}.$$

By Lemma 2.2 it follows that

$$\begin{aligned} [\phi_z(x,y),[u,v]] + [[x,v],\phi_z(u,y)] &= [[x,y],\phi_z(u,v)] + [\phi_z(x,v),[u,y]] \\ [\phi(x,y,z),[u,v]] + [[x,v],\phi(u,y,z)] &= [[x,y],\phi(u,v,z)] + [\phi(x,v,z),[u,y]]. \end{aligned} \tag{3.1}$$

Since ϕ is 3-Lie biderivation

$$\begin{aligned} \phi(e,e,m) &= \phi(e,e,[e,m]) \\ &= [\phi(e,e,e),m] + [e,\phi(e,e,m)] \\ &= \phi(e,e,e)m - m\phi(e,e,e) + e\phi(e,e,m) - \phi(e,e,m)e \\ &= e\phi(e,e,e)em - mf\phi(e,e,e)f + e\phi(e,e,m)f. \end{aligned}$$

Multiplying by e from right and f from left, we get $e\phi(e,e,e)em = mf\phi(e,e,e)f$ for all $m \in \mathrm{M}$. Therefore, we have $e\phi(e,e,e)e + f\phi(e,e,e)f \in \mathfrak{Z}(\mathfrak{T})$.

Suppose that $\phi(e,e,e) \neq 0$ and the map $\zeta : \mathfrak{T}\times\mathfrak{T}\times\mathfrak{T} \to \mathfrak{T}$ by $\zeta(x,y,z) = [x,[y,[z,\phi(e,e,e)]]]$ for all $x,y,z \in \mathfrak{T}$, is an extremal 3-derivation of $\mathfrak{T}$. Since $\phi(e,e,e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$. Note that

$$\begin{aligned} \zeta(e,e,e) &= [e,[e,[e,\phi(e,e,e)]]] \\ &= [e,[e,[e,e\phi(e,e,e)e + e\phi(e,e,e)f + f\phi(e,e,e)f]]] \\ &= e\phi(e,e,e)f. \end{aligned}$$

Then $\phi(e,e,e)-\zeta(e,e,e)=e\phi(e,e,e)e+f\phi(e,e,e)f\in\mathfrak{Z}(\mathfrak{T})$. Set $\phi-\zeta=\varphi$, it is easy to check that φ is a 3-Lie biderivation satisfying $\varphi(e,e,e)\in\mathfrak{Z}(\mathfrak{T})$.

For $x,y\in\mathfrak{T}$ and $m\in\mathrm{M}$

$$\begin{aligned}\varphi(x,y,m)&=\varphi(x,y,[e,m])\\&=[\varphi(x,y,e),m]+[e,\varphi(x,y,m)]\\&=\varphi(x,y,e)m-m\varphi(x,y,e)+e\varphi(x,y,m)-\varphi(x,y,m)e\\&=e\varphi(x,y,e)em-mf\varphi(x,y,e)f+e\varphi(x,y,m)f.\end{aligned}$$

Multiplying by e from right and f from left, we get $e\varphi(x,y,e)em=mf\varphi(x,y,e)f$ for all $m\in\mathrm{M}$. Hence

$$\varphi(x,y,m)\in\mathrm{M}\quad\text{and}\quad e\varphi(x,y,e)e+f\varphi(x,y,e)f\in\mathfrak{Z}(\mathfrak{T})\tag{3.2}$$

for all $x,y\in\mathfrak{T}$ and $m\in\mathrm{M}$.

It is enough to show that $\varphi(x,y,z)\in\mathfrak{Z}(\mathfrak{T})$ for all $x,y,z\in\mathfrak{T}$. We shall do this by following three claims.

Claim 1. For any $x,y\in\mathrm{A}\cup\mathrm{B}$ and $m\in\mathrm{M}$, we have

$$\begin{aligned}\varphi(x,y,m)=\varphi(y,x,m)=\varphi(x,m,y)&=0;\\ \varphi(y,m,x)=\varphi(m,x,y)=\varphi(m,y,x)&=0.\end{aligned}$$

