Operators and Matrices
Volume 16, Number 3 (2022), 623–643 doi:10.7153/oam-2022-16-46

k–QUASI–A–PARANORMAL OPERATORS IN SEMI–HILBERTIAN SPACES

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(*Communicated by R. Curto*)

Abstract. In this paper, we introduce and analyze a new class of generalized paranormal operators, namely k-quasi-A-paranormal operators for a bounded linear operator acting on a complex Hilbert space $\mathscr{H}$ when an additional semi-inner product induced by a positive operator A is considered. After establishing the basic properties of such operators. We extend some results obtained in several papers related to this class on a Hilbert space. In addition, we characterize the spectra and tensor product of these operators.

1. Introduction

Throughout this paper, $\mathscr{H}$ denotes a non trivial complex Hilbert space with inner product $\langle .,.\rangle$ and associated norm $\|.\|$. Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. Let the symbol I stand for the identity operator on $\mathscr{H}$. For every operator $S\in\mathscr{B}(\mathscr{H})$, $\mathscr{N}(S)$, $\mathscr{R}(S)$ and $\overline{\mathscr{R}(S)}$ stand for respectively, the null space, the range and the closure of the range of S and its adjoint by S^*.

For the sequel, it is useful to point out the following facts. Let $\mathscr{B}(\mathscr{H})^+$ be the cone of positive (semi-definite) operators i.e.;

$$\mathscr{B}(\mathscr{H})^+=\{A\in\mathscr{B}(\mathscr{H}):\langle Ax,x\rangle\geqslant 0,\forall\, x\in\mathscr{H}\}.$$

Any positive operator $A\in\mathscr{B}(\mathscr{H})^+$ defines a positive semi-definite sesquilinear form

$$\langle .,.\rangle_A:\mathscr{H}\times\mathscr{H}\to\mathbb{C},\;\; \langle x,y\rangle_A=\langle Ax,y\rangle.$$

Naturally, this semi-inner product induces a semi-norm $\|.\|_A$ defined by

$$\|x\|_A=\sqrt{\langle x,x\rangle_A}=\left\|A^{\frac{1}{2}}x\right\|,\forall\, x\in\mathscr{H}.$$

Observe that $\|x\|_A=0$ if and only if $x\in\mathscr{N}(A)$. Then $\|.\|_A$ is a norm on $\mathscr{H}$ if and only if A is injective operator and the semi-normed space $(\mathscr{B}(\mathscr{H}),\|.\|_A)$ is

Mathematics subject classification (2020): 47A05, 47A55, 47B15.

Keywords and phrases: A-adjoint, semi-Hilbertian spaces, A-positive operators, paranormal operators, quasi-paranormal operators.

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complete if and only if $\mathscr{R}(A)$ is closed. The above semi-norm induces a semi-norm on the subspace

$$\mathscr{B}^A(\mathscr{H}) = \{S \in \mathscr{B}(\mathscr{H}) \mid \exists\, c > 0,\ \|Sx\|_A \leqslant c\|x\|_A,\ \forall\, x \in \overline{\mathscr{R}(A)}\}.$$

For these operators the following identities hold.

$$\begin{aligned} \|S\|_A &:= \sup_{\substack{x \in \overline{\mathscr{R}(A)} \\ x \neq 0}} \frac{\|Sx\|_A}{\|x\|_A} \\ &= \sup_{\substack{x \in \overline{\mathscr{R}(A)} \\ \|x\|_A = 1}} \|Sx\|_A . \end{aligned}$$

It was observed that $\mathscr{B}^A(\mathscr{H})$ is not a subalgebra of $\mathscr{B}(\mathscr{H})$ ([8, Example 2.1]) and that $\|S\|_A = 0$ if and only if $ASA = 0$.

For $S \in \mathscr{B}(\mathscr{H})$, an operator $T \in \mathscr{B}(\mathscr{H})$ is called an A-adjoint of S if for every $x, y \in \mathscr{H}$

$$\langle Sx, y\rangle_A = \langle x, Ty\rangle_A,$$

i.e.; $AT = S^*A$. S is called A-selfadjoint if $AS = S^*A$, and it is called A-positive, and we write $S \geqslant_A 0$ if AS is positive (see [1]).

The existence of an A-adjoint operator is not guaranteed. The set of all operators which admit A-adjoints is denoted by $\mathscr{B}_A(\mathscr{H})$. By Douglas theorem [7], we get

$$\begin{aligned} \mathscr{B}_A(\mathscr{H}) &= \{S \in \mathscr{B}(\mathscr{H}) : \mathscr{R}(S^*A) \subset \mathscr{R}(A)\} \\ &= \{S \in \mathscr{B}(\mathscr{H}) : \exists\, c > 0; \|ASx\| \leqslant c\,\|Ax\|\,, \forall\, x \in \mathscr{H}\}. \end{aligned}$$

Note that $\mathscr{B}_A(\mathscr{H})$ is a subalgebra of $\mathscr{B}(\mathscr{H})$ which is neither closed nor dense in $\mathscr{B}(\mathscr{H})$. If $S \in \mathscr{B}_A(\mathscr{H})$ then S admits an A-adjoint operator. Moreover, there exists a distinguished A-adjoint operator of S, namely the reduced solution of the equation $AX = S^*A$, i.e., $S^\# = A^\dagger S^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A. The A-adjoint operator $S^\#$ verifies

$$AS^\# = S^*A,\ \mathscr{R}\left(S^\#\right) \subset \overline{\mathscr{R}(A)} \text{ and } \mathscr{N}\left(S^\#\right) = \mathscr{N}(S^*A).$$

Again, by applying Douglas theorem ([7]), we can see that

$$\mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H}) = \{S \in \mathscr{B}(\mathscr{H}) : \exists\, c > 0; \|Sx\|_A \leqslant c\,\|x\|_A\,, \forall\, x \in \mathscr{H}\}.$$

Any operator in $\mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is called A-bounded operator. Moreover, it was proved in [3] that if $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$, then

$$\begin{aligned} \|S\|_A &:= \sup_{x \notin \mathscr{N}(A)} \frac{\|Sx\|_A}{\|x\|_A} = \sup_{\|x\|_A = 1} \|Sx\|_A \\ &= \sup_{\|x\|_A \leqslant 1} \|Sx\|_A . \end{aligned}$$

In addition, if S is A-bounded, then $S(\mathcal{N}(A)) \subset \mathcal{N}(A)$ and

$$\|Sx\|_A \leqslant \|S\|_A \|x\|_A, \forall x \in \mathcal{H}.$$

Note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$ (see [2]). Moreover, the following inclusions

$$\mathcal{B}_A(\mathcal{H}) \subset \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subset \mathcal{B}^A(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}),$$

hold with equality if A is injective and has a closed range.

In the following theorem, we collect some interesting properties of $S^\#$.

THEOREM 1.1. ([1, 2, 3]) *Let $S \in \mathcal{B}_A(\mathcal{H})$. Then, the following statements hold:*

(1) *$S^\# \in \mathcal{B}_A(\mathcal{H})$, $(S^\#)^\# = P_{\overline{\mathcal{R}(A)}} S P_{\overline{\mathcal{R}(A)}}$ and $\left((S^\#)^\#\right)^\# = S^\#$, where $P_{\overline{\mathcal{R}(A)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$.*

(2) *$S^\# S$ and $SS^\#$ are A-self-adjoint and A-positive operators.*

(3) *If $T \in \mathcal{B}_A(\mathcal{H})$, then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^\# = S^\# T^\#$.*

(4) *$\|S\|_A = \|S^\#\|_A = \|S^\# S\|_A^{\frac{1}{2}} = \|SS^\#\|_A^{\frac{1}{2}}$.*

From now on, to simplify notation, we write P instead of $P_{\overline{\mathcal{R}(A)}}$.

