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# ON THE SOLVABILITY OF GENERALIZED SYLVESTER OPERATOR EQUATIONS

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*Abstract.* In this paper, some necessary and sufficient solvability conditions are established for the generalized Sylvester operator equations $AXB-CXD=E$, $AXB-CYD=E$ and $AX+YB+CZ=E$ on Hilbert spaces, respectively. Moreover, we give a solvability condition for the $*$-Sylvester operator equation $AX-X^*B=C$, which holds for finite matrices due to Wimmer (1994).

## 1. Introduction

The Sylvester matrix equation has broad applications in many fields such as neural network, feed back control and robust control, which has attracted the interest of many authors. Some methods have been developed to investigate solvability of the Sylvester equation and generalized Sylvester equation, including the rank, generalized inverse, equivalence and generalized singular-value decomposition of matrices [1, 2, 4, 10, 11, 20].

In 1952, Roth [15] proved that the Sylvester matrix equations $AX-XB=C$ and $AX-YB=C$ have solutions if and only if $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are, respectively, similar and equivalent, which are called Roth theorem. It has been extended to the generalized Sylvester matrix equation and systems of matrix equations. For example, Wimmer [22] obtained that the matrix equation $X-AXB=C$ has a solution if and only if $\left(\begin{pmatrix} A & C \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ and $\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ are simultaneously equivalent. In 1994, Wimmer [23] extended Roth theorem to a pair of generalized Sylvester equations $A_iX-YB_i=C_i$ $(i=1,2)$, and the system has a simultaneous solution if and only if $\begin{pmatrix} A_i & C_i \\ 0 & B_i \end{pmatrix}$ and $\begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix}$ are simultaneously equivalent. In 2012, by using equivalence relations of the block matrices, Lee and Vu [10] investigated some solvability conditions for systems of matrix equations $A_iX-XB_i=C_i$, $A_iX-YB_i=C_i$, and

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$A_iX - DXB_i = C_i$ $(i \in \mathscr{I})$, respectively. In 2015, Dmytryshyns [8] obtained Roth-type theorem for systems of matrix equations including Sylvester and $*$-Sylvester equations, which includes most of present results. Moreover, a necessary and sufficient solvability condition is also given in the paper for the systems of matrix equations $A_iX_kK_i - L_iX_jB_i = C_i, F_{i'}X_{k'}M_{i'} - N_{i'}X^*_{j'}G_{i'} = H_{i'}$. Finally, these results were extended to the systems of complex matrix equations by Dmytryshyn [9] in 2017.

In [12, 14], counter-examples are given to show that Roth theorem does not hold in the infinite dimensional space. However, Rosenblum [14] proved that Roth theorem is valid when $A$ and $B$ are selfadjoint on Hilbert spaces. In 1982, Schweinsberg [16] extended the result to normal operators and finite rank operators on Hilbert spaces. In 1986, Tong [17] gave some necessary and sufficient solvability conditions of the operator equation $AXB - X = C$ for normal operators and finite rank operators, and these conditions are different from Roth theorem. One of the results is that the operator equation $AXB - X = C$ has a solution if and only if there exists an invertible operator $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ such that

$$\begin{pmatrix} A & C \\ 0 & I \end{pmatrix}\begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

for normal operators $A$ and $B$ on Hilbert spaces. For some more recent references which study Sylvester equation, see [6, 7].

Motivated by Roth theorem and the paper [17], we investigate the generalized Sylvester operator equations $AXB - CXD = E$, $AXB - CYD = E$ and $AX + YB + CZ = E$, respectively. In this paper, we first give some necessary and sufficient solvability conditions for the operator equation $AXB - CXD = E$. Then, the solvability conditions are also presented for the operator equations $AXB - CYD = E$ and $AX + YB + CZ = E$ by using equivalence relations of block operator matrices, respectively. In 1994, Wimmer [24] gave a necessary and sufficient solvability condition for the $*$-Sylvester equation $AX - X^*B = C$ for finite matrices. Here, the result is extended to normal operators on Hilbert spaces. On the study of the $*$-Sylvester equation, we refer the reader to [3, 5, 18, 19, 22] and their references.

