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ON OPERATORS SATISFYING $T^*(T^{*2}T^2)^pT \geqslant T^*(T^2T^{*2})^pT$

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Abstract. An operator $T \in B(H)$ is called square-p-quasihyponormal if

$$T^*(T^{*2}T^2)^pT \geqslant T^*(T^2T^{*2})^pT \quad \text{for} \quad p \in (0,1],$$

which is a further generalization of normal operator. In this paper, we give a sufficient condition for an injective square-p-quasihyponormal operator to be self-adjoint, and we obtain that every square-p-quasihyponormal operator has a scalar extension. As a consequence, we prove that if T is a quasiaffine transform of square-p-quasihyponormal, then T satisfies Weyl's theorem. Finally some examples are presented.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H. If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range space of T, and also, write $\sigma(T)$, $\sigma_e(T)$ and $\omega(T)$ for the spectrum, the essential spectrum and the Weyl spectrum of T, respectively.

An operator $T \in B(H)$ is said to be p-hyponormal for $p \in (0,1]$ if $(T^*T)^p \geqslant (TT^*)^p$ where T^* is the adjoint of T. If $p = 1$, T is called hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia [25], and p-hyponormal operators were introduced by Aluthge [3]. An operator $T \in B(H)$ is called p-quasihyponormal for $p \in (0,1]$ if $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$. 1-quasihyponormal is called quasihyponormal (see [5]). An operator $T \in B(H)$ is called paranormal if $||T^2x|| \geqslant ||Tx||^2$ for unit vector x. Clearly hyponormal operators are quasihyponormal operators, p-hyponormal operators are p-quasihyponormal and p-quasihyponormal operators are paranormal. It is well-known that p-hyponormal operators are q-hyponormal if $0 < q \leqslant p$, however, it is not necessarily true that p-quasihyponormal operators are q-quasihyponormal even if $0 < q < p$.

An operator $T \in B(H)$ is normal and 2-normal if $T^*T = TT^*$ and $T^*T^2 = T^2T^*$, respectively. By Fuglede-Putnam theorem, it is easy to see that T is 2-normal if and only if T^2 is normal (see [4]). In [17] an operator $T \in B(H)$ is called kth root of p-hyponormal for $p \in (0,1]$ if T^k is p-hyponormal for some positive integer k. If $k = 2$, T is said to be square-p-hyponormal, i.e., $(T^{*2}T^2)^p \geqslant (T^2T^{*2})^p$, in particular

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for $k=2$ and $p=1$, T is said to be square hyponormal [8]. Now we are going to consider an extension of the notion of square-p-hyponormal operator, similar in spirit to the extension of the notion of p-hyponormality to p-quasihyponormality.

DEFINITION 1.1. An operator $T\in B(H)$ is called square-p-quasihyponormal if

$$T^*(T^{*2}T^2)^pT \geqslant T^*(T^2T^{*2})^pT \quad \text{for } p\in(0,1].$$

It is clear that

$$\begin{aligned}\text{normal} \Rightarrow \text{2-normal} &\Rightarrow \text{square hyponormal}\\ &\Rightarrow \text{square-}p\text{-hyponormal}\\ &\Rightarrow \text{square-}p\text{-quasihyponormal.}\end{aligned}$$

2-normal operator and square-p-hyponormal operator have been studied by many authors and it is known that they have many interesting properties similar to those of normal operator (see [7, 8, 9, 16, 17]).

In general, the conditions $S^{-1}TS=T^*$ and $0\notin\overline{W(S)}$ do not imply that T is normal, where $W(S)=\{\langle Sx,x\rangle : \|x\|=1\}$. For example (see [24]), if $T=SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS=T^*$ and $0\notin\overline{W(S)}$, but T is not normal. Therefore the following question arises naturally.

QUESTION 1.2. Suppose that T is an operator for which there is an operator S with $0\notin\overline{W(S)}$ such that $S^{-1}TS=T^*$. When does it follow that necessarily T is normal?

