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MEAN ERGODICITY OF MULTIPLICATION OPERATORS ON THE BLOCH AND BESOV SPACES

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Abstract. In this paper, the power boundedness and mean ergodicity of multiplication operators are investigated on the Bloch space $\mathscr{B}$, the little Bloch space $\mathscr{B}_0$ and the Besov Space $\mathscr{B}_p$. We completely characterize power bounded, mean ergodic and uniformly mean ergodic multiplication operators on $\mathscr{B}$ and $\mathscr{B}_0$.

1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{U})$ be the space of all holomorphic functions on $\mathbb{U}$. The Bloch space $\mathscr{B}$ is defined to be the space of all functions in $H(\mathbb{U})$ such that

$$\beta_f = \sup_{z\in\mathbb{U}}(1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathscr{B}_0$ is the closed subspace of $\mathscr{B}$ consisting of all functions $f \in \mathscr{B}$ with

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is easy to check that the Bloch and little Bloch, $\mathscr{B}$ and $\mathscr{B}_0$ are Banach spaces under the norm

$$||f||_{\mathscr{B}} = |f(0)| + \beta_f.$$

It is well known that $\mathscr{B}_0^* = A^1(\mathbb{U})$ and $(A^1(\mathbb{U}))^* = \mathscr{B}$ under the complex integral pairing $\langle f,g\rangle = \int_{\mathbb{U}} f(z)\overline{g(z)}dA(z)$, where $dA(z)$ is lebesgue area measure on $\mathbb{U}$ and $A^1(\mathbb{U})$ is the space of all analytic functions f on $\mathbb{U}$ such that $||f|| = \int_{\mathbb{U}}|f(z)|dA(z) < \infty$, see [4, Theorem 2.4].

Another space we dealt with in this paper is the Besov space $\mathscr{B}_p$ $(1 < p < \infty)$ which is defined to be the space of holomorphic functions f on $\mathbb{U}$ such that

$$\begin{aligned}\gamma_f^p &= \int_{\mathbb{U}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z)\\ &= \int_{\mathbb{U}} |f'(z)|^p(1-|z|^2)^p d\lambda(z) < \infty,\end{aligned}$$

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Paper OaM-16-49

where $d\lambda(z)$ is the Möbius invariant measure on U, with definition

$$d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

For $p=1$, the Besov space $\mathscr{B}_1$ consists of all holomorphic functions f on $\mathbb{U}$ whose second derivatives are integrable,

$$\mathscr{B}_1 = \{f \in H(\mathbb{U}) : ||f||_{\mathscr{B}_1} = \int_{\mathbb{U}} |f''(z)|dA(z) < \infty\}.$$

For $1<p<\infty$, $||f||_p = |f(0)| + \gamma_f$ is a norm on $\mathscr{B}_p$ which makes it a Banach space. $\mathscr{B}_p$ is reflexive space (while $\mathscr{B}_1$ is not) and polynomials are dense in it, [26, Theorem 5.24]. Furthermore, for each $1<q<p<\infty$, $\mathscr{B}_1 \subset \mathscr{B}_q \subset \mathscr{B}_p \subset \mathscr{B}$ and $\mathscr{B}_1$ is a subset of the little Bloch space $\mathscr{B}_0$ [28, Page 388]. Also remember that the Besov space $\mathscr{B}_2$ is known as the classical Dirichlet Space $\mathscr{D}$ and $\mathscr{B}_\infty$ is the Bloch space $\mathscr{B}$, see [26, Page 115]. Moreover, the following two lemmas determine that norm convergence implies pointwise convergence in the Bloch and Besov spaces $\mathscr{B}_p$ $(1<p<\infty)$. We state them here without proof.

LEMMA 1.1. *For all $f \in \mathscr{B}$ and for each $z \in \mathbb{U}$, we have*

$$|f(z)| \leqslant ||f||_{\mathscr{B}} \log \frac{2}{1-|z|^2}.$$

Proof. See page 82, (3.5) of [27]. □

LEMMA 1.2. *For each $f \in \mathscr{B}_p$ $(1<p<\infty)$ and for every $z \in \mathbb{U}$, there is $C \geqslant 0$ (depends only on p) such that*

$$|f(z)| \leqslant C||f||_p (\log \frac{2}{1-|z|^2})^{1-1/p}.$$

Proof. See Theorem 8 of [25]. □

Bloch and Besov spaces and their properties specially from an operator and geometric view were studied extensively in previous years in [6, 25, 27] and more recently in [14, 17].

If ψ is a holomorphic function on $\mathbb{U}$, the *multiplication operator* M_ψ on $H(\mathbb{U})$ is defined by

$$M_\psi(f) = \psi f.$$

We recall that for a set Ω, $H^\infty(\Omega) = \{f \in H(\Omega) : ||f||_\infty = \sup_{z\in\Omega} |f(z)| < \infty\}$. For every $z \in \Omega$ the linear operator $e_z : X \to \mathbb{C}$ which is defined by $e_z(f) = f(z)$ for $f \in X$ is called a point evaluation at z. A set X of complex-valued functions on a set Ω is called functional Banach space if each point evaluation is a bounded linear functional and there is no point in Ω that all functions in X vanish.