Since $\varphi(x,y,m)\in\mathrm{M}$ for all $m\in\mathrm{M}$ and define the map $\mathfrak{f}:\mathrm{M}\to\mathrm{M}$ by $\mathfrak{f}(m)=\varphi(e,e,m)$ for all $m\in\mathrm{M}$. Then $\mathfrak{f}$ is a bimodule homomorphism. For all $a\in\mathrm{A}$, $b\in\mathrm{B}$, $m\in\mathrm{M}$, we get

$$\begin{aligned}\mathfrak{f}(amb)&=\varphi(e,e,amb)\\&=e\varphi(e,e,[a,mb])f\\&=e[a,\varphi(e,e,mb)]f+e[\varphi(e,e,a),mb]f\\&=a\varphi(e,e,[m,b])f\\&=a[\varphi(e,e,m),b]f+a[m,\varphi(e,e,b)]f\\&=a\varphi(e,e,m)b=a\mathfrak{f}(m)b.\end{aligned}$$

In view of assumption (3), $\mathfrak{f}$ has standard form $\mathfrak{f}(m)=a_0m+mb_0$ for all $a_0\in\mathfrak{Z}(\mathrm{A})$, $b_0\in\mathfrak{Z}(\mathrm{B})$. From assumption (1), we see that $a_0\in\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T}))$ and $b_0\in\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T}))$. We may write

$$\varphi(e,e,m)=\mathfrak{f}(m)=(a_0+\eta^{-1}(b_0))m=\alpha_0m\text{ for all }m\in\mathrm{M},\tag{3.3}$$

where $\alpha_0=a_0+\eta^{-1}(b_0)\in\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T}))$. From (3.1) and (3.2), for $x=e$, $y=e$, $z=m$ and $u=a_1$, $v=a_2$, we find that

$$\begin{aligned}[\varphi(e,e,m),[a_1,a_2]]+[[e,a_2],\varphi(a_1,e,m)]&=[[e,m],\varphi(a_1,e,a_2)]+[\varphi(e,e,a_2),[e,m]]\\ [\varphi(e,e,m),[a_1,a_2]]&=0\end{aligned}$$

$$\begin{aligned}
[\alpha_0 m,[a_1,a_2]] &= 0\\
[a_1,a_2]\alpha_0 \mathrm{M} &= 0.
\end{aligned}$$

Since M is faithful as a left A-module, it follows that $[a_1,a_2]\alpha_0 = 0$ and then by assumption (2), (4), we have $\alpha_0 = 0$ and hence $\varphi(e,e,m) = 0$ for all $m \in \mathrm{M}$. For any $y \in \mathrm{A}\cup\mathrm{B}$, we see that

$$\begin{aligned}
0 &= \varphi(e,[e,y],m)\\
&= [e,\varphi(e,y,m)] + [\varphi(e,e,m),y]\\
&= e\varphi(e,y,m)f = \varphi(e,y,m).
\end{aligned}$$

Again for any $x \in \mathrm{A}\cup\mathrm{B}$, we find that

$$\begin{aligned}
0 &= \varphi([e,x],y,m)\\
&= [e,\varphi(x,y,m)] + [\varphi(e,y,m),x]\\
&= e\varphi(x,y,m)f.
\end{aligned}$$

Thence $\varphi(x,y,m) = 0$ for all $x,y \in \mathrm{A}\cup\mathrm{B}$, and $m \in \mathrm{M}$. Analogously, we can prove the remaining cases.

Claim 2. $\varphi(x,y,z) \in \mathfrak{Z}(\mathfrak{T})$ for all $x,y,z \in \mathrm{A}\cup\mathrm{B}$.
Since $\varphi(e,e,e) \in \mathfrak{Z}(\mathfrak{T})$, we find that

$$\begin{aligned}
\varphi(0,a_2,a_3) &= \varphi([e,a_1],a_2,a_3)\\
0 &= [\varphi(e,a_2,a_3),a_1] + [e,\varphi(a_1,a_2,a_3)]\\
\Longrightarrow\ e\varphi(a_1,a_2,a_3)f &= a_1\varphi(e,a_2,a_3)f.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
e\varphi(a_1,a_2,a_3)f &= a_2\varphi(a_1,e,a_3)f\\
e\varphi(a_1,a_2,a_3)f &= a_3\varphi(a_1,a_2,e)f.
\end{aligned}$$