An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be normal if $S^*S = SS^*$, hyponormal if $S^*S \geqslant SS^*$, k-quasi-hyponormal if $S^{*k}(S^*S - SS^*)S^k \geqslant 0$ ([6]), paranormal if $\|Sx\|^2 \leqslant \|S^2x\|\|x\|$, for all $x \in \mathcal{H}$ ([9]) and k-quasi-paranormal if $\|S^{k+1}x\|^2 \leqslant \|S^{k+2}x\|\|S^kx\|$, for all $x \in \mathcal{H}$ and for some positive integer k ([10]).

Many authors has extended some of these concepts to the semi-Hilbertian operators.

An operator $S \in \mathcal{B}_A(\mathcal{H})$ is said to be A-normal if $S^\# S = SS^\#$ ([13], A-hyponormal if $S^\# S \geqslant_A SS^\#$ ([14], k-quasi-A-hyponormal if $S^{\# k}(S^\# S - SS^\#)S^k \geqslant_A 0$ ([14]) and A-paranormal if $\|Sx\|_A^2 \leqslant \|S^2x\|_A \|x\|_A$, for all $x \in \mathcal{H}$ ([11]).

This paper is devoted to the study of some classes of operators on the semi-Hilbertian space $(\mathcal{H}, \langle\, .\, \rangle_A)$ which is a generalization of A-normal, A-hyponormal and A-paranormal operators. More precisely, we introduce a new class of operators which is called the class of k-quasi-A-paranormal operator. It is proved in Example 2.1 that there is an operator which is k-quasi-A-paranormal but not A-paranormal for some positive integer k, and thus, the proposed new class of operators contains the class of A-paranormal operators as a proper subset. In the course of our study, we have proven that some properties of A-paranormal operators remain true of k- quasi-A-paranormal operators. In Section 2, we prove an equivalent condition for an operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ to be k-quasi-A-paranormal (Theorem 2.1). Several properties are proved by exploiting this characterization (Theorem 2.4, Theorem 2.8, Lemma 4.1). In particular, we prove that if $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is an k-quasi-A-paranormal then its power is k-quasi-A-paranormal. Section 3, is devoted to describe some properties concerning the A-spectral radius and approximate spectrum of an k-quasi-A-paranormal operator (Theorem 2.7).

2. Properties of k-quasi-A-paranormal operators

In this section, we define the class of k-quasi-A-paranormal operators in semi-Hilbertian spaces and we investigate some properties of such operator.

Firstly, we start with the definition of this class.

DEFINITION 2.1. An operator $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is called an k-quasi-A-paranormal if for a positive integer k,

$$\left\|S^{k+1}x\right\|_A^2 \leqslant \left\|S^{k+2}x\right\|_A \left\|S^k x\right\|_A,$$

for all $x \in \mathscr{H}$.

Before we move on, we state the following remark.

REMARK 2.1. (1) When $k = 0$ we get the class of A-paranormal operators introduced in [11].

(2) If $k = 1$, we say that S is quasi-A-paranormal operator.

(3) αS is k-quasi-A-paranormal for all $\alpha \in \mathbb{C}$.

(4) If $A = I$, then every k-quasi-A-paranormal is k-quasi-paranormal operators ([10]).

(5) It is not difficult to verify the following inclusions:

$$\begin{aligned} A\text{-paranormal} \subseteq \text{quasi-A-paranormal} &\subseteq k\text{-quasi-A-paranormal} \\ &\subseteq (k+1)\text{-quasi-A-paranormal.} \end{aligned}$$

In the following example, we give an operator $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ that is k-quasi-A-paranormal for some positive integer k but not A-paranormal.

EXAMPLE 2.1. Let $S = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. A direct calculation shows that S satisfying the following conditions

$$\begin{cases} \|Sx\|_A \leqslant \dfrac{1}{\sqrt{2}}\|x\|_A, \ \forall\, x \in \mathbb{C}^3 \\ \|S^4x\|_A^2 \leqslant \|S^5x\|_A \|S^3x\|_A, \ \forall\, x \in \mathbb{C}^3 \\ \|Sx_0\|_A^2 \geqslant \|S^2x_0\|_A \ \text{ for some } x_0 \in \mathbb{C}^3 \text{such that } \|x_0\|_A = 1. \end{cases}$$

Therefore, $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ and S is a 3-quasi-A-paranormal but not A-paranormal.

In [11] it has been shown that $S \in \mathscr{B}_A(\mathscr{H})$ is A-paranormal if and only if

$$S^{\#2}S^2 - 2\lambda P S^{\#} S + \lambda^2 P \geqslant_A 0, \ \text{ for all } \lambda > 0.$$

Similarly, we have the following characterization for the members of the class of k-quasi-A-paranormal operators. It is similar to [10, Theorem 2.1] for Hilbert space operators.

THEOREM 2.1. *Let $S \in \mathscr{B}_A(\mathscr{H})$. Then S is k-quasi-A-paranormal if and only if*

$$\left(S^{\#}\right)^k \left(S^{\#^2} S^2 - 2\lambda S^{\#} S + \lambda^2 P\right) S^k \geqslant_A 0, \tag{2.1}$$

for all $\lambda > 0$.

Proof. Notice that if S is A-quasi-k-paranormal, then we have the following inequality

$$\left\langle S^{\#^{k+2}} S^{k+2} x, x\right\rangle_A^{\frac{1}{2}} \left\langle S^{\#^k} S^k x, x\right\rangle_A^{\frac{1}{2}} \geqslant \left\langle S^{\#^{k+1}} S^{k+1} x, x\right\rangle_A^{\frac{1}{2}},$$

for all $x \in \mathscr{H}$. By generalized arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}\left\langle S^{\#^{k+2}} S^{k+2} x, x\right\rangle_A^{\frac{1}{2}} \left\langle S^{\#^k} S^k x, x\right\rangle_A^{\frac{1}{2}} &= \left(\lambda^{-1} \left\langle S^{\#^{k+2}} S^{k+2} x, x\right\rangle_A\right)^{\frac{1}{2}} \left(\lambda \left\langle S^{\#^k} S^k x, x\right\rangle_A\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\lambda^{-1} \left\langle S^{\#^{k+2}} S^{k+2} x, x\right\rangle_A + \frac{1}{2}\lambda \left\langle S^{\#^k} S^k x, x\right\rangle_A,\end{aligned}$$

for all $x \in \mathscr{H}$ and $\lambda > 0$. Thus, we get

$$\frac{1}{2}\lambda^{-1} \left\langle S^{\#^{k+2}} S^{k+2} x, x\right\rangle_A + \frac{1}{2}\lambda \left\langle S^{\#^k} S^k x, x\right\rangle_A \geqslant \left\langle S^{\#^{k+1}} S^{k+1} x, x\right\rangle_A, \tag{2.2}$$

for all $x \in \mathscr{H}$ and $\lambda > 0$. So that implies the following inequality

$$\left\langle S^{\#^k} \left(S^{\#^2} S^2 - 2\lambda S^{\#} S + \lambda^2 P\right) S^k x, x\right\rangle_A \geqslant 0,$$

for all $x \in \mathscr{H}$ and $\lambda > 0$. Therefore, we deduce the desired inequality.