PROPOSITION 1.3. *Let X be a functional Banach space on the set Ω and suppose ψ is a complex-valued function on Ω such that $\psi X \subset X$. Then the operator M_ψ is a bounded operator on X and $|\psi(x)| \leqslant ||M_\psi||$ for all $x \in \Omega$. In particular, $\psi \in H^\infty(\Omega)$.*

Proof. See page 57, Lemma 1.1 of [16]. □

According to Lemma 1.1. and Lema 1.2. in all three spaces, $\mathscr{B}$, $\mathscr{B}_0$ and $\mathscr{B}_p$ every point evaluations are bounded linear functionals, in fact in $\mathscr{B}$, $\mathscr{B}_0$ we have $||e_z|| \leqslant \log \frac{2}{1-|z|^2}$ and in $\mathscr{B}_p$, $||e_z|| \leqslant C(\log \frac{2}{1-|z|^2})^{1-\frac{1}{p}}$, for all $z \in \mathbb{U}$. Since constant functions are in three spaces, the second property is also valid and they are functional Banach spaces so by above proposition in all of them we have the following inequality:

$$||\psi||_\infty \leqslant ||M_\psi||. \tag{1.1}$$

Investigating the boundedness or compactness and other properties of multiplication operators on the Bloch and Besov spaces have been done by many authors (see [6, 3, 28]). Arazy in [5] proved that the multiplication operator M_ψ is bounded on the Bloch space if and only if $\psi \in H^\infty(\mathbb{U})$ and $\sigma_\psi < \infty$ where

$$\sigma_\psi := \sup_{z \in \mathbb{U}} \frac{1}{2}(1-|z|^2)|\psi'(z)| \log \frac{1+|z|}{1-|z|}.$$

Also Brown and Shields in [12, Theorem 1], proved that M_ψ is bounded on Bloch space if and only if it is bounded on little Bloch space. In [3, Corollary 2.1], Allen and Colonna showed that if $\psi \in H(\mathbb{U})$ induces a bounded multiplication operator M_ψ on the Bloch space, then

$$\max\{||\psi||_{\mathscr{B}}, ||\psi||_\infty\} \leqslant ||M_\psi|| \leqslant \max\{||\psi||_{\mathscr{B}}, ||\psi||_\infty + \sigma_\psi\}. \tag{1.2}$$

In the case of Besov Space, the situation is somewhat different and the issue is not as straightforward as the Bloch Space. A function $\psi \in H(\mathbb{U})$ is said to be *multiplier* of $\mathscr{B}_p$ if $M_\psi(\mathscr{B}_p) \subseteq \mathscr{B}_p$. If the space of multipliers on $\mathscr{B}_p$ in to itself represented by $M(\mathscr{B}_p)$, then by Closed Graph theorem $\psi \in M(\mathscr{B}_p)$ if and only if M_ψ is a bounded operator on $\mathscr{B}_p$. Stegenga [22], Characterized multipliers of the Dirichlet space $\mathscr{D}$ in to itself. Characterization of multipliers of the Besov space $\mathscr{B}_p$ $(1 < p < \infty)$, based on capacities and Carleson measures type conditions was given by Wu [23] and Arcozzi [7]. Zorboska in [28, Corollary 3.2] proved that for $1 < p < \infty$ if $\psi \in M(\mathscr{B}_p)$, then $\psi \in H^\infty(\mathbb{U})$ and

$$\sup_{z \in \mathbb{U}} (1-|z|^2)|\psi'(z)| \left(\log \frac{2}{1-|z|^2}\right)^{1-1/p} < \infty.$$

Following proposition is another applied result of Zorboska about multiplication operators on the Besov spaces.

PROPOSITION 1.4. *Suppose that $1 < p < \infty$ and $\psi \in H^\infty(\mathbb{U})$.*

(i) If $\psi \in M(\mathscr{B}_p)$ *and* $0 < r < 1$*, then*

$$\sup_{\omega \in D} \int_{D(\omega,r)} (1-|z|^2)^{p-2} |\psi'(z)|^p \left(\log \frac{2}{1-|z|^2}\right)^{p-1} dA(z) < \infty,$$

where $D(\omega, r) = \{z \in \mathbb{U} : \beta(z,\omega) < r\}$ *is the hyperbolic disk with radius r,* $\beta(z,\omega) = \log \frac{1+|\psi_z(\omega)|}{1-|\psi_z(\omega)|}$ *and* $\psi_z(\omega) = \frac{z-\omega}{1-\bar{z}\omega}$ *for all* $z, \omega \in \mathbb{U}$.