Let $\mathscr{H}$ be a Hilbert space. The set $\mathscr{B}(\mathscr{H})$ consists of all bounded linear operators on $\mathscr{H}$, $A^*$ is the adjoint of $A \in \mathscr{B}(\mathscr{H})$, and $I$ is the identity operator on $\mathscr{H}$.

We collect some lemmas, which are important in later proofs.

LEMMA 1.1. [13, Putnam-Fuglede theorem] *Let $A, B \in \mathscr{B}(\mathscr{H})$ be normal operators. If $AX = XB$ for $X \in \mathscr{B}(\mathscr{H})$, then $A^*X = XB^*$.*

An extension of the Putnam-Fuglede theorem is as follows.

LEMMA 1.2. [21] *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$ be normal operators, $A$ commutes with $C$ and $B$ commutes with $D$. If $AXB = CXD$ for $X \in \mathscr{B}(\mathscr{H})$, then $A^*XB^* = C^*XD^*$.*

LEMMA 1.3. [16] *Let $M,N,P,Q \in \mathscr{B}(\mathscr{H})$. If the operator matrix $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is invertible, then $PP^* + QQ^*$ and $M^*M + P^*P$ are invertible.*

## 2. Solvability of the operator equation $AXB - CXD = E$

In this section, we consider the solvability conditions of the generalized Sylvester operator equation $AXB - CXD = E$.

THEOREM 2.1. *Let $A,B,C,D \in \mathscr{B}(\mathscr{H})$ be normal operators and $E \in \mathscr{B}(\mathscr{H})$. Then the equation $AXB - CXD = E$ has a solution $X \in \mathscr{B}(\mathscr{H})$ if and only if there exist invertible operators $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ $(i = 1,2,3)$ with $M_1 = M_2$ and $P_1 = P_2 = P_3$ such that*

$$U_1 \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} U_2, \tag{2.1}$$

$$U_1 \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} U_3, \tag{2.2}$$

$$U_3 \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} U_2. \tag{2.3}$$

*Proof.* First, let $X \in \mathscr{B}(\mathscr{H})$ be a solution of $AXB - CXD = E$. Then the operators $U_1 = \begin{pmatrix} I & CX \\ 0 & I \end{pmatrix}$, $U_2 = \begin{pmatrix} I & XB \\ 0 & I \end{pmatrix}$ and $U_3 = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ are invertible and satisfy (2.1)–(2.3).

Conversely, assume that $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix}$ $(i = 1,2,3)$ are invertible operators such that (2.1)–(2.3) hold, where $M_1 = M_2$ and $P_1 = P_2 = P_3$. Then we have

$$M_1A = AM_2,\ M_1E + N_1D = AN_2,\ P_1A = DP_2,\ P_1E + Q_1D = DQ_2, \tag{2.4}$$

$$M_1C = CM_3,\ N_1 = CN_3,\ P_1C = P_3,\ Q_1 = Q_3, \tag{2.5}$$

$$M_3 = M_2,\ N_3B = N_2,\ P_3 = BP_2,\ Q_3B = BQ_2. \tag{2.6}$$

The Putnam-Fuglede theorem implies

$$M_1^*A = AM_1^*,\ \ M_1^*C = CM_1^*,\ \ P_1^*D = AP_1^*,\ \ P_1^* = CP_1^*,\ \ P_1^* = P_1^*B$$

by the first and third equalities in (2.4)–(2.6). So, from the second and fourth equalities in (2.4)–(2.6), it follows that

$$\begin{aligned} M_1^*M_1E &= M_1^*AN_2 - M_1^*N_1D \\ &= M_1^*AN_3B - M_1^*CN_3D \\ &= AM_1^*N_3B - CM_1^*N_3D \end{aligned}$$

and

$$\begin{aligned} P_1^*P_1E &= P_1^*DQ_2 - P_1^*Q_1D \\ &= AP_1^*Q_2 - CP_1^*Q_1D \\ &= AP_1^*BQ_2 - CP_1^*Q_1D \\ &= AP_1^*Q_1B - CP_1^*Q_1D. \end{aligned}$$