Conversely, it is easily checked that (2.1) is equivalent to the inequality (2.2). Again by generalized arithmetic-geometric mean inequality, we have

$$\left\langle S^{\#^{k+1}} S^{k+1} x, x\right\rangle_A^{\frac{1}{2}} \leqslant \left\langle S^{\#^{k+2}} S^{k+2} x, x\right\rangle_A^{\frac{1}{2}} \left\langle S^{\#^k} S^k x, x\right\rangle_A^{\frac{1}{2}}.$$

So, that implies

$$\left\|S^{k+1} x\right\|_A^2 \leqslant \left\|S^{k+2} x\right\|_A \left\|S^k x\right\|_A,$$

for all $x \in \mathscr{H}$. Hence, S is k-quasi-A-paranormal operator. □

In [11], the authors proved that if $S \in \mathscr{B}_A(\mathscr{H})$ such that $\left(S^{\#}\right)^2 S^2 \geqslant_A \left(S^{\#} S\right)^2$, then S is A-paranormal. In the following theorem we give a similar condition for k-quasi-A-paranormal operator.

THEOREM 2.2. *Let $S \in \mathscr{B}_A(\mathscr{H})$ such that $\|S\|_A \leqslant 1$ and S satisfies the following inequality*

$$S^{\#^k}\left(\left(S^{\#}\right)^2 S^2 - S^{\#}S\right)S^k \geqslant_A 0, \tag{2.3}$$

for a positive integer k. Then, S is an k-quasi-A-paranormal operator.

Proof. Assume that S satisfies (2.3). Let $x \in \mathscr{H}$, then we have

$$\begin{aligned}
\left\|S^{k+1}x\right\|_A^2 &= \left\langle S^{k+1}x, S^{k+1}x\right\rangle_A \\
&= \left\langle S^{\#}SS^k x, S^k x\right\rangle_A \\
&= \left\langle S^{\#k}S^{\#}SS^k x,\ x\right\rangle_A \\
&\leqslant \left\langle S^{\#k}S^{\#2}S^2S^k x,\ x\right\rangle_A \\
&= \left\langle S^{\#2}S^2S^k x,\ S^k x\right\rangle_A \\
&\leqslant \|S^{\#2}S^2S^k x\|_A \|S^k x\|_A \\
&\leqslant \|S^{k+2}x\|_A \|S^k x\|_A.
\end{aligned}$$

So, we have that

$$\left\|S^{k+1}x\right\|_A^2 \leqslant \left\|S^{k+2}x\right\|_A \left\|S^k x\right\|_A,$$

for all $x \in \mathscr{H}$. Therefore, S is an k-quasi-A-paranormal operator. □

PROPOSITION 2.1. *Let $S \in \mathscr{B}_A(\mathscr{H})$ be k-quasi-A-paranormal. If $\overline{\mathscr{R}(S^k)} = \mathscr{H}$, then S is A-paranormal.*

Proof. Since S is k-quasi-A-paranormal it follows by Theorem 2.1

$$S^{\#k}\left(S^{\#2}S^2 - 2\lambda S^{\#}T + \lambda^2 P\right)S^k \geqslant_A 0,$$

for all $x \in \mathscr{H}$ and for all $\lambda > 0$. This means that

$$\left\langle \left(S^{\#2}S^2 - 2\lambda S^{\#}S + \lambda^2 P\right)S^k x,\ S^k x\right\rangle_A \geqslant 0,$$

for all $x \in \mathscr{H}$ and for all $\lambda > 0$. Therefore

$$S^{\#2}S^2 - 2\lambda S^{\#}S + \lambda^2 P \geqslant_A 0 \text{ on } \overline{\mathscr{R}(S^k)} = \mathscr{H}.$$

Consequently, S is A-paranormal. □

Let $T, S \in \mathscr{B}(\mathscr{H})$ we say that S is A-unitary equivalent to T if there exists an A-unitary operator $U \in \mathscr{B}_A(\mathscr{H})$ such that $S = UTU^{\sharp}$.

THEOREM 2.3. *Let $T \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ be an k-quasi-A-paranormal operator such that $\mathscr{N}(A)^{\perp}$ is invariant subspace of T. If $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is A-unitarily equivalent to T, then S is k-quasi-A-paranormal operator.*

Proof. Since S is A-unitary equivalent to T, there exists an A-unitary operator U such that $S = UTU^{\sharp}$.

Under the assumption that $\mathscr{N}(A)$ is a reducing subspace of T, it follows that $TP = PT$ and $PA = AP = A$. Furthermore, it is not difficult to verify that

$$S^n = \left(UTU^{\sharp}\right)^n = UPT^nU^{\sharp},$$

for a positive integer n.

On the other hand, we have

$$\begin{aligned}
\left\|S^{k+1}x\right\|_A^2 &= \left\|\left(UTU^{\sharp}\right)^{k+1}x\right\|_A^2 \\
&= \left\|UPT^{k+1}U^{\sharp}x\right\|_A^2 \\
&= \left\|PT^{k+1}U^{\sharp}x\right\|_A^2 \quad \left(\text{since } \|Ux\|_A = \|x\|_A,\ \forall\, x \in \mathscr{H}\right) \\
&= \left\|T^{k+1}\left(U^{\sharp}x\right)\right\|_A^2 \\
&\leqslant \left\|T^{k+2}\left(U^{\sharp}x\right)\right\|_A \left\|T^{k}\left(U^{\sharp}x\right)\right\|_A \\
&= \left\|PT^{k+2}\left(U^{\sharp}x\right)\right\|_A \left\|PT^{k}\left(U^{\sharp}x\right)\right\|_A \ (\text{since } T\left(\overline{\mathscr{R}(\mathrm{A})}\right) \subseteq \overline{\mathscr{R}(\mathrm{A})}) \\
&= \left\|UPT^{k+2}\left(U^{\sharp}x\right)\right\|_A \left\|UPT^{k}\left(U^{\sharp}x\right)\right\|_A \\
&= \left\|S^{k+2}x\right\|_A \left\|S^{k}x\right\|_A.
\end{aligned}$$

Hence,

$$\left\|S^{k+1}x\right\|_A^2 \leqslant \left\|S^{k+2}x\right\|_A \left\|S^{k}x\right\|_A, \text{ for all } x \in \mathscr{H}.$$

Therefore, S is k-quasi-A-paranormal operator. $\square$

PROPOSITION 2.2. *Let $S \in \mathscr{B}_{\mathrm{A}}(\mathscr{H})$, then the following assertions hold:*

(1) *If S is A-self-adjoint, then S is k-quasi-A-paranormal operator.*

(2) *If S is A-normal, then S and $S^{\#}$ are k-quasi-A-paranormal operators.*

(3) *If S is A-hyponormal operator, then S is k-quasi-A-paranormal operator.*

(4) *If S is k-quasi-A-hyponormal operator, then S is k-quasi-A-paranormal operator.*

Proof. (1) Assume that S is A-self-adjoint, then $AS = S^*A$. Let $x \in \mathscr{H}$ and for a positive integer k. We have

$$\begin{aligned}
\|S^{k+1}x\|_A^2 &= \langle S^{k+1}x, S^{k+1}x\rangle_A \\
&= \langle AS^{k+1}x, S^{k+1}x\rangle \\
&= \langle S^*AS^{k+1}x, S^kx\rangle \\
&= \langle AS^{k+2}x, S^kx\rangle \\
&= \langle A^{\frac{1}{2}}S^{k+2}x, A^{\frac{1}{2}}S^kx\rangle \\
&\leqslant \|A^{\frac{1}{2}}S^{k+2}x\| \| A^{\frac{1}{2}}S^kx\| \\
&= \left\|S^{k+2}x\right\|_A \left\|S^kx\right\|_A.
\end{aligned}$$

So, S is k-quasi-A-paranormal.