In Section 2, we show that if T is an injective square-p-quasihyponormal operator and S is an arbitrary operator for which $0\notin\overline{W(S)}$ and $ST=T^*S$, then T is a 2-normal operator. A bounded linear operator T on H is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras $\Phi: C_0^m(\mathbb{C})\to B(H)$ such that $\Phi(z)=T$, where z stands for the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0\leqslant m\leqslant\infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. In 1984, Putinar [22] proved that every hyponormal operator has a scalar extension, which has been extended from hyponormal operators to p-hyponormal operators [18], to analytic roots of hyponormal operators [16], to analytic extensions of M-hyponormal operators [19], and to kth roots of p-hyponormal operators [17]. In Section 3, we show that every square-p-quasihyponormal operator is subscalar. As a consequence, we prove that every square-p-quasihyponormal operator with rich spectrum has a nontrivial invariant subspace. In Section 4, we also obtain that every F-square-p-quasihyponormal operator has a scalar extension. Finally, we give some examples of square-p-quasihyponormal operator in Section 5.

2. Operators similar to their adjoints

Before we state main theorems, we need several preliminary results.

LEMMA 2.1. (Hansen inequality [14]) *If* $A, B \in B(H)$ *satisfy* $A \geqslant 0$ *and* $\| B \| \leqslant 1$, *then*

$$(B^*AB)^{\delta} \geqslant B^*A^{\delta}B \quad \textit{for all} \quad \delta \in (0,1].$$

LEMMA 2.2. (Löwner-Heinz inequality [13]) $A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$.

LEMMA 2.3. *Suppose that* $T \in B(H)$ *is a square-* p *-quasihyponormal operator and* $R(T)$ *is not dense. Then*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \quad \textit{on} \quad H = \overline{R(T)} \oplus N(T^*),$$

where A *is a square-* p *-hyponormal operator and* $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. The spectral inclusion relations are clear and it is sufficient to show that A is square- p -hyponormal. Let P be the orthogonal projection onto $\overline{R(T)}$. Then

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is a square- p -quasihyponormal operator, we have

$$P((T^{*2}T^2)^p - (T^2T^{*2})^p)P \geqslant 0.$$

Then

$$\begin{aligned} P(T^*T^*TT)^pP &\leqslant (PT^*T^*TTP)^p \quad \text{(by lemma 2.1)} \\ &= (PT^*PT^*TPTP)^p \\ &= \begin{pmatrix} (A^{*2}A^2)^p & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} P(TTT^*T^*)^pP &\geqslant P(TTPT^*T^*)^pP \quad \text{(by lemma 2.2)} \\ &= \begin{pmatrix} (A^2A^{*2})^p & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{pmatrix} (A^{*2}A^2)^p & 0 \\ 0 & 0 \end{pmatrix} \geqslant P(T^{*2}T^2)^pP \geqslant P(T^2T^{*2})^pP \geqslant \begin{pmatrix} (A^2A^{*2})^p & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., A is a square- p -hyponormal operator. $\square$

LEMMA 2.4. ([24, Theorem 1]) *If $T \in B(H)$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

In Lemma 2.4, the condition, $0 \notin \overline{W(S)}$, is essential. For example ([24, Example 1]), let W is the bilateral shift on l^2 which is defined by $We_n = e_{n+1}$, where $\{e_n\}_{n=-\infty}^{\infty}$ is the canonical orthonormal basis for l^2, and let S be the unitary operator defined by $Se_n = e_{-n}$. Then $S^{-1}WS = W^*$, but the spectrum of W is not real. Actually, the spectrum of W is the unit circle.