(ii) If $\int_{\mathbb{U}} (1-|z|^2)^{p-2} |\psi'(z)|^p (\log \frac{2}{1-|z|^2})^{p-1} dA(z) < \infty$*, then* $\psi \in M(\mathscr{B}_p)$.

Proof. See Theorem 3.1 of [28]. □

Notice that Galanopoulos in [17, Theorem 1.3] has shown the existence of $\psi \in M(\mathscr{B}_p)$ such that $\int_{\mathbb{U}} (1-|z|^2)^{p-2} |\psi'(z)|^p (\log \frac{2}{1-|z|^2})^{p-1} dA(z) = \infty$, i.e. the reverse of (ii) in preceding proposition is not correct.

Let $L(X)$ be the space of all linear bounded operators from locally convex Hausdorff space X into itself and $T \in L(X)$, the Cesáro means of T is defined by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \; n \in \mathbb{N}.$$

An operator T is (uniformly) mean ergodic if $\{T_{[n]}\}_{n=0}^{\infty}$ is a convergent sequence in (norm) strong topology and is called *power bounded* if the sequence $\{T^n\}_{n=0}^{\infty}$ is bounded in $L(X)$.

It is easy to check that for all $n \in \mathbb{N}$, $\frac{1}{n} T^n = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$, where $T_{[0]} = I$ is the identity operator. From this we get if T is mean ergodic, then for all $x \in X$, $\lim_{n \to \infty} \frac{1}{n} T^n x = 0$ and in the uniform mean ergodic case, $\lim_{n \to \infty} \frac{1}{n} ||T^n|| = 0$. The study of mean ergodicity of linear operators on Banach spaces goes back to 1931, when Von Neumann proved that for a unitary operator T on a Hilbert space H, there is a projection P on H, such that $T_{[n]}$ converges to P in the strong operator topology. In 1939 Lorch demonstrated that for reflexive Banach spaces, power bounded operators are mean ergodic, [1, Page 401]. Dunford in 1943 stated the connection between the spectral properties of an operator and its uniform mean ergodicity, [15, Theorem 3.16]. There are a lot of references about dynamical properties of different linear bounded operators on Banach, Fréchet and locally convex spaces. One of the best, is a book written by Bayart and Matheron [9]. Additionally, [1, 2, 19] and the references therein give more details about mean ergodic and power bounded operators on locally convex spaces. Bonet and Ricker [11], characterized the mean ergodicity of multiplication operators in weighted spaces of holomorphic functions and recently Bonet, Jordá and Rodríguez-Arenas [10] extended the results to the weighted space of continuous functions. In this paper, we look for conditions under which the multiplication operator M_ψ is power bounded and its Cesáro means is convergent or uniformly convergent on the Bloch space $\mathscr{B}$, little Bloch space $\mathscr{B}_0$ and the Besov Space $\mathscr{B}_p$.

2. The Bloch space $\mathscr{B}$ and The little Bloch space $\mathscr{B}_0$

The following Lemma provides the necessary condition for multiplication operators to be power bounded, uniform mean ergodic and mean ergodic on functional Banach spaces and especially on $\mathscr{B}$ or $\mathscr{B}_0$.

LEMMA 2.1. *Let X be a functional Banach space and let ψ be a multiplier for X. Then $||\psi||_\infty \leqslant 1$ whenever $\{\frac{M_{\psi^n}}{n}\}_n$ is a bounded sequence. Consequently, $||\psi||_\infty \leqslant 1$ when M_ψ is power bounded or mean ergodic on X.*

Proof. Since in a functional Banach space for every $z \in \Omega$ and $n \in \mathbb{N}$, $|\frac{\psi^n(z)}{n}| \leqslant ||\frac{M_{\psi^n}}{n}||$ the result follows immediately.

In the mean ergodic case, for all $f \in X$, $||\frac{M_{\psi^n} f}{n}|| \to 0$, as $n \to \infty$, see page 5 of this paper. So by Uniform Boundedness Principle $\{\frac{M_{\psi^n}}{n}\}$ is a bounded sequence. In the power bounded case, there is nothing to prove. □

The following Lemma is proved in [12, Lemma 1].