(2) If S is A-normal, then we know that $\|Sx\|_A = \left\|S^{\#}x\right\|_A$ for all $x \in \mathscr{H}$. We have

$$\begin{aligned}
\|S^{k+1}x\|_A^2 &= \langle S^{k+1}x, S^{k+1}x\rangle_A \\
&= \langle AS^{k+1}x, S^{k+1}x\rangle \\
&= \langle (S^*A)\, S^{k+1}x,\ S^kx\rangle \\
&= \langle \left(AS^{\#}\right) S^{k+1}x, S^kx\rangle \\
&= \langle A^{\frac{1}{2}}S^{\#}S^{k+1}x, A^{\frac{1}{2}}S^kx\rangle \\
&\leqslant \left\|A^{\frac{1}{2}}S^{\#}S^{k+1}x\right\| \left\|A^{\frac{1}{2}}S^kx\right\| \\
&= \left\|S^{\#}\left(S^{k+1}x\right)\right\|_A \left\|S^kx\right\|_A \\
&= \left\|S\left(S^{k+1}x\right)\right\|_A \left\|S^kx\right\|_A \text{(since S is A-normal)} \\
&= \left\|S^{k+2}x\right\|_A \left\|S^kx\right\|_A.
\end{aligned}$$

Therefore,

$$\left\|S^{k+1}x\right\|_A^2 \leqslant \left\|S^{k+2}x\right\|_A \left\|S^kx\right\|_A.$$

Now, we prove that $S^{\#}$ is k-quasi-A-paranormal. We have

$$\begin{aligned}
\|S^{\#(k+1)}x\|_A^2 &= \langle S^{\#(k+1)}x,\ S^{\#(k+1)}x\rangle_A \\
&= \langle AS^{\#(k+1)}x, S^{\#(k+1)}x\rangle \\
&= \langle S^{\#(k+1)}x, \left(AS^{\#}\right) S^{\#k}x\rangle \\
&= \langle S^{\#(k+1)}x, S^*AS^{\#k}x\rangle \\
&= \langle ASS^{\#(k+1)}x, S^{\#k}x\rangle \\
&= \langle A^{\frac{1}{2}}SS^{\#(k+1)}x, A^{\frac{1}{2}}S^{\#k}x\rangle
\end{aligned}$$

$$\begin{aligned}&\leqslant \left\|A^{\frac{1}{2}}SS^{\#k+1}x\right\| \left\|A^{\frac{1}{2}}S^{\#k}x\right\| \\ &= \left\|SS^{\#(k+1)}x\right\|_A \left\|S^{\#k}x\right\|_A.\end{aligned}$$

Since S is A-normal, then

$$\left\|SS^{\#k+1}x\right\|_A = \left\|S^{\#k+2}x\right\|_A, \text{ for all } x \in \mathscr{H}.$$

Therefore, we get $\|S^{\#k+1}x\|_A^2 \leqslant \left\|S^{\#(k+2)}x\right\|_A \|S^{\#k}x\|_A$.

(3) If S is A-hyponormal, it follows that

$$\left\|S^{\#}x\right\|_A \leqslant \|Sx\|_A,$$

for all $x \in \mathscr{H}$. We have

$$\begin{aligned}\|S^{k+1}x\|_A^2 &= \langle S^{k+1}x, S^{k+1}x\rangle_A \\ &= \langle AS^{k+1}x, S^{k+1}x\rangle \\ &= \langle S^*AS^{k+1}x, S^kx\rangle \\ &= \langle AS^{\#}S^{k+1}x, S^kx\rangle \\ &= \langle A^{\frac{1}{2}}S^{\#}S^{k+1}x, A^{\frac{1}{2}}S^kx\rangle \\ &\leqslant \|A^{\frac{1}{2}}S^{\#}S^{k+1}x\| \|A^{\frac{1}{2}}S^kx\| \\ &= \left\|S^{\#}\left(S^{k+1}x\right)\right\|_A \left\|S^kx\right\|_A \\ &\leqslant \left\|S\left(S^{k+1}x\right)\right\|_A \left\|S^kx\right\|_A \text{ (since } S \text{ is } A\text{-hyponormal)} \\ &= \left\|S^{k+2}x\right\|_A \left\|S^kx\right\|_A.\end{aligned}$$

So, we get

$$\left\|S^{k+1}x\right\|_A^2 \leqslant \left\|S^{k+2}x\right\|_A \left\|S^kx\right\|_A,$$

for all $x \in \mathscr{H}$. Therefore, S is k-quasi-A-paranormal operator.

(4) Suppose that S is k-quasi-A-hyponormal, then $\left\|S^{\#}S^kx\right\|_A \leqslant \left\|S^{k+1}x\right\|_A$ for a positive integer k. Let $x \in \mathscr{H}$. From (3) we found

$$\|S^{k+1}x\|_A^2 \leqslant \left\|S^{\#}\left(S^{k+1}x\right)\right\|_A \left\|S^kx\right\|_A.$$

Since S is k-quasi-A-hyponormal, so $\left\|S^{\#}\left(S^{k+1}x\right)\right\|_A \leqslant \left\|S^{k+2}x\right\|_A$. Consequently, we infer that S is k-quasi-A-paranormal. □

In the following theorem, we give sufficient conditions for which the product of an k-quasi-A-paranormal operator with an A-isometric operator is an k-quasi-A-paranormal operator.

THEOREM 2.4. *Let $T,S \in \mathscr{B}_A(\mathscr{H})$ be such that T is an k-quasi-A-paranormal and S is an A-isometry. If $TS = ST$ and $ST^{\sharp} = T^{\sharp}S$, then TS is an k-quasi-A-paranormal operator.*

Proof. In view of Theorem 2.1, we need to prove that

$$(TS)^{\#k}\left((TS)^{\#2}(TS)^2 - 2\lambda(TS)^{\#}(TS) + \lambda^2 P\right)(TS)^k \geqslant_A 0,$$

for all $\lambda > 0$. In fact, since S is A-isometric ($S^{\#}S = P$), $TS = ST$ and $ST^{\sharp} = T^{\sharp}S$, it follows that

$$\begin{aligned}(TS)^{\#2}(TS)^2 - 2\lambda(TS)^{\#}(TS) + \lambda^2 P &= S^{\#2}T^{\#2}S^2T^2 - 2\lambda S^{\#}T^{\#}TS + \lambda^2 P\\ &= S^{\#2}S^2T^{\#2}T^2 - 2\lambda S^{\#}ST^{\#}T + \lambda^2 P\\ &= P\left(T^{\#2}T^2 - 2\lambda T^{\#}T + \lambda^2 P\right).\end{aligned}$$

On the other hand, we have for all $x \in \mathscr{H}$,

$$\begin{aligned}&\left\langle (TS)^{\#k}\left((TS)^{\#2}(TS)^2 - 2\lambda(TS)^{\#}(TS) + \lambda^2 P\right)(TS)^k x, x\right\rangle_A\\ &= \left\langle (TS)^{\#k} P\left(T^{\#2}T^2 - 2\lambda T^{\#}T + \lambda^2 P\right)(TS)^k x, x\right\rangle_A\\ &= \left\langle A(TS)^{\#k} P\left(T^{\#2}T^2 - 2\lambda T^{\#}T + \lambda^2 P\right)(TS)^k x, x\right\rangle\\ &= \left\langle (TS)^{*k} AP\left(T^{\#2}T^2 - 2\lambda T^{\#}T + \lambda^2 P\right)(TS)^k x, x\right\rangle\\ &= \left\langle A\left(T^{\#2}T^2 - 2\lambda T^{\#}T + \lambda^2 P\right)(TS)^k x, (TS)^k x\right\rangle\\ &= \left\langle \left(T^{\#2}T^2 - 2\lambda T^{\#}T + \lambda^2 P\right)T^kS^k x,\ T^kS^k x\right\rangle_A\\ &\geqslant 0.\end{aligned}$$

Consequently, we obtain

$$(TS)^{\#k}\left((TS)^{\#2}(TS)^2 - 2\lambda(TS)^{\#}(TS) + \lambda^2 P\right)(TS)^k \geqslant_A 0,$$

for all $\lambda > 0$. This shows that TS is k-quasi-A-paranormal operator. □

In the following theorem we give a sufficient condition under which the product of two k-quasi-A-paranormal operators is k -quasi-A-paranormal.