Conseqently,

$$(M_1^*M_1 + P_1^*P_1)E = A(M_1^*N_3 + P_1^*Q_1)B - C(M_1^*N_3 + P_1^*Q_1)D. \tag{2.7}$$

Note that

$$\begin{aligned} P_1^*P_1A &= P_1^*DP_2 = AP_1^*BP_2 = AP_1^*P_1, \\ P_1^*P_1C &= P_1^*P_1 = CP_1^*P_1, \\ M_1^*M_1A &= M_1^*AM_1 = AM_1^*M_1, \\ M_1^*M_1C &= M_1^*CM_1 = CM_1^*M_1. \end{aligned}$$

Obviously, $M_1^*M_1 + P_1^*P_1$ commutes with $A$ and $C$. In combination with the invertibility of $M_1^*M_1 + P_1^*P_1$ shown by Lemma 1.3, it is immediate that $(M_1^*M_1 + P_1^*P_1)^{-1}$ commutes with $A$ and $C$. Therefore, from (2.7), we arrive at

$$E = A(M_1^*M_1 + P_1^*P_1)^{-1}(M_1^*N_3 + P_1^*Q_1)B - C(M_1^*M_1 + P_1^*P_1)^{-1}(M_1^*N_3 + P_1^*Q_1)D,$$

which means that $X = (M_1^*M_1 + P_1^*P_1)^{-1}(M_1^*N_3 + P_1^*Q_1)$ is a solution of the equation $AXB - CXD = E$. $\square$

If, in addition, $B$ commutes with $D$ in Theorem 2.1, then the equation will allow more relaxed solvability conditions.

THEOREM 2.2. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$ be normal operators, $E \in \mathscr{B}(\mathscr{H})$ and $BD = DB$. Then the equation $AXB - CXD = E$ has a solution $X \in \mathscr{B}(\mathscr{H})$ if and only if there exist invertible operators $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ $(i = 1, 2, 3)$ with $M_1 = M_2$ and $P_1 = P_3$ such that (2.1)–(2.3) hold.*

*Proof.* We only need to note the following fact. From the third equalities in (2.6) and (2.4) and $BD = DB$, we infer that

$$DP_1 = DBP_2 = BDP_2 = BP_1A,$$

which shows that

$$P_1^*D = AP_1^*B$$

by Lemma 1.2. The rest of the proof proceeds similar to Theorem 2.1. $\square$

In Theorem 2.2, it is clear that $U_1 = U_3$ for $C = I$, which is collected as a corollary.

COROLLARY 2.1. *Let* $A,B,D \in \mathscr{B}(\mathscr{H})$ *be normal operators and* $BD = DB$. *Then the equation* $AXB - XD = E$ *has a solution* $X \in \mathscr{B}(\mathscr{H})$ *if and only if there exist invertible operators* $U_1, U_2 \in \mathscr{B}(\mathscr{H})$ *such that*

$$U_1 \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} U_2, \quad U_1 \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} U_2. \tag{2.8}$$

*Note that the condition (2.8) is equivalent to*

$$U_1 \begin{pmatrix} A - \lambda I & E \\ 0 & D - \lambda B \end{pmatrix} = \begin{pmatrix} A - \lambda I & 0 \\ 0 & D - \lambda B \end{pmatrix} U_2,$$

*i.e.,* $\begin{pmatrix} A - \lambda I & E \\ 0 & D - \lambda B \end{pmatrix}$ *and* $\begin{pmatrix} A - \lambda I & 0 \\ 0 & D - \lambda B \end{pmatrix}$ *are equivalent for any complex number* $\lambda$.