THEOREM 2.5. *Let T be a square-p-hyponormal operator. If T is a paranormal operator, S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

Proof. Suppose that T is a square-p-hyponormal operator. Since $\sigma(S) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T$. Then $\sigma(T) \subseteq \mathbb{R}$ by Lemma 2.4. Hence $m_2(\sigma(T)) = 0$ for the planar Lebesgue measure m_2. Now apply Putnam's inequality for p-hyponormal operators to T^2 (depending upon which is p-hyponormal) to get

$$||(T^{*2}T^2)^p - (T^2T^{*2})^p|| \leqslant \frac{1}{\pi} m_2(\sigma(T^2)) = 0.$$

It follows that T is 2-normal. Since a 2-normal paranormal operator is normal by [23, Theorem 4.6], we have T is an normal operator, apply [24, Theorem], thus T is self-adjoint. □

THEOREM 2.6. *Let T be an injective square-p-quasihyponormal operator. If T is a paranormal operator, S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

Proof. Since T is a square-p-quasihyponormal operator, we have the following matrix representation by Lemma 2.3:

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T)} \oplus N(T^*),$$

where A is a square-p-hyponormal operator and $\sigma(T) = \sigma(A) \cup \{0\}$. Let $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$. Then from $0 \notin \overline{W(S)}$ and $ST = T^*S$, we have $0 \notin \overline{W(S_1)}$ and $S_1A = A^*S_1$. Therefore A is 2-normal by Theorem 2.5. Now let P be the orthogonal projection of H onto $\overline{R(T)}$. Then we have

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP,$$

$$\begin{aligned} P(T^*T^*TT)^pP &\leqslant (PT^*T^*TTP)^p \quad \text{(by lemma 2.1)} \\ &= (PT^*PT^*TPTP)^p \\ &= \begin{pmatrix} (A^{*2}A^2)^p & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} P(TTT^*T^*)^pP &\geqslant P(TTPT^*T^*)^pP \quad \text{(by lemma 2.2)} \\ &= \begin{pmatrix} (A^2A^{*2})^p & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since T is a square- p-quasihyponormal operator,

$$\begin{pmatrix} (A^{*2}A^2)^p & 0 \\ 0 & 0 \end{pmatrix} \geqslant P(T^{*2}T^2)^pP \geqslant P(T^2T^{*2})^pP \geqslant \begin{pmatrix} (A^2A^{*2})^p & 0 \\ 0 & 0 \end{pmatrix},$$

and hence we may write

$$(T^2T^{*2})^p = \begin{pmatrix} (A^{*2}A^2)^p & M \\ M^* & N \end{pmatrix}.$$

Let $(T^2T^{*2})^{\frac{p}{2}} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$. Then

$$\begin{aligned} \begin{pmatrix} (A^{*2}A^2)^{\frac{p}{2}} & 0 \\ 0 & 0 \end{pmatrix} &= (P(T^2T^{*2})^pP)^{\frac{1}{2}} \\ &\geqslant P(T^2T^{*2})^{\frac{p}{2}}P \\ &= \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \\ &\geqslant P(T^2PT^{*2})^{\frac{p}{2}}P \\ &= \begin{pmatrix} (A^{*2}A^2)^{\frac{p}{2}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$X = (A^{*2}A^2)^{\frac{p}{2}}.$$

On the other hand, a straightforward calculation shows

$$(T^2T^{*2})^p = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}^2 = \begin{pmatrix} X^2+YY^* & XY+YZ \\ Y^*X+ZY^* & Y^*Y+Z^2 \end{pmatrix}.$$

Hence

$$(A^{*2}A^2)^p = X^2+YY^* = X^2.$$

This implies $Y=0$ and

$$(T^2T^{*2})^{\frac{p}{2}} = \begin{pmatrix} (A^{*2}A^2)^{\frac{p}{2}} & 0 \\ 0 & Z \end{pmatrix}.$$