THEOREM 2.5. *Let $T,S \in \mathscr{B}_A(\mathscr{H})$ be k-quasi-A-paranormal operators. If $(TS)^2 \geqslant_A (TS)^{\#}(TS)$, then TS is k-quasi-A-paranormal operator.*

Proof. Let $x \in \mathscr{H}$, we have

$$\begin{aligned}\|(TS)^{k+1}x\|_A^2 &= \langle (TS)^{k+1}x, (TS)^{k+1}x\rangle_A\\ &= \langle A(TS)^{k+1}x, (TS)^{k+1}x\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle A(TS)^{k+1}x, (TS)(TS)^k x\rangle \\
&= \langle (TS)^* A(TS)^{k+1}x, (TS)^k x\rangle \\
&= \langle (TS)^{\#}(TS)(TS)^k x, (TS)^k x\rangle_A \\
&\leqslant \langle (TS)^2(TS)^k x, (TS)^k x\rangle \\
&= \langle (TS)^{k+2}x, (TS)^k x\rangle_A \\
&\leqslant \|(TS)^{k+2}x\|_A \|(TS)^k x\|_A .
\end{aligned}
$$

Hence,

$$\|(TS)^{k+1}x\|_A^2 \leqslant \|(TS)^{k+2}x\|_A \|(TS)^k x\|_A,$$

for all positive integer k. Therefore, TS is k-quasi-A-paranormal. $\square$

The following theorem is a remarkable extension of [11, Proposition 3].

THEOREM 2.6. *Let $T \in \mathscr{B}_A(\mathscr{H})$ and $S \in \mathscr{B}_A(\mathscr{H})$ be two commuting k-quasi-A-paranormal for some positive integer k. If T and S satisfy the following condition*

$$\max\left\{\|T^{k+2}S^k x\|_A^2,\ \|S^{k+2}T^k x\|_A^2\right\} \leqslant \|\left(TS\right)^{k+2}x\|_A \|\left(TS\right)^k x\|_A,$$

for all $x \in \mathscr{H}$, then TS is k-quasi-A-paranormal.

Proof. Since $TS = ST$ and T,S are k-quasi-A-paranormal, it follows that

$$
\begin{aligned}
\|\left(TS\right)^{k+1}x\|_A^2 &= \|T^{k+1}S^{k+1}x\|_A^2 \\
&\leqslant \|T^{k+2}S^{k+1}x\|_A \|T^k S^{k+1}\|_A \\
&= \|S^{k+1}T^{k+2}x\|_A \|S^{k+1}T^k\|_A \\
&\leqslant \|S^{k+2}T^{k+2}x\|_A^{\frac12} \|S^k T^{k+2}x\|_A^{\frac12} \|S^{k+2}T^k x\|_A^{\frac12} |S^k T^k x\|_A^{\frac12} \\
&= \left(\|S^{k+2}T^{k+2}x\|_A^{\frac12} |S^k T^k x\|_A^{\frac12}\right)\left(\|S^k T^{k+2}x\|_A^{\frac12} \|S^{k+2}T^k x\|_A\right) \\
&\leqslant \|S^{k+2}T^{k+2}x\|_A |S^k T^k x\|_A \\
&= \|\left(TS\right)^{k+2}x\|_A |\left(TS\right)^k x\|_A, \ \forall\, x \in \mathscr{H}.
\end{aligned}
$$

This means that TS is k-quasi-A-paranormal. $\square$

PROPOSITION 2.3. *Let $S \in \mathscr{B}_A(\mathscr{H})$ such that S^2 is an A-isometry. If S satisfies the following inequality*

$$2\|S^{k+1}x\|_A^2 \leqslant \|S^{k+2}x\|_A^2 + \|S^k x\|_A^2,$$

for all $x \in \mathscr{H}$ and for some positive integer k, then S is k-quasi-A-paranormal.

Proof. Let $x \in \mathscr{H}$, we have

$$
\begin{aligned}
2\|S^{k+1}x\|_A^2 &\leqslant \|S^{k+2}x\|_A^2 + \|S^k x\|_A^2 \\
&= \left(\|S^{k+2}x\|_A - \|S^k x\|_A\right)^2 + 2\|S^{k+2}x\|_A \|S^k x\|_A
\end{aligned}
$$

By the assumption that S^2 is A-isometry, we have $\|S^{k+2}x\|_A = \|S^kx\|_A$, $\forall\, x \in \mathscr{H}$ and hence

$$\|S^{k+1}x\|_A^2 \leqslant \|S^{k+2}x\|_A \|S^kx\|_A,\ \forall\, x \in \mathscr{H}.$$

Therefore, S is k-quasi-A-paranormal. $\square$

In [11], the authors proved that a power of an A-paranormal operator is A-paranormal. However, it was proved in [12, Theorem 3.8] that a power of k-quasi-paranormal operator is again an k-quasi-paranormal. Next we show that the corresponding result is true for the class of k-quasi-A-paranormal operators. Its proof is inspired from [12].

THEOREM 2.7. *Let $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$. If S is an k-quasi-A-paranormal operator, then S^n is also k-quasi-A-paranormal operator for every integer $n \geqslant 1$.*

Proof. We will prove that

$$\left\|S^{n(k+1)}x\right\|_A^2 \leqslant \left\|S^{n(k+2)}x\right\|_A \left\|S^{nk}x\right\|_A, \tag{2.4}$$

for all $x \in \mathscr{H}$. Notice first that if $S^kx \in \mathscr{N}(A)$, then (2.4) holds. Now, assume that $S^jx \notin \mathscr{N}(A)$ for all $j \geqslant k$ and $x \in \mathscr{H}$. From the assumption that S is an k-quasi-A-paranormal, we get

$$\frac{\left\|S^{k+1}x\right\|_A}{\|S^kx\|_A} \leqslant \frac{\left\|S^{k+2}x\right\|_A}{\|S^{k+1}x\|_A}.$$

Hence, it follows that

$$\frac{\left\|S^{k+i+1}x\right\|_A}{\|S^{k+i}x\|_A} \leqslant \frac{\left\|S^{k+i+2}x\right\|_A}{\|S^{k+i+1}x\|_A},$$

for all non-negative integer i.

Therefore,

$$\begin{aligned}
\frac{\left\|S^{nk+n}x\right\|_A}{\|S^{nk}x\|_A} &= \frac{\left\|S^{nk+1}x\right\|_A}{\|S^{nk}x\|_A} \cdot \frac{\left\|S^{nk+2}x\right\|_A}{\|S^{nk+1}x\|_A} \cdots \frac{\left\|S^{nk+n}x\right\|_A}{\|S^{nk+n-1}x\|_A} \\
&\leqslant \frac{\left\|S^{nk+2}x\right\|_A}{\|S^{nk+1}x\|_A} \cdot \frac{\left\|S^{nk+3}x\right\|_A}{\|S^{nk+2}x\|_A} \cdots \frac{\left\|S^{nk+n+1}x\right\|_A}{\|S^{nk+n}x\|_A} \\
&\leqslant \frac{\left\|S^{nk+n+1}x\right\|_A}{\|S^{nk+n}x\|_A} \cdot \frac{\left\|S^{nk+n+2}x\right\|_A}{\|T^{nk+n+1}x\|_A} \cdots \frac{\left\|S^{nk+n+n}x\right\|_A}{\left\|S^{nk+n+(n-1)}x\right\|_A} \\
&= \frac{\left\|S^{nk+2n}x\right\|_A}{\left\|S^{nk+n}x\right\|_A}.
\end{aligned}$$

Consequently, we infer that (2.4) holds for all $x \in \mathscr{H}$. Hence, S^n is k-quasi-A-paranormal. This finishes the proof. $\square$

PROPOSITION 2.4. *Let $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$. If $(S_n)_n \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is a sequence of k-quasi-A-paranormal operators such that $\lim\limits_{n\to\infty} \|S_n - S\| = 0$, then S is k-quasi-A-paranormal.*

Proof. Since the product of operators is sequentially continuous in the strong topology, one concludes that

$$S_n \longrightarrow S, \quad S_n^k \longrightarrow S^k \text{ and } A^{\frac{1}{2}}S_n^k \longrightarrow A^{\frac{1}{2}}S^k,$$

for each positive integer k.