If $B = C = I$, then, from Theorem 2.1, the result in [16] is derived.

COROLLARY 2.2. [16] *Let* $A,B \in \mathscr{B}(\mathscr{H})$ *be normal operators. Then the equation* $AX - XD = E$ *has a solution* $X \in \mathscr{B}(\mathscr{H})$ *if and only if there exists an invertible operator* $U \in \mathscr{B}(\mathscr{H})$ *such that* $U \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} U$, *i.e.,* $\begin{pmatrix} A & E \\ 0 & D \end{pmatrix}$ *and* $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ *are similar.*

In the following, we give other necessary and sufficient solvability conditions of the equation $AXB - CXD = E$, which is different from Roth theorem.

THEOREM 2.3. *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$ *be normal operators and* $BD = DB$. *Then the equation* $AXB - CXD = E$ *has a solution* $X \in \mathscr{B}(\mathscr{H})$ *if and only if there exists an invertible operator* $\begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ *with* $BP = P$ *such that*

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} M & CN \\ P & Q \end{pmatrix} \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} M & N \\ P & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \tag{2.9}$$

$$\begin{pmatrix} M & CN \\ P & Q \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M & N \\ P & Q \end{pmatrix}. \tag{2.10}$$

*Proof.* Let $X \in \mathscr{B}(\mathscr{H})$ be a solution of $AXB - CXD = E$. Then the invertible operator $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ satisfies (2.9) and (2.10).

Conversely, assume that $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is an invertible operator such that (2.9) and (2.10) hold and $BP = P$. Then

$$MA = AM, \quad ME + CND = ANB, \quad BPA = DP, \tag{2.11}$$
$$BPE + BQD = DQB, \quad MC = CM, \quad PC = P.$$

By the third equality in (2.11) and the assumption $BD = DB$, we infer from Lemma 1.2 that

$$P^*D = AP^*B.$$

Similar to the proof of Theorem 2.1, we conclude

$$E = A(M^*M + P^*P)^{-1}(M^*N + P^*BQ)B - C(M^*M + P^*P)^{-1}(M^*N + P^*BQ)D.$$

The proof is completed. □

According to Theorem 2.3, the following result is obvious for $C = I$.

COROLLARY 2.3. *Let $A, B, D \in \mathscr{B}(\mathscr{H})$ be normal operators and $BD = DB$. Then the equation $AXB - XD = E$ has a solution $X \in \mathscr{B}(\mathscr{H})$ if and only if there exists an invertible operator $\begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ with $BP = P$ such that*

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}.$$

For the operator equation $AXB - XD = E$, we also have the following solvability conditions.

THEOREM 2.4. *Let $A, B, D \in \mathscr{B}(\mathscr{H})$ be normal operators and $BD = DB$. Then the equation $AXB - XD = E$ has a solution $X \in \mathscr{B}(\mathscr{H})$ if and only if there exists an invertible operator $\begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ with $BQ = QB$ and $DP = P$ such that*

$$\begin{pmatrix} A & E \\ 0 & D \end{pmatrix}\begin{pmatrix} M & NB \\ P & Q \end{pmatrix} = \begin{pmatrix} M & N \\ BP & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \tag{2.12}$$

*Proof.* Suppose that $X \in \mathscr{B}(\mathscr{H})$ is a solution of $AXB - XD = E$. Then the invertible operator $\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$ satisfies (2.12).

Conversely, assume that $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is an invertible operator such that (2.12) holds, where $BQ = QB$ and $DP = P$. Then

$$AM + EP = MA, \quad ANB + EQ = ND, \quad DP = BPA, \quad DQ = QD.$$

Similar to the proof of Theorem 2.1, it turns out that

$$E = -A(NQ^* + MAP^*)(PP^* + QQ^*)^{-1}B - (NQ^* + MAP^*)(PP^* + QQ^*)^{-1}D,$$

as a result. □

In Theorem 2.4, when $D = I$, we obtain the result in [17].