LEMMA 2.2. *If $\{f_n\} \subseteq \mathscr{B}_0$ then $f_n \to 0$ weakly if and only if $f_n(z) \to 0$ for all $z \in \mathbb{U}$ and $\sup_n ||f_n||_{\mathscr{B}} < \infty$.*

REMARK 2.3. Consider that for $\psi \in H^\infty(\mathbb{U})$ Schwarz-Pick Lemma [13, Theorem 2.39] implies that $\beta_\psi \leqslant ||\psi||_\infty$. In fact if $\psi \not\equiv 0$, then for $\varphi = \frac{\psi}{||\psi||}$ we have:

$$\frac{\beta_\psi}{||\psi||_\infty} = \beta_\varphi = \sup_{z\in\mathbb{U}}(1-|z|^2)|\varphi'(z)| \leqslant \sup_{z\in\mathbb{U}}(1-|\varphi(z)|^2) \leqslant 1.$$

THEOREM 2.4. *Suppose ψ is a non-constant holomorphic function on $\mathbb{U}$ inducing a bounded multiplication operator M_ψ on $\mathscr{B}$ $(\mathscr{B}_0)$, if $||\psi||_\infty \leqslant 1$ then $\{M_{\psi^n}\}$ is convergent to zero in the weak operator topology on $\mathscr{B}_0$. Hence M_ψ is power bounded $\mathscr{B}$ $(\mathscr{B}_0)$ and mean ergodic on $\mathscr{B}_0$.*

Proof. We recall that $\mathscr{B}_0^{**} = \mathscr{B}$ [4, Theorem 2.4] and $(M_\psi|_{\mathscr{B}_0})^{**} = M_\psi|_{\mathscr{B}}$. (By $M_\psi|_{\mathscr{B}_0}$ and $M_\psi|_{\mathscr{B}}$ we mean the multiplication on $\mathscr{B}_0$ and $\mathscr{B}$, respectively). So $||(M_\psi|_{\mathscr{B}_0})|| = ||M_\psi|_{\mathscr{B}}||$ and clearly power boundedness of M_ψ on $\mathscr{B}$ implies it on $\mathscr{B}_0$ and vice versa. For all $z \in \mathbb{U}$ by Maximum Modules Principle, we have $|\psi(z)| < 1$. Fix $f \in \mathscr{B}_0$ and let $L : \mathscr{B}_0 \to \mathbb{C}$ be a bounded linear functional. There exists $g \in A^1(\mathbb{U})$ such that $L(M_{\psi^n} f) = \int_{\mathbb{U}} \psi^n(z) f(z)\overline{g(z)} dA(z)$. Clearly, for all $n \in \mathbb{N}$ the function $\psi^n(z) f(z)\overline{g(z)}$ is integrable and Since $|\psi^n(z)| < 1$, it converges pointwise to zero and by Lebesgue Convergence Theorem, $L(M_{\psi^n} f) \to 0$. By Lemma 2.2 and Uniform B oundedness Principle M_ψ is power bounded. Also by Yosida mean ergodic Theorem [1, Theorem 2.2] it is mean ergodic. □

PROPOSITION 2.5. *If $||\psi||_\infty < 1$ and M_ψ is a bounded multiplication operator on $\mathscr{B}(\mathscr{B}_0)$, then M_{ψ^n} is convergent to zero in operator norm on $\mathscr{B}(\mathscr{B}_0)$.*

Proof. Consider inequality (1.2). If for $n\in\mathbb{N}$, $||M_{\psi^n}||\leqslant||\psi^n||_{\mathscr{B}}$, then

$$\begin{aligned}||M_{\psi^n}||\leqslant||\psi^n||_{\mathscr{B}}&=|\psi^n(0)|+\beta_{\psi^n}\\&\leqslant|\psi^n(0)|+||\psi^n||_\infty\\&=|\psi^n(0)|+||\psi||_\infty^n,\end{aligned}\tag{2.1}$$

and if $n\in\mathbb{N}$, $||M_{\psi^n}||\leqslant||\psi^n||_\infty+\sigma_{\psi^n}$, then

$$\begin{aligned}||M_{\psi^n}||&\leqslant||\psi^n||_\infty+\frac{n}{2}\sup_{z\in\mathbb{U}}(1-|z|^2)|\psi'(z)||\psi(z)|^{n-1}\log\frac{1+|z|}{1-|z|}\\&\leqslant||\psi^n||_\infty+n||\psi||_\infty^{n-1}\left(\frac{1}{2}\sup_{z\in\mathbb{U}}(1-|z|^2)|\psi'(z)|\log\frac{1+|z|}{1-|z|}\right)\\&=||\psi||_\infty^n+n||\psi||_\infty^{n-1}\sigma_\psi.\end{aligned}\tag{2.2}$$

So by (2.1) and (2.2) for all $n\in\mathbb{N}$, $||M_\psi^n||\leqslant\max\{|\psi^n(0)|+||\psi||_\infty^n,||\psi||_\infty^n+n||\psi||_\infty^{n-1}\sigma_\psi\}$. Both sequences in the right side are convergent to zero whenever $||\psi||_\infty<1$. □

THEOREM 2.6. *Suppose that M_ψ is a bounded operator on $\mathscr{B}(\mathscr{B}_0)$. If $\psi\equiv\xi$ where $\xi\in\partial\mathbb{U}$, then M_ψ is uniformly mean ergodic (and hence mean ergodic) on $\mathscr{B}(\mathscr{B}_0)$.*

Proof. Suppose $\psi\equiv\xi$, $|\xi|=1$. If $\xi=1$, then for all $n\in\mathbb{N}$ and $f\in\mathscr{B}$, $(M_\psi)_{[n]}f=f$ and clearly M_ψ is uniformly mean ergodic on $\mathscr{B}(\mathscr{B}_0)$ and if $\xi\neq1$, then $M_{\psi^n}f=\psi^nf=\xi^nf$ and $(M_\psi)_{[n]}f=\frac{\xi+\xi^2+\ldots+\xi^n}{n}f=\frac{f}{n}\frac{\xi(1-\xi^{n+1})}{1-\xi}$.