Taking any $x \in \mathscr{H}$, a direct computation shows that

$$\begin{aligned}
\|S^{k+1}x\|_A^2 &= \|A^{\frac{1}{2}}S^{k+1}x\|^2 \\
&= \lim_{n\longrightarrow\infty} \|A^{\frac{1}{2}}S_n^{k+1}x\|^2 \\
&= \lim_{n\longrightarrow\infty} \|S_n^{k+1}x\|_A \\
&\leqslant \lim_{n\longrightarrow\infty} \left(\|S_n^{k+2}x\|_A\|S_n^k x\|_A\right) \\
&= \lim_{n\longrightarrow\infty} \left(\|S_n^{k+2}x\|_A\right) \lim_{n\longrightarrow\infty} \left(\|S_n^k x\|_A\right) \\
&= \|A^{\frac{1}{2}}S^{k+2}x\|\|A^{\frac{1}{2}}S^k x\| \\
&= \|S^{k+2}x\|_A\|S^k x\|_A.
\end{aligned}$$

Hence,

$$\|S^{k+1}x\|_A^2 \leqslant \|S^{k+2}x\|_A\|S^k x\|_A,$$

for all $x \in \mathscr{H}$. Thus shows that S is k-quasi-A-paranormal. $\square$

LEMMA 2.1. *Let $(S_{ij})_{1\leqslant i,\, j\leqslant 2}$ where $S_{ij} \in \mathscr{B}_A(\mathscr{H})$ for all $i, j = 1, 2$. Then $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathscr{B}_{A_0}(\mathscr{H} \oplus \mathscr{H})$ where $A_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Furthermore, $S^{\#_{A_0}} = \begin{pmatrix} S_{11}^{\#} & S_{21}^{\#} \\ S_{12}^{\#} & S_{22}^{\#} \end{pmatrix}$.*

Proof. The proof follows from [5, Lemma 3.1]. $\square$

THEOREM 2.8. *Let $S_1, S_2 \in \mathscr{B}(\mathscr{H})$ and let S be the operator on $\mathscr{B}_{A_0}(\mathscr{H} \oplus \mathscr{H})$ defined as*

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & 0 \end{pmatrix}.$$

If S_1 is A-paranormal, then S is quasi-A_0-paranormal.

Proof. From Lemma 2.1 we have $S^{\#_{A_0}} = \begin{pmatrix} S_1^{\#} & 0 \\ S_2^{\#} & 0 \end{pmatrix}$ and with simple calculation we show that

$$S^{\#}\left(S^{\#^2}S^2 - 2\lambda S^{\#}S + \lambda^2 P\right)S$$
$$= \begin{pmatrix} S_1^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_1 & S_1^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_2 \\ S_2^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_1 & S_2^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_2 \end{pmatrix},$$

for all $\lambda > 0$.

Let $u = x \oplus y \in \mathscr{H} \oplus \mathscr{H}$. Then, we have

$$\begin{aligned} &\left\langle S^{\#}\left(S^{\#^2}S^2 - 2\lambda S^{\#}S + \lambda^2 P\right)Su, u\right\rangle_A \\ &= \left\langle S_1^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_1x, x\right\rangle_A + \left\langle S_1^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_2y, x\right\rangle_A \\ &\quad + \left\langle S_2^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_1x, y\right\rangle_A + \left\langle S_2^{\#}\left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)S_2y, y\right\rangle_A \\ &= \left\langle \left(S_1^{\#^2}S_1^2 - 2\lambda S_1^{\#}S_1 + \lambda^2 P\right)(S_1x + S_2y), (S_1x + S_2y)\right\rangle_A \geqslant 0 \end{aligned}$$

(since S_1 is A-paranormal). □

3. Spectral properties of k-quasi-A-paranormal operators

In this section, we describe some spectral properties of an k-quasi-A-paranormal operator. The introduction of the concept of spectral radius and numerical radius of transformation in Hilbert spaces yielded a flow of papers generalizing this concept both in Hilbert and Banach spaces. Recently, many authors extended these concepts to operators in semi-Hilbertian spaces. For an operator $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$, the A-spectral radius of S is defined by

$$r_A(S) = \inf_{n\in\mathbb{N}} \|S^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty} \|S^n\|_A^{\frac{1}{n}},$$

([8]) and its A-numerical radius is defined by

$$\omega_A(S) = \sup\{|\langle Sx\,,\,x\rangle_A|,\ x \in \mathscr{H} :\ \|x\|_A = 1\},$$

(see [13]).

The following theorem extends [10, Theorem 2.4].

THEOREM 3.1. *Let $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ be an k-quasi-A-paranormal operator, then the following assertions hold:*

(1) $\|S^{m+1}\|_A^2 \leqslant \|S^{m+2}\|_A\|S^m\|_A$ *for all positive integers* $m \geqslant k$.
(2) *If* $\|S^m\|_A = 0$ *for some positive integer* $m \geqslant k$, *then* $\|S^{k+1}\|_A = 0$.
(3) $\|S^m\|_A \leqslant \|S^{m-1}\|_A r_A(S)$ *for all positive integer* $m \geqslant k+1$.

Proof. (1) This statement follows from Definition 2.1 and the fact that every k-quasi-A-paranormal operator is $(k+1)$-quasi-A-paranormal operator.

(2) This is a direct consequence of (1) above.

(3) Observe that if $\|S^{m-1}\|_A = 0$, the desired inequality is satisfied. Now assume that $\|S^j\|_A \neq 0$ for all $j \geqslant k$ and we need to prove

$$r_A(S) \geqslant \frac{\|S^m\|_A}{\|S^{m-1}\|_A} \ \ \forall\, m \geqslant k+1.$$

From the hypothesis that S is k-quasi-A-paranormal, it follows that

$$\frac{\|S^{k+m}\|_A}{\|S^{k+m-1}\|_A} \geqslant \frac{\|S^{k+1}\|_A}{\|s^k\|_A}.$$

This implies that

$$\begin{aligned}
\|S^{k+m}\|_A &\geqslant \frac{\|S^{k+1}\|_A}{\|S^k\|_A}\|S^{k+m-1}\|_A \\
&\geqslant \left(\frac{\|S^{k+1}\|_A}{\|S^k\|_A}\right)^2 \|S^{k+m-2}\|_A \\
&\geqslant \vdots \\
&\geqslant \left(\frac{\|S^{k+1}\|_A}{\|S^k\|_A}\right)^m \|S^k\|_A.
\end{aligned}$$

This yields that

$$\frac{\|S^{k+m}\|_A}{\|S^k\|_A} \geqslant \left(\frac{\|S^{k+1}\|_A}{\|S^k\|_A}\right)^m.$$

Hence,

$$\|S^m\|_A \geqslant \frac{\|S^{k+m}\|_A}{\|S^k\|_A} \geqslant \left(\frac{\|S^{k+1}\|_A}{\|S^k\|_A}\right)^m.$$

So we get

$$\|S^m\|_A^{\frac{1}{m}} \geqslant \frac{\|S^{k+1}\|_A}{\|S^k\|_A}.$$

According to [8, Theorem 1], we have

$$r_A(S) = \lim_{m\to\infty} \|S^m\|_A^{\frac{1}{m}} \geqslant \frac{\|S^{k+1}\|_A}{\|S^k\|_A}.$$

Repeating the above process we can prove that

$$r_A(S) \geqslant \frac{\|S^{k+2}\|_A}{\|S^{k+1}\|_A},$$

and furthermore,

$$r_A(S) \geqslant \frac{\|S^m\|_A}{\|S^{m-1}\|_A},$$

for all positive integer $m \geqslant k+1$. $\square$

Next, we need to introduce the following definition.