COROLLARY 2.4. [17] *Let* $A,B \in \mathscr{B}(\mathscr{H})$ *be normal operators. Then the equation* $AXB - X = E$ *has a solution* $X \in \mathscr{B}(\mathscr{H})$ *if and only if there exists an invertible operator* $\begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ *with* $BQ = QB$ *such that*

$$\begin{pmatrix} A & E \\ 0 & I \end{pmatrix} \begin{pmatrix} M & NB \\ P & Q \end{pmatrix} = \begin{pmatrix} M & N \\ BP & Q \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Similar to Theorem 2.4, we have

THEOREM 2.5. *Let* $A,B,D \in \mathscr{B}(\mathscr{H})$ *be normal operators and* $BD = DB$. *Then the equation* $AXB - XD = E$ *has a solution* $X \in \mathscr{B}(\mathscr{H})$ *if and only if there exists an invertible operator* $\begin{pmatrix} M & N \\ P & Q \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ *with* $BQ = QB$ *and* $BP = P$ *such that*

$$\begin{pmatrix} M & N \\ BP & Q \end{pmatrix} \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} M & BN \\ P & Q \end{pmatrix}.$$

## 3. Solvability of the operator equation $AXB - CYD = E$

This section is devoted to the solvability conditions of the operator equations $AXB - CYD = E$ and $AX + YB + CZ = E$.

THEOREM 3.1. *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$ *be normal operators and* $E \in \mathscr{B}(\mathscr{H})$. *Then the equation* $AXB - CYD = E$ *has a solution* $(X,Y)$ *if and only if there exist invertible operators* $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ $(i = 1,2,3,4)$ *with* $M_1 = M_2 = M_3$ *and* $P_1 = P_2 = P_3 = P_4$ *such that*

$$U_1 \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} U_2, \tag{3.1}$$

$$U_1 \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} U_3, \tag{3.2}$$

$$U_4 \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} U_2, \tag{3.3}$$

*Proof.* Similar to the proof of Theorem 2.1, we notice that (2.6) becomes

$$M_4 = M_2, \;\; N_4 B = N_2, \;\; P_4 = BP_2, \;\; Q_4 B = BQ_2. \tag{3.4}$$

Then

$$E = A(M_1^* M_1 + P_1^* P_1)^{-1}(M_1^* N_4 + P_1^* Q_4)B - C(M_1^* M_1 + P_1^* P_1)^{-1}(M_1^* N_3 + P_1^* Q_1)D,$$

So, $E = AXB - CYD$ for $X = (M_1^* M_1 + P_1^* P_1)^{-1}(M_1^* N_4 + P_1^* Q_4)$ and $Y = (M_1^* M_1 + P_1^* P_1)^{-1}(M_1^* N_3 + P_1^* Q_1)$, which completes the proof. □

In particular, we have the result as follows for $B = C = I$ in Theorem 3.1.

COROLLARY 3.1. *Let $A,D \in \mathscr{B}(\mathscr{H})$ be normal operators and $E \in \mathscr{B}(\mathscr{H})$. Then the equation $AX - YD = E$ has a solution $(X,Y)$ if and only if there exist invertible operators $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ $(i = 1,2)$ with $M_1 = M_2$ and $P_1 = P_2$ such that*

$$U_1 \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} U_2.$$

According to the proof of Theorem 3.1, we can conclude the corresponding result on the system of operator equations.

THEOREM 3.2. *Let $A_k, B_k, C_k, D_k \in \mathscr{B}(\mathscr{H})$ be normal operators and $E_k \in \mathscr{B}(\mathscr{H})$, $k = 1,2,\cdots,n$. Then the system of equations $A_kXB_k - C_kYD_k = E_k$ has a solution $(X,Y)$ if and only if there exist invertible operators $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ $(i = 1,2,3,4)$ with $M_1 = M_2 = M_3$ and $P_1 = P_2 = P_3 = P_4$ such that*

$$U_1 \begin{pmatrix} A_k & E_k \\ 0 & D_k \end{pmatrix} = \begin{pmatrix} A_k & 0 \\ 0 & D_k \end{pmatrix} U_2,$$
$$U_1 \begin{pmatrix} C_k & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C_k & 0 \\ 0 & I \end{pmatrix} U_3,$$
$$U_4 \begin{pmatrix} I & 0 \\ 0 & B_k \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B_k \end{pmatrix} U_2.$$

We now consider solvability of the equation $AX + YB + CZ = E$. We first give a lemma similar to Lemma 1.3.