In this case for $f\in\mathscr{B}(\mathscr{B}_0)$ with $||f||_{\mathscr{B}}\leqslant1$, we have

$$\begin{aligned}||(M_\psi)_{[n]}f||_{\mathscr{B}}&=\left|\frac{f(0)}{n}\frac{\xi(1-\xi^{n+1})}{1-\xi}\right|+\frac{1}{n}\sup_{z\in\mathbb{U}}(1-|z|^2)|f'(z)|\left|\frac{1-\xi^{n+1}}{1-\xi}\right|\\&\leqslant\frac{2}{n|1-\xi|}+||f||_{\mathscr{B}}\frac{2}{n|1-\xi|}\leqslant\frac{4}{n|1-\xi|},\end{aligned}$$

so $||(M_\psi)_{[n]}||\to0$ when $n\to\infty$ and M_ψ is uniformly mean ergodic on $\mathscr{B}(\mathscr{B}_0)$. □

LEMMA 2.7. *Let M_ψ be a bounded operator on $\mathscr{B}$ $(\mathscr{B}_0)$, then $I-M_\psi$ is an isomorphism of $\mathscr{B}$ $(\mathscr{B}_0)$ if and only if $\frac{1}{1-\psi}\in H^\infty(\mathbb{U})$.*

Proof. If $I-M_\psi$ is invertible, then clearly, $(I-M_\psi)^{-1}=(M_{1-\psi})^{-1}=M_{\frac{1}{1-\psi}}$, so by [6] we must have $\frac{1}{1-\psi}\in H^\infty(\mathbb{U})$. Conversely, suppose $\frac{1}{1-\psi}\in H^\infty(\mathbb{U})$.

$$\sigma_{\frac{1}{1-\psi}}=\sup_{z\in\mathbb{U}}\frac{1}{2}(1-|z|^2)\frac{|\psi'(z)|}{|1-\psi(z)|^2}\log\frac{1+|z|}{1-|z|}\leqslant\left|\left|\frac{1}{1-\psi}\right|\right|_\infty\sigma_\psi<\infty,$$

so by [6], $M_{\frac{1}{1-\psi}}$ is bounded on $\mathscr{B}$ $(\mathscr{B}_0)$, i. e. $I-M_\psi$ is invertible. □

PROPOSITION 2.8. *Let* ψ *be a non constant analytic function on* $\mathbb{U}$ *with* $||\psi||_\infty = 1$. *If* M_ψ *is bounded on* $\mathscr{B}_0$ *then it is mean ergodic. Moreover, it is uniformly mean ergodic on* $\mathscr{B}_0$ *if and only if* $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.

Proof. Fix $f \in \mathscr{B}_0$. By Theorem 2.4, M_ψ is power bounded. Let $M = \sup_{n\in\mathbb{N}} ||M_{\psi^n}||$. Then $||(M_\psi)_{[n]}|| \leqslant M$ and also $\lim_{n\to\infty} \frac{1}{n} M_{\psi^n} f = 0$. Maximum Modules Principle implies that for all $z \in \mathbb{U}$, $|\psi(z)| < 1$, so $\{(M_\psi)_{[n]}\}_n$ is a bounded sequence that converges pointwise to zero. We show it converges weakly to zero. Let $L \in \mathscr{B}_0^*$, fore some $g \in A^1(\mathbb{U})$, $L((M_\psi)_{[n]} f) = \int_{\mathbb{U}} (M_\psi)_{[n]} f(z)\overline{g(z)}dA(z)$. By Lebesgue Convergence Theorem we can deduce that $L((M_\psi)_{[n]} f) \to 0$. The proof of proposition is completed by using [19, Theorem 1.1].

Suppose M_ψ is uniformly mean ergodic. Via the above proof we must have $||(M_\psi)_{[n]}|| \to 0$. One can easily see that $ker(I - M_\psi) = 0$ and since M_ψ is power bounded, $\frac{1}{n}||M_{\psi^n}|| \to 0$. So by proposition 2.16 of [1], M_ψ is uniformly mean ergodic if and only if $I - M_\psi = M_{1-\psi}$ is an isomorphism of $\mathscr{B}_0$, which is equivalent to $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$. □

PROPOSITION 2.9. *Suppose* $\psi \in H(\mathbb{U})$ *is non constant and* $||\psi||_\infty = 1$ *and* M_ψ *is a bounded operator on* $\mathscr{B}$, *then* M_ψ *is mean ergodic if and only if it is uniformly mean ergodic if and only if* $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.