DEFINITION 3.1. ([8]) An operator $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is said to be A-normaloid if

$$r_A(S) = \|S\|_A.$$

In [11], the authors proved that if $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ is A-paranormal, then $r_A(S) = \|S\|_A$ i.e, S is A-normaloid. It was observed in [10] that in general an k-quasi-A-paranormal operator is not A-normaloid for $A = I$. Now, in view of Theorem 3.1, we drive a sufficient condition for which an k-quasi-A-paranormal operator to be A-normaloid.

COROLLARY 3.1. *Let $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ be an k-quasi-A-paranormal operator for a positive integer k.*

(1) *If $\|S^{n+1}\|_A = \|S^n\|_A\|S\|_A$ for some positive integer $n \geqslant k$, then S is A-normaloid.*
(2) *If $\|S^{n+1}\|_A = \|S\|_A^{n+1}$ for some positive integer $n \geqslant k$, then S is A-normaloid.*

Proof. (1) By the statement (3) of Theorem 3.1, it follows that

$$\|S^{n+1}\|_A \leqslant \|S^n\|_A r_A(S).$$

If $\|S^{n+1}\|_A = \|S^n\|_A\|S\|_A$, we obtain

$$\|S\|_A \leqslant r_A(S).$$

In view of [4, Proposition 2.5] and [8, Theorem 3], we get

$$r_A(S) \leqslant \omega_A(S) \leqslant \|S\|_A.$$

Consequently, $r_A(S) = \|S\|_A$ and therefore S is A-normaloid.

(2) From the statement (1) we get

$$\|S\|_A = \|S^n\|_A^{\frac{1}{n}} \leqslant r_A(S) \leqslant \|S\|_A.$$

Therefore, S is A-normaloid. $\square$

THEOREM 3.2. *Let $S \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ be an k-quasi-A-paranormal operator. If S^p is A-normaloid for $p \geqslant k$, then S^{p+m} is A-normaloid for $m = 1, 2, \cdots$.*

Proof. We need to prove by induction on m that S^{p+m} is A-normaloid for all $m = 1, 2\cdots$. Firstly, we observe that if $\|S^p\|_A = 0$, then

$$r_A(S^{m+p}) \leqslant r_A(S^p) r_A(S^m) = 0,$$

and

$$\|S^{m+p}\|_A \leqslant \|S^p\|_A\|S^m\|_A = 0.$$

Hence $r_A(S^{m+p}) = \|S^{m+p}\|_A$ and the result is true.

Assume that $\|S^j\|_A \neq 0$ for all $j \geqslant p$. We prove that S^{p+1} is A-normaloid.

From the fact that S is k-quasi-A-paranormal, it follows in view of Theorem 3.1

$$\frac{\|S^{k+j}\|_A}{\|S^{k+j-1}\|_A} \geqslant \frac{\|S^{k+j-1}\|_A}{\|S^{k+j-2}\|_A} \geqslant \cdots \geqslant \frac{\|S^{k+1}\|_A}{\|S^k\|_A},$$

and in particular

$$\frac{\|S^{2p}\|_A}{\|S^{2p-1}\|_A} \geqslant \frac{\|S^{p+1}\|_A}{\|S^p\|_A}.$$

Since S^p is A-normaloid, we have

$$\frac{\left\|S^{2p}\right\|_A}{\|S^{2p-1}\|_A} = \frac{\|S^p\|_A^2}{\|S^{2p-1}\|_A} \geqslant \frac{\left\|S^{p+1}\right\|_A}{\|S^p\|_A}.$$

Hence, we obtain

$$\|S^p\|_A^3 \geqslant \left\|S^{2p-1}\right\|_A \left\|S^{p+1}\right\|_A.$$

It follows that

$$(r_A(S))^{3p} \geqslant \left\|S^{p+1}\right\|_A (r_A(S))^{2p-1},$$

so we get

$$r_A\left(S^{p+1}\right) \geqslant \left\|S^{p+1}\right\|_A,$$

and always we have

$$r_A\left(S^{p+1}\right) \leqslant \left\|S^{p+1}\right\|_A.$$

Hence, we get

$$r_A\left(S^{p+1}\right) = \left\|S^{p+1}\right\|_A.$$

So, the result is true for $m = 1$.

Now assume that S^{p+m} is A-normaloid and prove that S^{p+m+1} is A-normaloid.

In fact, since S^{p+m} is A-normaloid we deduce from the above calculation that

$$\frac{\left\|S^{2(p+m)}\right\|_A}{\left\|S^{2(p+m)-1}\right\|_A} \geqslant \frac{\|S^{p+m+1}\|_A}{\|S^{p+m}\|_A},$$

or equivalently

$$\left\|S^{p+m}\right\|_A^3 \geqslant \left\|S^{p+m+1}\right\|_A \left\|S^{2(p+m)-1}\right\|_A.$$

This in turn gives,

$$(r_A(S))^{3p+3m} \geqslant \left\|S^{p+m+1}\right\|_A (r_A(S))^{2p+2m-1}.$$

Hence,

$$r_A\left(S^{p+m+1}\right) \geqslant \left\|S^{p+m+1}\right\|_A,$$

and so that, $r_A\left(S^{p+m+1}\right) = \left\|S^{p+m+1}\right\|_A$. Therefore, S^{p+m+1} is A-normaloid as required. $\square$

DEFINITION 3.2. Let $S \in \mathscr{B}_A(\mathscr{H})$ we say that S is A-regular operator if S is invertible and $S^{-1} \in \mathscr{B}_A(\mathscr{H})$.

THEOREM 3.3. *Let $S \in \mathscr{B}_A(\mathscr{H})$ be A-regular k-quasi-A-paranormal operator. If $0 \notin \sigma_a(A)$, then*

$$\sigma_a(S) \subseteq \left\{ \lambda \in \mathbb{C} : \frac{1}{\|(S^{-1})^{k+1}\|_A \sqrt{\|S^{k+1}\|_A \|S^{k-1}\|_A}} \leqslant |\lambda| \right\},$$

where $\sigma_a(S)$ is the approximate spectrum of S.

Proof. Let $x \in \mathscr{H}$ such that $\|x\|_A = 1$. Since S is A-regular k-quasi-A-paranormal, it follows that

$$\begin{aligned}
\|x\|_A^2 &= \|(S^{-1})^{k+1} S^{k+1} x\|_A^2 \\
&\leqslant \|(S^{-1})^{k+1}\|_A^2 \|S^{k+1} x\|_A^2 \\
&\leqslant \|(S^{-1})^{k+1}\|_A^2 \|S^{k+2} x\|_A \|S^k x\|_A \\
&\leqslant \|(S^{-1})^{k+1}\|_A^2 \|S^{k+1}\|_A \|S^{k-1}\|_A \|Sx\|_A^2.
\end{aligned}$$

So,

$$\|Sx\|_A \geqslant \frac{1}{\|(S^{-1})^{k+1}\|_A \sqrt{\|S^{k+1}\|_A \|S^{k-1}\|_A}}.$$

Assume that $\lambda \in \sigma_a(S)$. Since $0 \notin \sigma_a(A)$, there exists a sequence $(x_n)_n \in \mathscr{H}$: $\|x_n\| = 1$ satisfying $(S - \lambda)x_n \longrightarrow 0$ and $\|Ax_n\| \geqslant \delta$ for some $\delta > 0$.