Then

$$T^2T^{*2} = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{*2} & 0 \\ B^*A^* & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A^2A^{*2} + ABB^*A^* & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A^{*2}A^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $ABB^*A^* = 0$. Since T is an injective square-p-quasihyponormal operator, A is an injective square-p-hyponormal operator, hence $B = 0$, T must be 2-normal. Since T is a paranormal operator, it follows that T is an normal operator, apply [24, Theorem], thus T is self-adjoint. □

COROLLARY 2.7. *Let T be an injective square-p-quasihyponormal operator. If S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is 2-normal.*

Proof. This is a consequence of Theorem 2.6. □

THEOREM 2.8. *Let T be a square-p-quasihyponormal operator and M be its invariant subspace. Then the restriction $T|_M$ of T to M is also a square-p-quasihyponormal operator.*

Proof. Let E be the orthogonal projection onto M. Thus we can reprsent T as the following 2×2 operator matrix with respect to the decomposition $M \oplus M^\perp$,

$$T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Put $A = T|_M$. Then $TE = ETE$ and $A = (ETE)|_M$. Since T is a square-p-quasihyponormal operator, we have

$$ET^*(T^{*2}T^2)^pTE \geqslant ET^*(T^2T^{*2})^pTE.$$

Since

$$\begin{aligned} ET^*(T^{*2}T^2)^pTE &= ET^*E(T^{*2}T^2)^pETE \\ &\leqslant ET^*(ET^{*2}T^2E)^pTE \quad \text{(by lemma 2.1)} \\ &= ET^*E(ET^{*2}EET^2E)^pETE \\ &= \begin{pmatrix} A^*(A^{*2}A^2)^pA & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} ET^*(T^2T^{*2})^pTE &= ET^*E(T^2T^{*2})^pETE \\ &\geqslant ET^*E(T^2ET^{*2})^pETE \quad \text{(by lemma 2.2)} \\ &= ET^*E(ET^2EET^{*2}E)^pETE \\ &= \begin{pmatrix} A^*(A^2A^{*2})^pA & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

we have

$$\begin{pmatrix} A^*(A^{*2}A^2)^pA & 0 \\ 0 & 0 \end{pmatrix} \geqslant \begin{pmatrix} A^*(A^2A^{*2})^pA & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that A is a square- p -quasihyponormal operator. □

3. Subscalarity

For a Banach space $\mathscr{X}$, let $\xi(U,\mathscr{X})$ (resp., $\mathscr{O}(U,\mathscr{X})$) denote the Fréchet space of all infinite differentiable $\mathscr{X}$-value functions on U (resp., of all analytic $\mathscr{X}$-value functions on U). An operator $T \in B(\mathscr{X})$ is said to have property $(\beta)_\varepsilon$ at $\lambda \in \mathbb{C}$ if there exists a neighbourhood D of λ such that for every open subset U of D and $\mathscr{X}$-value functions sequence $\{f_n\}$ in $\xi(U,\mathscr{X})$, $(T-zI)f_n(z) \to 0$ in $\xi(U,\mathscr{X}) \Rightarrow f_n(z) \to 0$ in $\xi(U,\mathscr{X})$, and $T \in B(\mathscr{X})$ is said to have property (β) at $\lambda \in \mathbb{C}$ if there exists an $r > 0$ such that for every subset U of the open disc $D(\lambda;r)$ of radius r centered at λ and sequence $\{f_n\}$ of $\mathscr{X}$-value functions in $\mathscr{O}(U,\mathscr{X})$, $(T-zI)f_n(z) \to 0$ in $\mathscr{O}(U,\mathscr{X}) \Rightarrow f_n(z) \to 0$ in $\mathscr{O}(U,\mathscr{X})$. An operator $T \in B(H)$ is said to have property $(\beta)_\varepsilon$ (resp., (β)) if T has property $(\beta)_\varepsilon$ (resp., (β)) at every point $\lambda \in \mathbb{C}$. In this section we show that every square- p -quasihyponormal operator has a scalar extension, we need the following lemma.