Proof. As we said before, $||(M_\psi)_{[n]}|_{\mathscr{B}}|| = ||(M_\psi)_{[n]}|_{\mathscr{B}_0}||$, so by above proposition M_ψ is uniformly mean ergodic if and only if $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$. On the other hand Bloch space is a Grothendieck Banach space which satisfies the Dunford Pettis property (GDP space) which Lotz in [21, Theorem 5] proved that mean ergodicity and uniform mean ergodicity are equivalent in these spaces. □

The forthcoming example is a direct consequence of previous propositions.

EXAMPLE 2.10. M_z is power bounded operator on both $\mathscr{B}$ and $\mathscr{B}_0$. But it is not uniformly mean ergodic nor is it mean ergodic on $\mathscr{B}$. It is mean ergodic on $\mathscr{B}_0$, but it is not uniformly mean ergodic. These statement are also true for M_ψ, where ψ is an automorphism of the unit disk.

The following theorems are direct consequences of this sections:

THEOREM 2.11. *Let* ψ *be a non-constant analytic function on* $\mathbb{U}$ *and* M_ψ *be a bounded multiplication operator on* $\mathscr{B}_0$, *then the following are equivalent:*

1. $||\psi||_\infty \leqslant 1$.
2. M_ψ *is convergent to zero in the weak operator topology.*
3. M_ψ *is power bounded.*

4. M_ψ *is mean ergodic.*

THEOREM 2.12. *Let* M_ψ *be a bounded multiplication operator on* $\mathscr{B}$ $(\mathscr{B}_0)$, *then the following are equivalent:*

1. $||\psi||_\infty < 1$.
2. $\{M_{\psi^n}\}$ *converges to zero in operator norm.*

THEOREM 2.13. *Let* M_ψ *be a bounded multiplication operator on* $\mathscr{B}$, *then the following are equivalent:*

1. $||\psi||_\infty \leqslant 1$ *and either,* $\psi \equiv \xi$, *where* $\xi \in \partial\mathbb{U}$ *or* $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.
2. M_ψ *is mean ergodic.*
3. M_ψ *is uniformly mean ergodic.*

THEOREM 2.14. *Let* M_ψ *be a bounded multiplication operator on* $\mathscr{B}_0$, *then the following are equivalent:*

1. $||\psi||_\infty \leqslant 1$ *and either,* $\psi \equiv \xi$, *where* $\xi \in \partial\mathbb{U}$ *or* $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.
2. M_ψ *is uniformly mean ergodic.*

3. Besov Space $\mathscr{B}_p$ $(1 < p < \infty)$

Before starting this section, it is necessary to remind that a Banach space X is said to be *mean ergodic* if each power bounded operator is mean ergodic. Lorch by extending the result of Rizes, showing that L_p spaces are mean ergodic, proved that the reflexive spaces are also mean ergodic, see [1, Pages 401 and 402]. According to the introduction, for $1 < p < \infty$ Besov Spaces $\mathscr{B}_p$ are reflexive spaces and therefore power boundedness of an operator implies mean ergodicity. In this section we only consider the case $1 < p < \infty$.

THEOREM 3.1. *Suppose* $\psi \in H(\mathbb{U})$ *and* M_ψ *is a bounded operator on Besov space* $\mathscr{B}_p$. *If* $\{\frac{M_{\psi^n}}{n}\}$ *is a bounded sequence, then* $||\psi||_\infty \leqslant 1$. *So* $||\psi||_\infty \leqslant 1$ *whenever* M_ψ *is power bounded, mean ergodic or uniformly mean ergodic operator on* $\mathscr{B}_p$.

Proof. Since $\mathscr{B}_p$ is a functional Banach space, this theorem is a direct consequence of Lemma 2.1. $\square$

From now on, we assume that analytic function ψ holds in the following condition:

$$\int_{\mathbb{U}} (1-|z|^2)^{p-2} |\psi'(z)|^p \left(\log \frac{2}{1-|z|^2}\right)^{p-1} dA(z) < \infty. \tag{3.1}$$

Let B be an infinite Blaschke product whose zeros belong to a stolz angel with vertex at 1, then For all $\alpha \geqslant 2$, $\psi(z) = (1-z)^{\alpha}B(z)$ satisfies condition (3.1). For $M > 1$ the stolz angle with vertex 1 is defined as the points of the unit disk satisfying $|z-1| < M(1-|z|)$, see Theorem 7.1 of [14]. Also by the Theorem 7.3 of previous reference condition (3.1) is true for the function $\psi(z) = (1-z)exp\left(\frac{z+1}{z-1}\right)$.