LEMMA 3.1. *If the operator $\mathscr{A} = \begin{bmatrix} M & N & R \\ P & Q & S \\ K & L & T \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H})$ is invertible, then $PP^* + QQ^* + SS^*$ is invertible on $\mathscr{H}$.*

*Proof.* Suppose that $PP^* + QQ^* + SS^*$ is not invertible. Since $PP^* + QQ^* + SS^*$ is self-adjoint, we have

$$\sigma(PP^* + QQ^* + SS^*) = \sigma_a(PP^* + QQ^* + SS^*).$$

Then there exists a sequence $\{x_n\} \subset H$ such that

$$\|x_n\| = 1 \text{ and } \lim_{n \to +\infty} \|(PP^* + QQ^* + SS^*)x_n\| = 0.$$

Thus

$$\begin{aligned}
&\lim_{n \to +\infty} \|\mathscr{A}(0 \oplus x_n \oplus 0)\|^2 \\
&= \lim_{n \to +\infty} \langle \mathscr{A}\mathscr{A}^*(0 \oplus x_n \oplus 0), (0 \oplus x_n \oplus 0) \rangle \\
&= \lim_{n \to +\infty} \langle (PP^* + QQ^* + SS^*)x_n, x_n \rangle \\
&= 0,
\end{aligned}$$

which shows that $\mathscr{A}$ is not invertible, a contradiction. □

THEOREM 3.3. *Let $A,B,C \in \mathscr{B}(H)$ be normal operators and $E \in \mathscr{B}(\mathscr{H})$. Then the operator equation*

$$AX + YB + CZ = E \tag{3.5}$$

*has a solution $(X,Y,Z)$ if and only if there exist invertible operators $U_i = \begin{bmatrix} M_i & N_i & R_i \\ P_i & Q_i & S_i \\ K_i & L_i & T_i \end{bmatrix}$ $(i = 1,2)$ with $P_1 = P_2 = S_2$ and $Q_1 = Q_2$ such that*

$$\mathrm{U}_1 \begin{bmatrix} A & 0 & C \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A & E & C \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{U}_2. \tag{3.6}$$

*Proof.* Let $(X,Y,Z)$ be a solution of (3.5). Then the invertible operators $U_1 = \begin{bmatrix} I & Y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and $U_2 = \begin{bmatrix} I & -X & 0 \\ 0 & I & 0 \\ 0 & -Z & I \end{bmatrix}$ satisfy (3.6).

Conversely, assume that (3.6) holds, where $U_1$ and $U_2$ are invertible with $P_1 = P_2 = S_2$ and $Q_1 = Q_2$. Then

$$\begin{aligned} &M_1A = AM_2 + EP_2 + CK_2, \\ &N_1B = AN_2 + EQ_2 + CL_2, \\ &M_1C = AR_2 + ES_2 + CT_2, \\ &P_2A = BP_2, \;\; Q_2B = BQ_2, \;\; S_2C = BS_2. \end{aligned}$$