Hence, we infer that $e\varphi(a_1,a_2,a_3)f = a_1a_2a_3\varphi(e,e,e)f = 0$ for all $a_1,a_2,a_3 \in \mathrm{A}$. Similarly, we can find that $e\varphi(b_1,b_2,b_3)f = e\varphi(e,e,e)b_1b_2b_3 = 0$ for all $b_1,b_2,b_3 \in \mathrm{B}$. Further,

$$\begin{aligned}
\varphi(0,a_2,a_3) &= \varphi([e,b],a_2,a_3)\\
0 &= [\varphi(e,a_2,a_3),b] + [e,\varphi(b,a_2,a_3)]\\
\Longrightarrow\ e\varphi(b,a_2,a_3)f &= -e\varphi(e,a_2,a_3)b\\
&= -a_2a_3\varphi(e,e,e)b = 0
\end{aligned}$$

for all $a_2,a_3 \in \mathrm{A}$ and $b \in \mathrm{B}$. For any $m \in \mathrm{M}$, we obtain

$$\begin{aligned}
\varphi(m,a_2,a_3) &= \varphi([e,m],a_2,a_3)\\
&= [\varphi(e,a_2,a_3),m] + [e,\varphi(m,a_2,a_3)]\\
&= e\varphi(e,a_2,a_3)em - mf\varphi(e,a_2,a_3)f + e\varphi(m,a_2,a_3)f.
\end{aligned}$$

Multiplying by e from right and f from left, we get $e\varphi(e,a_2,a_3)em = mf\varphi(e,a_2,a_3)f$ for all $m \in \mathrm{M}$. This implies that $e\varphi(e,a_2,a_3)e + f\varphi(e,a_2,a_3)f \in \mathfrak{Z}(\mathfrak{T})$. From (3.1) and Claim 1, we have

$$\begin{aligned}[]
[\varphi(a_1,a_2,a_3),[e,m]]+[[a_1,m],\varphi(e,a_2,a_3)] &= [[a_1,a_2],\varphi(e,m,a_3)]+[\varphi(a_1,m,a_3),[e,a_2]]\\
[\varphi(a_1,a_2,a_3),m] &= 0
\end{aligned}$$

and hence $\varphi(a_1,a_2,a_3) \in \mathfrak{Z}(\mathfrak{T})$ for all $a_1,a_2,a_3 \in \mathrm{A}$. On similar lines, we can complete other cases.

Claim 3. $\varphi(x,m',m) = 0$ for all $x \in \mathfrak{T}$ and $m',m \in \mathrm{M}$.

Since $\varphi(x,y,m) \in \mathrm{M}$. Fix $m \in \mathrm{M}$ and define a map $\mathfrak{h} : \mathrm{M} \to \mathrm{M}$ by $\mathfrak{h}(m) = \varphi(e,m_0,m)$. Then $\mathfrak{h}$ is bimodule homomorphism and we see that

$$\begin{aligned}
\mathfrak{h}(amb) &= \varphi(e,m_0,amb)\\
&= e\varphi(e,m_0,[am,b])f\\
&= e[\varphi(e,m_0,am),b]f + e[am,\varphi(e,m_0,b)]f\\
&= e\varphi(e,m_0,[a,m])b\\
&= e[a,\varphi(e,m_0,m)]b + e[\varphi(e,m_0,a),m]b\\
&= a\varphi(e,m_0,m)b = a\mathfrak{h}(m)b.
\end{aligned}$$

In view of assumption (1) and (3), there exist $\beta_0 \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T}))$ such that $\varphi(e,m_0,m) = \beta_0 m$ for all $m \in \mathrm{M}$. Without loss of generality, we assume that A is a noncommutative algebra and let $a_1,a_2 \in \mathrm{A}$ be fixed elements with nonzero commutator. By (3.1) and Claim 1, we have

$$\begin{aligned}[]
[\varphi(e,m_0,m),[a_1,a_2]] &+ [[e,a_2],\varphi(a_1,m_0,m)]\\
&= [[e,m_2],\varphi(a_1,a_2,m)] + [\varphi(e,a_2,m),[a_1,m_0]]\\
[\varphi(e,m_0,m),[a_1,a_2]] &= 0\\
[\beta_0 m,[a_1,a_2]] &= 0\\
[a_1,a_2]\beta_0\mathrm{M} &= 0.
\end{aligned}$$

Since M is faithful, so that $[a_1,a_2]\beta_0 = 0$ and hence by assumption (4), it follows $\beta_0 = 0$. Therefore, $\varphi(e,m_0,m) = \beta_0 m = 0$ for all $m \in \mathrm{M}$. For any $x \in \mathrm{A}\cup\mathrm{B}$, we see that

$$\begin{aligned}
0 &= \varphi([e,x],m_0,m)\\
&= [e,\varphi(x,m_0,m)] + [\varphi(e,m_0,m),x]\\
&= e\varphi(x,m_0,m)f = \varphi(x,m_0,m).
\end{aligned}$$