THEOREM 3.2. *Suppose that* $\psi \in H(\mathbb{U})$ *and condition* (3.1) *is met.* M_ψ *is a bounded operator on the Besov space* $\mathscr{B}_p$. *If* M_ψ *is non-constant and* $||\psi||_\infty \leqslant 1$, *then* $\{M_{\psi^n}\}_n$ *is convergent to zero pointwise on* $\mathscr{B}_p$ *and* M_ψ *is power bounded. Also if* $||\psi||_\infty < 1$, *then* $\{M_{\psi^n}\}_n$ *converges to zero in operator norm, hence* M_ψ *is uniformly mean ergodic.*

Proof. We have $|\psi(z)| < 1$ for all $z \in \mathbb{U}$. Let $f \in \mathscr{B}_p$ with $||f||_p \leqslant 1$. In this case, the followings can be deduced;

1. $|\psi^n(0)f(0)| \to 0$ when $n \to \infty$, since $|\psi(0)| < 1$.
2. $\int_{\mathbb{U}} |f'(z)|^p |\psi^n(z)|^p (1-|z|^2)^{p-2} dA(z) \to 0$, as $n \to \infty$, since $|f'(z)\psi^n(z)|^p(1-|z|^2)^{p-2} \leqslant |f'(z)^p(1-|z|^2)^{p-2}$, and $f \in \mathscr{B}_p$ gives us that
$$\int_{\mathbb{U}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty,$$
then $|f'(z)\psi^n(z)|^p(1-|z|^2)^{p-2}$ is integrable for all $n \in \mathbb{N}$. By using Lebesgue Convergence theorem the result is obtained.
3. $\int_{\mathbb{U}} |n\psi'(z)\psi^{n-1}(z)f(z)|^p(1-|z|^2)^{p-2}dA(z) \to 0$, since by lemma (1.2):
$$\begin{aligned} &|n\psi'(z)\psi^{n-1}(z)f(z)|^p(1-|z|^2)^{p-2} \\ &\leqslant n^p C^p ||f||_p^p (1-|z|^2)^{p-2} |\psi'(z)|^p \left(\log \frac{2}{1-|z|^2}\right)^{p-1}, \end{aligned}$$
by hypothesis the right side of the last inequality is integrable for all $n \in \mathbb{N}$ and so is $|n\psi'(z)\psi^{n-1}(z)f(z)|^p(1-|z|^2)^{p-2}$. Lebesgue Converges theorem gives the desired result.

Consider that in the case $||\psi||_\infty < 1$ the three above limits are independent from f so $||M_{\psi^n}|| \to 0$ when $n \to \infty$. $\square$

COROLLARY 3.3. *Suppose that* $\psi \in H(\mathbb{U})$ *and condition* (3.1) *is met.* M_ψ *is a bounded operator on the Besov space* $\mathscr{B}_p$ *the following statements are equivalent:*

(i) $||\psi||_\infty < 1$.

(ii) $||M_{\psi^n}|| \to 0$ *when* $n \to \infty$.

THEOREM 3.4. *Suppose that* $\psi \in H(\mathbb{U})$ *and condition* (3.1) *is met. If* M_ψ *is a bounded operator on the Besov space* $\mathscr{B}_p$, *then the following statements are equivalent.*

(i) $||\psi||_\infty \leqslant 1$.

(ii) M_ψ *is power bounded.*

(iii) M_ψ *is mean ergodic.*

Proof. According to the initial interpretations of the section, it is sufficient to show that (i) and (ii) are equivalent. Also by previous theorem we only consider the case $\psi(z) = \lambda$ where $|\lambda| = 1$. In this case $\psi' \equiv 0$. So for $f \in \mathscr{B}_p$ and $||f||_p = 1$ we have

$$\begin{aligned} ||M_{\psi^n} f||_p &= |\psi^n(0)f(0)| + \gamma_{\psi^n f} \\ &\leqslant |f(0)| + \gamma_f = ||f||_p, \end{aligned}$$

and M_ψ is power bounded on $\mathscr{B}_p$, in fact $||M_{\psi^n}|| \leqslant 1$, for all $n \in \mathbb{N}$. □

Recall that by $\sigma(T)$ (spectrum of T) we mean the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

LEMMA 3.5. *Suppose* $\psi \in H(\mathbb{U})$ *which satisfies condition* (3.1) *and* M_ψ *is a bounded operator on the Besov space* $\mathscr{B}_p$*, then* $\overline{\psi(\mathbb{U})} = \sigma(M_\psi)$*, (* $\overline{\psi(\mathbb{U})}$ *means the norm closure of* $\psi(\mathbb{U})$*).*