LEMMA 3.1. ([18, Lemma 1]) *For $T \in B(\mathscr{X})$, the following statements are equivalent:*

(i) T is subscalar;

(ii) T has property $(\beta)_\varepsilon$.

THEOREM 3.2. *Suppose that T is a square- p -quasihyponormal operator. Then T is subscalar.*

Proof. Assume that $R(T)$ is dense. Then T is a square- p -hyponormal operator, it is subscalar of order 8 by [17, Theorem 3.6]. So we may assume that T does not have dense range. Then by Lemma 2.3 the operator T can be decomposed as follows: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$, where A is a square- p -hyponormal operator. Set $\sigma_{(\beta)_\varepsilon}(S) = \{\mu \in \sigma(S) : S \text{ doesn't satisfy property } (\beta)_\varepsilon \text{ at } \mu\}$. Recall from [6, Theorem 2.1] that given operators S and R, $\lambda \in \sigma_{(\beta)_\varepsilon}(RS) \Leftrightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(SR)$. Considering $T =$

$\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_1 & T_2 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}$, let $B = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$, $E = \begin{pmatrix} I_1 & T_2 \\ 0 & I_2 \end{pmatrix}$, $A = \begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}$. Then $T = BEA$. Suppose $\lambda \in \sigma_{(\beta)_\varepsilon}(T) \Leftrightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(BEA) = \sigma_{(\beta)_\varepsilon}(EAB)$. Hence, since E is invertible, $\lambda \in \sigma_{(\beta)_\varepsilon}(AB) = \sigma_{(\beta)_\varepsilon}(T_1 \oplus 0) \Rightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(T_1)$, contradiction. Thus T has property $(\beta)_\varepsilon$, i.e., T is subscalar. $\square$

COROLLARY 3.3. *Suppose that T is a square-p-quasihyponormal operator. Then T has Bishop's property (β).*

Proof. Since the Bishop's property (β) is transmitted from an operator to its restrictions to closed invariant subspace, we are reduced by Theorem 3.2 to the case of a scalar operator. Since every scalar operator has Bishop's property (β) [22], T has Bishop's property (β). $\square$

COROLLARY 3.4. *Let T be a square-p-quasihyponormal operator. If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.*

Proof. It suffices to apply Theorem 3.2 and [11]. $\square$

COROLLARY 3.5. *Suppose that T is a quasinilpotent square-p-quasihyponormal operator. Then T is nilpotent.*

Proof. Since a quasinilpotent subscalar operator is nilpotent. It follows by Theorem 3.2 that T is nilpotent. $\square$

DEFINITION 3.6. An operator $T \in B(H)$ is said to belong to the class $H(p)$ if there exists a natural number $p := p(\lambda)$ such that

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \lambda \in \mathbb{C},$$

where $H_0(\lambda I - T) := \{x \in H : \lim_{n\to\infty} ||(\lambda I - T)^n x||^{\frac{1}{n}} = 0\}$.

THEOREM 3.7. [20] *Every subscalar operator $T \in B(H)$ is $H(p)$.*

Classical examples of subscalar operators are hyponormal operators. In this paper we will show that other important classes of operators are $H(p)$.

DEFINITION 3.8. An operator $T \in B(H)$ is said to be polaroid if every $\lambda \in \text{iso}\,\sigma(T)$ is a pole of the resolvent of T, where $\text{iso}\,\sigma(T)$ denotes the isolated points of the spectrum.

The condition of being polaroid may be characterized by means of the quasinilpotent part:

THEOREM 3.9. [2] *An operator $T \in B(H)$ is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \quad \lambda \in \operatorname{iso}\sigma(T).$$

Note that every $H(p)$ operator is polaroid. By using Theorem 3.2 and Theorem 3.7, we deduce the following corollaries.