We observe that

$$\begin{aligned}
\left\| (S-\lambda) \frac{x_n}{\|Ax_n\|} \right\|_A &\geqslant \left\| S \frac{x_n}{\|Ax_n\|} \right\|_A - |\lambda| \left\| \frac{x_n}{\|Ax_n\|} \right\|_A \\
&= \left\| S \frac{x_n}{\|Ax_n\|} \right\|_A - |\lambda| \\
&\geqslant \frac{1}{\|(S^{-1})^{k+1}\|_A \sqrt{\|S^{k+1}\|_A \|S^{k-1}\|_A}} - |\lambda|.
\end{aligned}$$

When $n \longrightarrow \infty$, we get

$$0 \geqslant \frac{1}{\|(S^{-1})^{k+1}\|_A \sqrt{\|S^{k+1}\|_A \|S^{k-1}\|_A}} - |\lambda|.$$

Therefore,

$$|\lambda| \geqslant \frac{1}{\|(S^{-1})^{k+1}\|_A \sqrt{\|S^{k+1}\|_A \|S^{k-1}\|_A}}.$$

Consequently,

$$\sigma_a(S) \subseteq \left\{ \lambda \in \mathbb{C} : \frac{1}{\|(S^{-1})^{k+1}\|_A \sqrt{\|S^{k+1}\|_A \|S^{k-1}\|_A}} \leqslant |\lambda| \right\}. \qquad \square$$

4. Tensor product of k-quasi-A-paranormal operators

In this section, we prove under suitable conditions that the tensor product of an k-quasi-A-paranormal operator and an A-isometry is an k-quasi-$A\otimes A$-paranormal operator (Proposition 4.1). However, the tensor product of an k-quasi-A-paranormal and an k-quasi-B-paranormal is an k-quasi $A\otimes B$-paranormal (Theorem 4.1).

LEMMA 4.1. *Let $S\in\mathscr{B}_A(\mathscr{H})$ be an k-quasi-A-paranormal, then the tensor product $S\otimes I$ and $I\otimes S$ are k-quasi-$A\otimes A$-paranormal.*

Proof. Let $\lambda>0$, we observe that

$$(S\otimes I)^{\#k}\Big((S\otimes I)^{\#2}(S\otimes I)^2-2\lambda(S\otimes I)^{\#}(S\otimes I)+\lambda^2P\Big)(S\otimes I)^k$$
$$=S^{\#k}\Big(S^{\#2}S^2-2\lambda SS^{\#}+\lambda^2P\Big)S^k\otimes P$$
$$\geqslant_{A\otimes A}0. \quad \square$$

PROPOSITION 4.1. *Let $T,S\in\mathscr{B}_A(\mathscr{H})$ such that $\mathscr{N}(A)^{\perp}$ is invariant for T. If T is an k-quasi-A-paranormal and S is an A-isometry, then $T\otimes S\in\mathscr{B}_{A\otimes A}(\mathscr{H}\,\overline{\otimes}\,\mathscr{H})$ is an k-quasi-$A\otimes A$-paranormal.*

Proof. It is well known that $T\otimes S=\big(T\otimes I\big)\big(I\otimes S\big)=\big(I\otimes S\big)\big(T\otimes I\big)$.

Under the condition that $\mathscr{N}(A)^{\perp}$ is invariant for T, we obtain $TP=PT$ and hence

$$\big(T\otimes I\big)\big(I\otimes S\big)^{\#}=\big(I\otimes S\big)^{\#}\big(T\otimes I\big).$$

Now, Since T is an k-quasi-A-paranormal and S is an A-isometry, it follows that $T\otimes I$ is an k-quasi-$A\otimes A$-paranormal (by Lemma 4.1) and $I\otimes S$ is an $A\otimes A$-isometry. Clearly $T\otimes I$ and $I\otimes S$ satisfy the conditions of Theorem 2.4 and therefore $T\otimes S$ is an k-quasi-$A\otimes A$-paranormal. $\square$

COROLLARY 4.1. *Let $T,S\in\mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ such that $\mathscr{N}(A)^{\perp}$ is invariant for T. If T is an k-quasi-A-paranormal and S is an A-isometry, then $T^p\otimes S^q\in\mathscr{B}_{A\otimes A}(\mathscr{H}\,\overline{\otimes}\,\mathscr{H})$ is an k-quasi-$A\otimes A$-paranormal.*

Proof. It is obvious that if S is an A-isometry so is S^q. On the other hand , since T is an k-quasi-A-paranormal, in view of Theorem 2.7, T^p is an k-quasi-A-paranormal. The desired conclusion follow from Proposition 4.1. $\square$

THEOREM 4.1. *Let $T\in\mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ and $S\in\mathscr{B}_{B^{\frac{1}{2}}}(\mathscr{H})$. If T is an k-quasi-A-paranormal and S is an k-quasi-B-paranormal, then $T\otimes S$ is k-quasi-$A\otimes B$-paranormal.*

Proof. Since T is an k-quasi-A-paranormal and S is an k-quasi-B-paranormal, it follows that

$$\|T^{k+1}u\|_A^2 \leqslant \|T^{k+2}u\|_A^2\|T^k u\|_A^2, \quad \forall\, u \in \mathscr{H},$$

and

$$\|S^{k+1}v\|_B^2 \leqslant \|S^{k+2}v\|_B^2\|S^k v\|_B^2, \quad \forall\, v \in \mathscr{H}.$$

This means that

$$\|T^{k+1}u\|_A^2\|S^{k+1}v\|_B^2 \leqslant \|T^{k+2}u\|_A^2\|S^{k+2}v\|_B^2\|T^k u\|_A^2\|S^k v\|_B^2, \quad \forall\, u,v \in \mathscr{H},$$

similarly,

$$\|T^{k+1}\otimes S^{k+1}(u\otimes v)\|_{A\otimes B}^2 \leqslant \|T^{k+2}\otimes S^{k+2}(u\otimes v)\|_{A\otimes B}^2\|T^k\otimes S^k(u\otimes v)\|_{A\otimes B}^2, \quad \forall\, u,v \in \mathscr{H},$$

or equivalently,

$$\|\big(T\otimes S\big)^{k+1}(u\otimes v)\|_{A\otimes B}^2 \leqslant \|\big(T\otimes S\big)^{k+2}(u\otimes v)\|_{A\otimes B}^2\|\big(T\otimes S\big)^{k}(u\otimes v)\|_{A\otimes B}^2, \quad \forall\, u,v \in \mathscr{H}.$$

Therefore, $T\otimes S$ is an k-quasi-$A\otimes B$-paranormal. $\square$

The following corollary is an immediate consequence of Theorem 2.7 and Theorem 4.1.

COROLLARY 4.2. *Let $T \in \mathscr{B}_{A^{\frac{1}{2}}}(\mathscr{H})$ and $S \in \mathscr{B}_{B^{\frac{1}{2}}}(\mathscr{H})$. If T is an k-quasi-A-paranormal and S is an k-quasi-B-paranormal, then $T^n\otimes S^m$ is k-quasi-$A\otimes B$-paranormal for all positive integers n and m.*

Acknowledgements. This work was funded by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-0333).

The authors express their sincere thanks to the reviewer for his/her valuable comments which helped us to improve this manuscript.

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(Received April 17, 2021)

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