Proof. First since $M_\psi - \lambda I = M_{\psi-\lambda}$, then $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi-\lambda}$ is not invertible. If $M_{\psi-\lambda}$ is invertible, then $(M_{\psi-\lambda})^{-1} = M_{(\psi-\lambda)^{-1}} = M_{\frac{1}{\psi-\lambda}}$. So if $\lambda \in \psi(\mathbb{U})$ then there exists $z_0 \in \mathbb{U}$ such that $\psi(z_0) = \lambda$ therefore $\frac{1}{\psi-\lambda} \notin H^\infty(\mathbb{U})$ and $M_{\psi-\lambda}$ is not invertible that means $\lambda \in \sigma(M_\psi)$ and $\psi(\mathbb{U}) \subseteq \sigma(M_\psi)$. But $\sigma(M_\psi)$ is closed so $\overline{\psi(\mathbb{U})} \subseteq \sigma(M_\psi)$. Now assume that (3.1) holds and $\lambda \notin \overline{\psi(\mathbb{U})}$, hence $\frac{1}{\psi(z)-\lambda} \in H^\infty(\mathbb{U})$. By (3.1)

$$\int_{\mathbb{U}} \frac{|\psi'(z)|^p}{|\psi(z)-\lambda|^{2p}} \log \frac{2}{1-|z|^2})^{p-1}(1-|z|^2)^{p-2} dA(z) < \infty.$$

Thus by proposition (1.4), $M_{\frac{1}{\psi-\lambda}}$ is bounded on $\mathscr{B}_p$ and $M_{\psi-\lambda}$ is invertible which means $\lambda \notin \sigma(M_\psi)$. □

Dunford in [15, Theorem 3.16] stated the connection between the spectral properties of an operator and its uniform mean ergodicity. The following Theorem represents Lin and Dunford Theorems together.

THEOREM 3.6. *If an operator* T *on a Banach space* X *is uniformly mean ergodic, if and only if both* $(||T^n||/n)_n$ *converges to* 0 *and either* $1 \in \mathbb{C} \setminus \sigma(T)$ *or* 1 *is a pole of order* 1 *of the resolvent* $R_T : \mathbb{C} \setminus \sigma(T) \to L(X), R_T(\lambda) = (T - \lambda I)^{-1}$*. Consequently if* 1 *is an accumulation of* $\sigma(T)$*, then* T *is not uniformly mean ergodic.*

Proof. See [15, Theorem 3.16] and [20, Theorem 2.7]. □

THEOREM 3.7. *Suppose* $\psi \in H(\mathbb{U})$ *which holds* (3.1) *and* M_ψ *is a bounded operator on the Besov space* $\mathscr{B}_p$, *then* M_ψ *is uniformly mean ergodic on* $\mathscr{B}_p$ *if and only if* $||\psi||_\infty \leqslant 1$ *and either* $\psi \equiv \xi$ *for some* $\xi \in \partial\mathbb{U}$ *or* $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.

Proof. Let $||\psi||_\infty \leqslant 1$. Consider that $(M_\psi)_{[n]} f(z) = \frac{f(z)}{n}\sum_{m=1}^{n}(\psi(z))^n$. So if $\psi \equiv 1$, we can easily see $||(M_\psi)_{[n]} - I|| \to 0$ when $n \to \infty$, where I is the identity operator on $\mathscr{B}_p$. In the case $\psi \equiv \xi$, where $\xi \neq 1$, we have $(M_\psi)_{[n]} = \frac{\xi+\xi^2+\ldots+\xi^n}{n} f = \frac{f}{n}\frac{\xi(1-\xi^{n+1})}{1-\xi}$ and clearly $||(M_\psi)_{[n]}|| \to 0$. If $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$, an apply of propositon (1.4) shows that the function $\frac{1}{1-\psi} \in M(\mathscr{B}_p)$ and $M_{\frac{1}{1-\psi}}$ is bounded on $\mathscr{B}_p$, it means that $1 \notin \sigma(M_\psi)$ and since M_ψ is power bounded, Dunford-Lin Theorem guaranties the uniform mean ergodicity of M_ψ on $\mathscr{B}_p$.

Conversely; assume that M_ψ is uniformly mean ergodic on $\mathscr{B}_p$. So by Theorems (3.1) and (3.2) it is power bounded and $||\psi||_\infty \leqslant 1$. Suppose ψ is not uni- modular constant function, so $|\psi(z)| < 1$ for all $z \in \mathbb{U}$, this get us that $||(M_\psi)_{[n]}|| \to 0$ as $n \to \infty$. Also $ker(I - M_\psi) = 0$ and power boundedness of M_ψ implies that $\lim_{n\to\infty} \frac{1}{n}||M_\psi^n|| = 0$, Proposition 2.16 of [1] confirms that $I - M_\psi$ is an isomorphism on $\mathscr{B}_p$, i.e. $M_{1-\psi}^{-1} = M_{\frac{1}{1-\psi}}$ is bounded on $\mathscr{B}_p$, thus $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$. □

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