COROLLARY 3.10. *Every square- p -quasihyponormal operator is $H(p)$.*

COROLLARY 3.11. *Every square- p -quasihyponormal operator is polaroid.*

Recall that an operator $X \in B(H_1, H_2)$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S \in B(H_1)$ is said to be a quasiaffine transform of $T \in B(H_2)$ if there is a quasiaffinity $X \in B(H_1, H_2)$ such that $XS = TX$. Furthermore, S and T are quasisimilar if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$.

COROLLARY 3.12. *Let T be a square- p -quasihyponormal operator. If S is a quasiaffine transform of T, then S satisfies Weyl's theorem (i.e., $\sigma(T) - \omega(T) = \pi_{00}(T)$, where $\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < N(T - \lambda I) < \infty\}$.*

Proof. If T is a square- p -quasihyponormal operator, then $H_0(\lambda I - T) = N(\lambda I - T)^p$ for some integer $p := p(\lambda) \geqslant 0$ and all $\lambda \in \mathbb{C}$. Suppose $US = TU$ with U injective and $x \in H_0(\lambda I - S)$. Then

$$||(\lambda I - T)^n Ux||^{\frac{1}{n}} = ||U(\lambda I - S)^n x||^{\frac{1}{n}} \leqslant ||U||^{\frac{1}{n}} ||(\lambda I - S)^n x||^{\frac{1}{n}},$$

for which we obtain that $Ux \in H_0(\lambda I - T) = N(\lambda I - T)^p$. Hence

$$U(\lambda I - S)^p x = (\lambda I - T)^p Ux = 0,$$

and since U injective this implies that $(\lambda I - S)^p x = 0$. Consequently $H_0(\lambda I - S) = N(\lambda I - S)^p$ for some integer $p := p(\lambda) \geqslant 0$ and all $\lambda \in \mathbb{C}$. By [1, Theorem 3.10] Weyl's theorem holds for S. □

COROLLARY 3.13. *Let T and S be square- p -quasihyponormal operators. If T and S are quasisimilar, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$.*

Proof. It follows by Corollary 3.3 and [21]. □

4. F-square-p-quasihyponormal operators

In this section we will define F-square-p-quasihyponormal operators, and we will present some properties of this class of operators.

DEFINITION 4.1. For $0 < p \leqslant 1$ an operator $T \in B(H)$ is said to be F-square-p-quasihyponormal if $F(T)^*(T^*(T^{*2}T^2)^pT - T^*(T^2T^{*2})^pT)F(T) \geqslant 0$ for some nonconstant analytic function F on some neighborhood of $\sigma(T)$, and q-square-p-quasihyponormal operators if there exist a nonconstant polynomial q such that

$$q(T)^*(T^*(T^{*2}T^2)^pT - T^*(T^2T^{*2})^pT)q(T) \geqslant 0.$$

In particular, if $q(z) = z^k$ for some positive integer k, then T is said to be k-square-p-quasihyponormal.

If $T \in B(H)$ is analytic, then $F(T) = 0$ for some nonconstant analytic function F on a neighborhood U of $\sigma(T)$. Since F cannot have infinitely many zeros in U, we write $F(z) = G(z)q(z)$ where the function G is analytic and does not vanish on U and q is a nonconstant polynomial with zeros in U. By Riesz-Dunford calculus, $G(T)$ is invertible and the invertibility of $G(T)$ induces that $q(T) = 0$, which means that T is algebraic (See [10]).

THEOREM 4.2. *If T is an F-square-p-quasihyponormal operator, then T is subscalar. In particular, every k-square-p-quasihyponormal operator is subscalar.*