Again, we define a map $\mathfrak{g} : \mathrm{M} \to \mathrm{M}$ by $\mathfrak{g}(m) = \varphi(m_1,m_2,m)$. It can be easily seen that $\mathfrak{g}$ is a bimodule homomorphism. Then, in view of assumption (1) and (3), there exist $\gamma_0 \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T}))$ such that $\varphi(m_1,m_2,m) = \gamma_0 m$ for all $m \in \mathrm{M}$. Without loss of generality, we assume that A is a noncommutative algebra and let $a_1, a_2 \in \mathrm{A}$ be fixed elements with nonzero commutator. Using (3.1) and Claim 1, we get

$$\begin{aligned}
[\varphi(m_1,m_0,m),[a_1,a_2]] &+ [[m_1,a_2],\varphi(a_1,m_0,m)] \\
&= [[e,m_2],\varphi(a_1,a_2,m)] + [\varphi(m_1,a_2,m),[a_1,m_0]] \\
[\varphi(m_1,m_0,m),[a_1,a_2]] &= 0 \\
[\gamma_0 m,[a_1,a_2]] &= 0 \\
[a_1,a_2]\gamma_0 \mathrm{M} &= 0.
\end{aligned}$$

By faithfulness of M, we find that $[a_1,a_2]\gamma_0 = 0$ and hence by assumption (4), it follows $\gamma_0 = 0$. Therefore, $\varphi(m_1,m_2,m) = 0$ for all $m \in \mathrm{M}$. Finally we have $\varphi(x,m',m) = 0$ for all $x \in \mathfrak{T}$ and $m', m \in \mathrm{M}$.

From the above three claims, we see that $\varphi(x,y,z) \in \mathfrak{Z}(\mathfrak{T})$ for all $x,y,z \in \mathfrak{T}$. Since φ is linear in each argument, we obtain that $\varphi \in \mathfrak{Z}(\mathfrak{T})$. Therefore, ϕ can be written as sum of extremal 3-derivation ζ and a 3-linear central map φ. □

Right away, we equip the main result of this article.

THEOREM 3.2. *Let $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ be a triangular algebra. If the following conditions hold:*

1. $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{B})$;
2. *at least one of the algebras* A *and* B *is noncommutative;*
3. *every bimodule homomorphism* $\mathfrak{f}: \mathrm{M} \to \mathrm{M}$ *is of the standard form;*
4. *If* $\alpha a = 0$, $\alpha \in \mathfrak{Z}(\mathfrak{T})$, $0 \neq a \in \mathfrak{T}$, *then* $\alpha = 0$.

Then every n-Lie derivation $\phi : \mathfrak{T} \times \mathfrak{T} \times \cdots \times \mathfrak{T} \to \mathfrak{T}$ can be written as $\phi = \zeta + \varphi$, where ζ is an extremal n-derivation such that

$$\zeta(x_1,x_2,\cdots,x_n) = [x_1,[x_2,\cdots,[x_n,\phi(e,e,\cdots,e)]\cdots]],$$

where $\phi(e,e,\cdots,e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$ and φ is a n-linear central map on $\mathfrak{T}$.

Proof. For $n = 3$ result follows from the Theorem 3.1. For $n \geqslant 4$ we apply induction method. Now fix $x_4,\cdots,x_n \in \mathfrak{T}$. Set

$$\phi_{x_4,\cdots,x_n}(x_1,x_2,x_3) = \phi(x_1,x_2,x_3,x_4,\cdots,x_n) \text{ for all } x_1,x_2,x_3 \in \mathfrak{T}.$$