Proof. Suppose that $T \in B(H)$ is F-square-p-quasihyponormal for some analytic function F on a neighborhood of $\sigma(T)$. If the range of $F(T)$ is norm dense in H, then T is square-p-quasihyponormal, hence T is subscalar. Now it suffices to assume that the range of $F(T)$ is not norm dense in H. Since $F(T)$ commutes with T, $\overline{R(F(T))}$ is a T-invariant subspace. Thus T can expressed as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

on $\overline{R(F(T))} \oplus N(F(T)^*)$; where $T_1 = T|_{\overline{R(F(T))}}$ and $T_3 = (I-P)T(I-P)|_{N(F(T)^*)}$, and P denotes the projection of H onto $\overline{R(F(T))}$. Note that $F(z) = G(z)q(z)$ where G is a nonvanishing analytic function on a neighborhood of $\sigma(T)$ and q is a nonconstant polynomial. Then $G(T)$ is invertible and thus we obtain that $N(F(T)^*) = N(q(T)^*)$. Since $q(T_3) = (I-P)q(T)(I-P)|_{N(F(T)^*)}$, it follows for any $x \in N(F(T)^*)$ that

$$\langle q(T_3)x; x\rangle = \langle q(T)x; x\rangle = \langle x; q(T)^*x\rangle = 0.$$

Hence $q(T_3) = 0$. Thus T_3 is algebraic. Since $P(T^*(T^{*2}T^2)^pT - T^*(T^2T^{*2})^pT)P \geqslant 0$. Hence $T_1^*(T_1^{*2}T_1^2)^pT_1 - T_1^*(T_1^2T_1^{*2})^pT_1 \geqslant 0$. This shows that T_1 is square-p-quasihyponormal. Therefore if T_3 is algebraic, then T is subscalar by Theorem 3.2. □

COROLLARY 4.3. *Every F-square-p-quasihyponormal operator has the Bishop's property* (β).

COROLLARY 4.4. *Every k-square-p-quasihyponormal operator has the Bishop's property* (β).

5. Examples

Consider unilateral weighted shift operator as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \ldots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on $H = l_2$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geqslant 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthonormal basis for l_2. We easily see that W_α can be never normal, and so in general it is used to giving some easy examples of non-normal operators. It is well known that W_α is p-quasihyponormal if and only if α is monotonically increasing (see [26, Example 2.3]).

LEMMA 5.1. *W_α belongs to square-p-quasihyponormal if and only if*

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \alpha_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\alpha_n \alpha_{n+1} \leqslant \alpha_{n+2} \alpha_{n+3} \quad (n = 1, 2, 3, \ldots).$$

Proof. By simple calculations,

$$W_\alpha^{*2} W_\alpha^2 = (\alpha_1^2 \alpha_2^2) \oplus (\alpha_2^2 \alpha_3^2) \oplus (\alpha_3^2 \alpha_4^2) + \ldots$$

and

$$W_\alpha^2 W_\alpha^{*2} = 0 \oplus 0 \oplus (\alpha_1^2 \alpha_2^2) \oplus (\alpha_2^2 \alpha_3^2) \oplus (\alpha_3^2 \alpha_4^2) + \ldots.$$

Hence

$$W_\alpha^* (W_\alpha^{*2} W_\alpha^2)^p W_\alpha = \alpha_1^2 (\alpha_2^{2p} \alpha_3^{2p}) \oplus \alpha_2^2 (\alpha_3^{2p} \alpha_4^{2p}) \oplus \alpha_3^2 (\alpha_4^{2p} \alpha_5^{2p}) + \ldots$$

and

$$W_\alpha^* (W_\alpha^2 W_\alpha^{*2})^p W_\alpha = 0 \oplus \alpha_2^2 (\alpha_1^{2p} \alpha_2^{2p}) \oplus \alpha_3^2 (\alpha_2^{2p} \alpha_3^{2p}) \oplus \alpha_4^2 (\alpha_3^{2p} \alpha_4^{2p}) + \ldots.$$

Thus W_α belongs to square-p-quasihyponormal if and only if

$$\alpha_n \alpha_{n+1} \leqslant \alpha_{n+2} \alpha_{n+3} \quad (n = 1, 2, 3, \ldots). \qquad \square$$

The following example provides an operator which is square-p-quasihyponormal but not p-quasihyponormal.

EXAMPLE 5.2. A square-p-quasihyponormal operator which is not p-quasihyponormal.

Proof. Let W_α be a unilateral weighted shift operator with weights $\alpha_n = 2$ $(n \neq 2)$ and $\alpha_2 = 1$. Simple calculations show that W_α is square-p-quasihyponormal, but W_α is non-p-quasihyponormal. □

Finally we give an example to show that the class of square-p-hyponormal operators is properly contained in the class of square-p-quasihyponormal operators. We need the following lemma.

LEMMA 5.3. *Let $K = \bigoplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. For given positive operators A and B on H, define the operator $T = T_{A,B}$ on K as follows:*

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & A & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the following assertions hold:

(1) *T belongs to square-p-hyponormal if and only if $B^{4p} \geqslant A^{4p}$ and $B^{4p} \geqslant (BA^2B)^p$.*

(2) *T belongs to square-p-quasihyponormal if and only if $A(B^{4p} - A^{4p})A \geqslant 0$ and $B(B^{4p} - (BA^2B)^p)B \geqslant 0$.*

Proof. Since

$$T^* = \begin{pmatrix} 0 & A & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

by simple calculations,

$$(T^{*2}T^2)^p = \begin{pmatrix} A^{4p} & 0 & 0 & 0 & 0 & \cdots \\ 0 & (AB^2A)^p & 0 & 0 & 0 & \cdots \\ 0 & 0 & B^{4p} & 0 & 0 & \cdots \\ 0 & 0 & 0 & B^{4p} & 0 & \cdots \\ 0 & 0 & 0 & 0 & B^{4p} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$(T^2T^{*2})^p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A^{4p} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & (BA^2B)^p & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B^{4p} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & B^{4p} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence

$$T^*(T^{*2}T^2)^pT = \begin{pmatrix} A(AB^2A)^pA & 0 & 0 & 0 & \cdots \\ 0 & AB^{4p}A & 0 & 0 & \cdots \\ 0 & 0 & BB^{4p}B & 0 & \cdots \\ 0 & 0 & 0 & BB^{4p}B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$T^*(T^2T^{*2})^pT = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & AA^{4p}A & 0 & 0 & \cdots \\ 0 & 0 & B(BA^2B)^pB & 0 & \cdots \\ 0 & 0 & 0 & BB^{4p}B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus T is square- p -hyponormal ($(T^{*2}T^2)^p \geqslant (T^2T^{*2})^p$) if and only if

$$\begin{cases} B^{4p} \geqslant A^{4p}, \\ B^{4p} \geqslant (BA^2B)^p. \end{cases}$$

Similarly, T is square- p -quasihyponormal ($T^*(T^{*2}T^2)^pT \geqslant T^*(T^2T^{*2})^pT$) if and only if

$$\begin{cases} AB^{4p}A \geqslant AA^{4p}A, \\ BB^{4p}B \geqslant B(BA^2B)^pB. \end{cases} \qquad \square$$

EXAMPLE 5.4. A square- 1 -quasihyponormal operator which is not square- 1 -hyponormal.

Proof. Let H be a two dimensional Hilbert space and $p = 1$. Take A and B as

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then

$$B^4 - A^4 = \begin{pmatrix} \frac{7}{16} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \ngeqslant 0.$$

Hence $T_{A,B}$ is a non-square-1-hyponormal operator.

On the other hand,

$$A(B^4 - A^4)A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{7}{16} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{7}{64} & 0 \\ 0 & 0 \end{pmatrix} \geqslant 0$$

and

$$B(B^4 - BA^2B)B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{7}{16} & \frac{7}{16} \\ \frac{7}{16} & \frac{7}{16} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{16} & \frac{7}{16} \\ \frac{7}{16} & \frac{7}{16} \end{pmatrix} \geqslant 0.$$

Thus $T_{A,B}$ is a square-1-quasihyponormal operator. □

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