Then $\phi_{x_4,\cdots,x_n}(x_1,x_2,x_3)$ is a 3-Lie derivation. By Theorem 3.1 it follows that

$$\phi_{x_4,\cdots,x_n}(x_1,x_2,x_3) = [x_1,[x_2,[x_3,\phi(e,e,e)]]] + \varphi(x_1,x_2,x_3) \text{ for all } x_1,x_2,x_3 \in \mathfrak{T},$$

where $\phi(e,e,e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$ (depending on $x_4,\cdots,x_n$) with the property $[\phi(e,e,e),[\mathfrak{T},\mathfrak{T}]] = 0$. Particularly, we have that $\phi_{x_4,\cdots,x_n}(e,e,e) = m + z$ and so $\phi(e,e,e,x_4,\cdots,x_n) = m + z$ for all $z \in \mathfrak{Z}(\mathfrak{T})$. Hence

$$\phi(x_1,x_2,\cdots,x_n) = [x_1,[x_2,[x_3,\phi(e,e,e,x_4,\cdots,x_n)]]] + \varphi(x_1,x_2,\cdots,x_n) \tag{3.4}$$

for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. It is clear that $\phi(e, x_2, x_3, \cdots, x_n)$ is a $(n-1)$-Lie derivation on $\mathfrak{T}$. By induction assumption we get

$$\phi(e, x_2, x_3, \cdots, x_n) = [x_2, [x_3, \cdots, [x_n, \phi(e, e, \cdots, e)] \cdots]] + \varphi(e, x_2, x_3, \cdots, x_n)$$

for all $x_2, \cdots, x_n \in \mathfrak{T}$, where $\phi(e, e, \cdots, e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$ and $[\phi(e, e, \cdots, e), [\mathfrak{T}, \mathfrak{T}]] = 0$. Particularly,

$$\phi(e, e, e, x_4, \cdots, x_n) = [x_4, [x_5, \cdots, [x_n, \phi(e, e, \cdots, e)] \cdots]] + \varphi(e, e, e, x_4, \cdots, x_n)$$

for all $x_2, \cdots, x_n \in \mathfrak{T}$, where we used that $\phi(e, e, \cdots, e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$. From (3.4) we have

$$\phi(x_1, x_2, x_3, \cdots, x_n) = [x_1, [x_2, \cdots, [x_n, \phi(e, e, \cdots, e)] \cdots]] + \varphi(x_1, x_2, x_3, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Hence we obtain the expected result. $\square$

In view of [6, Corollary 3.4], we can restate our main theorem with a weaker condition as follows:

COROLLARY 3.3. *Let $\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a triangular algebra. If the following conditions hold:*

1. $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{B})$;
2. *at least one of the algebras* A *and* B *is noncommutative;*
3. *each derivation of $\mathfrak{T}$ is inner;*
4. *If $\alpha a = 0$, $\alpha \in \mathfrak{Z}(\mathfrak{T})$, $0 \neq a \in \mathfrak{T}$, then $\alpha = 0$.*

Then every n-Lie derivation $\phi : \mathfrak{T} \times \mathfrak{T} \times \cdots \times \mathfrak{T} \to \mathfrak{T}$ can be written as $\phi = \zeta + \varphi$, where ζ is an extremal n-derivation such that

$$\zeta(x_1, x_2, \cdots, x_n) = [x_1, [x_2, \cdots, [x_n, \phi(e, e, \cdots, e)] \cdots]],$$

where $\phi(e, e, \cdots, e) \in \mathrm{M} + \mathfrak{Z}(\mathfrak{T})$ and φ is a n-linear central map on $\mathfrak{T}$.

At this position, we implement above result to some classical examples of triangular algebras (for details see [13]) as follows:

COROLLARY 3.4. *Let* R *be a commutative ring with unity. If $n \geqslant 3$, then each n-Lie derivation of (block) upper triangular matrix algebra $\mathfrak{B}_m^{\bar{k}}(\mathrm{R})$ is the sum of an extremal n-derivation and a n-linear central map. In particular, every n-Lie derivation of upper triangular matrix algebra $\mathfrak{T}_m(\mathrm{R})$ is the sum of an extremal n-derivation and an n-linear central map.*

COROLLARY 3.5. *Let* N *be a nest of a Hilbert space* H, *where* $\dim \mathrm{H} \geqslant 3$. *If $n \geqslant 3$, then each n-Lie derivation of nest algebra $\mathfrak{T}(\mathrm{N})$ is the sum of an extremal n-derivation and an n-linear central map.*

4. For further discussions

In this part, we make an effort to collect a few specific queries related to the literature of the article. But before that, we should bring up some basic notions of related subject matter as follows:

Let $\mathscr{A}$ be an algebra over a commutative ring R with unity. For any $a,b \in \mathscr{A}$, $a \circ b = ab + ba$ denotes the Jordan product (or anti-commutator). Recall that a linear map $\mathrm{J} : \mathscr{A} \to \mathscr{A}$ is said to be a Jordan derivation if $\mathrm{J}(a \circ b) = \mathrm{J}(a) \circ b + a \circ \mathrm{J}(b)$ for all $a,b \in \mathscr{A}$. An n-linear map $\mathscr{J} : \mathscr{A} \times \mathscr{A} \times \cdots \times \mathscr{A} \to \mathscr{A}$ is said to be an n-Jordan derivation if it is a Jordan derivation with respect to all components, i.e.,

$$\begin{aligned} &\mathscr{J}(a_1 \circ b, a_2, \cdots, a_n) = \mathscr{J}(a_1, a_2, \cdots, a_n) \circ b + a_1 \circ \mathscr{J}(b, a_2, \cdots, a_n) \\ \text{and } &\mathscr{J}(a_1, a_2 \circ b, \cdots, a_n) = \mathscr{J}(a_1, a_2, \cdots, a_n) \circ b + a_2 \circ \mathscr{J}(a_1, b, \cdots, a_n) \\ &\vdots \\ \text{and } &\mathscr{J}(a_1, a_2, \cdots, a_n \circ b) = \mathscr{J}(a_1, a_2, \cdots, a_n) \circ b + a_n \circ \mathscr{J}(a_1, a_2, \cdots, b) \end{aligned}$$

for all $a_1, a_2, \cdots, a_n, b \in \mathscr{A}$. Particularly, for $n = 2$ it is called as 2-Jordan derivation (or some author mentioned it as Jordan biderivation see [5]).

A *Morita context* consists of two unital R-algebras A and B, two bimodules (A,B)-bimodule M and (B,A)-bimodule N, and two bimodule homomorphisms called the bilinear pairings $\xi_{\mathrm{MN}} : \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \longrightarrow \mathrm{A}$ and $\Omega_{\mathrm{NM}} : \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \longrightarrow \mathrm{B}$ satisfying the following commutative diagrams:

$$\begin{tikzcd} \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \arrow[r, "\xi_{\mathrm{MN}} \otimes I_{\mathrm{M}}"] \arrow[d, "I_{\mathrm{M}} \otimes \Omega_{\mathrm{NM}}"] & \mathrm{A} \underset{\mathrm{A}}{\otimes} \mathrm{M} \arrow[d, "\cong"] \\ \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{B} \arrow[r, "\cong"] & \mathrm{M} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \arrow[r, "\Omega_{\mathrm{NM}} \otimes I_{\mathrm{N}}"] \arrow[d, "I_{\mathrm{N}} \otimes \xi_{\mathrm{MN}}"] & \mathrm{B} \underset{\mathrm{B}}{\otimes} \mathrm{N} \arrow[d, "\cong"] \\ \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{A} \arrow[r, "\cong"] & \mathrm{N}. \end{tikzcd}$$

Let us write this Morita context as $(\mathrm{A},\mathrm{B},\mathrm{M},\mathrm{N},\xi_{\mathrm{MN}},\Omega_{\mathrm{NM}})$. We refer the reader to [19] for the basic properties of Morita context. If $(\mathrm{A},\mathrm{B},\mathrm{M},\mathrm{N}, \xi_{\mathrm{MN}},\Omega_{\mathrm{NM}})$ is a Morita context, then the set

$$\begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in \mathrm{A}, m \in \mathrm{M}, n \in \mathrm{N}, b \in \mathrm{B} \right\}$$

forms an R-algebra under matrix addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an R-algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by

$$\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B}) = \begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix}.$$

This kind of algebra was first introduced by Morita in [19], where the author investigated Morita duality theory of modules and its applications to Artinian algebras. All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. Most common examples of generalized matrix algebras are full matrix algebras over a unital algebra and triangular algebras [20, 24]. Also, if $\mathrm{N} = 0$, then $\mathfrak{G}$ is called a triangular algebra. Now it is reasonable to raise the question as below:

QUESTION 4.1. *What is the most general form of n-Lie (Jordan) derivations on generalized matrix algebras and which constraints are needed to apply on generalized matrix algebras?*

5. Conclusions

This article initiates the study of n-Lie derivations on triangular algebras and we see that every n-Lie derivations seemingly split into an extremal n-derivation and an n-linear central map on triangular algebras. As an immediate outcome, we observe that the main result is true for upper triangular matrix algebras and nest algebras. At the end of this article, we call attention towards the investigation of n-Lie (Jordan) derivations in a more general background named as generalized matrix algebras.

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