The Putnam-Fuglede theorem implies

$$AP_2^* = P_2^*B, \;\; BQ_2^* = Q_2^*B, \;\; S_2^*B = CS_2^*.$$

Thus

$$\begin{aligned} &E(P_2P_2^* + Q_2Q_2^* + S_2S_2^*) \\ =&(M_1A - AM_2 - CK_2)P_2^* + (N_1B - AN_2 - CL_2)Q_2^* + (M_1C - AR_2 - CT_2)S_2^* \\ =&M_1P_2^*B - AM_2P_2^* - CK_2P_2^* + N_1Q_2^*B - AN_2Q_2^* - CL_2Q_2^* + M_1S_2^*B \\ &- AR_2S_2^* - CT_2S_2^* \\ =&-A(M_2P_2^* + N_2Q_2^* + R_2S_2^*) + (M_1P_2^* + N_1Q_2^* + M_1S_2^*)B - C(K_2P_2^* \\ &+ L_2Q_2^* + T_2S_2^*). \end{aligned}$$

According to Lemma 3.1, $P_2P_2^* + Q_2Q_2^* + S_2S_2^*$ is invertible. Note that its inverse commutes with $B$. So, we arrive at that $AX + YB + CZ = E$ for

$$\begin{aligned} X &= -(M_2P_2^* + N_2Q_2^* + R_2S_2^*)(P_2P_2^* + Q_2Q_2^* + S_2S_2^*)^{-1}, \\ Y &= (M_1P_2^* + N_1Q_2^* + M_1S_2^*)(P_2P_2^* + Q_2Q_2^* + S_2S_2^*)^{-1}, \\ Z &= -(K_2P_2^* + L_2Q_2^* + T_2S_2^*)(P_2P_2^* + Q_2Q_2^* + S_2S_2^*)^{-1}. \quad \square \end{aligned}$$

## 4. Solvability of the $*$-Sylvester equation $AX - X^*B = C$

For the $*$-Sylvester matrix equation $AX - X^*B = C$, Wimmer [24] obtained that it has a solution if and only if there exist invertible matrices $U_1, U_2$ such that

$$U_2 \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} U_1 \text{ and } U_2^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} U_1 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

For the $*$-Sylvester operator equation $AX - X^*B = C$, we find the result is valid for normal operators subject to some constraint on $U_1$ and $U_2$.

THEOREM 4.1. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be normal operators and $C \in \mathscr{B}(\mathscr{H})$. Then the equation $AX - X^*B = C$ has a solution $X \in \mathscr{B}(\mathscr{H})$ if and only if there exist invertible operators $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ $(i = 1, 2)$ with $M_1 = M_2$ and $P_1 = P_2$ such that*

$$U_2 \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} U_1, \tag{4.1}$$

$$U_2^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} U_1 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{4.2}$$

*Proof.* Let $X \in \mathscr{B}(\mathscr{H})$ be a solution of $AX - X^*B = C$. Then the invertible operators $U_1 = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ and $U_2 = \begin{pmatrix} I & X^* \\ 0 & I \end{pmatrix}$ satisfy (4.1)–(4.2).

Conversely, assume that there exist invertible operators $U_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix}$ $(i = 1, 2)$ such that (4.1)–(4.2) hold, where $M_1 = M_2$ and $P_1 = P_2$. Then, by (4.2),

$$U_2 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} (U_1^*)^{-1} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{4.3}$$

Substituting this into (4.1) gives

$$\begin{bmatrix} 0 & -B \\ A & C \end{bmatrix} = U_1^* \begin{bmatrix} 0 & -B \\ A & 0 \end{bmatrix} U_1, \tag{4.4}$$

and hence

$$\begin{bmatrix} 0 & A^* \\ -B^* & C^* \end{bmatrix} = U_1^* \begin{bmatrix} 0 & A^* \\ -B^* & 0 \end{bmatrix} U_1.$$

This and the equality (4.4) lead to

$$\begin{bmatrix} 0 & -B - \lambda A^* \\ A + \lambda B^* & C - \lambda C^* \end{bmatrix} = U_1^* \begin{bmatrix} 0 & -B - \lambda A^* \\ A + \lambda B^* & 0 \end{bmatrix} U_1$$

for any complex number $\lambda$, which yields

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A + \lambda B^* & C - \lambda C^* \\ 0 & B + \lambda A^* \end{bmatrix} = U_1^* \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A + \lambda B^* & 0 \\ 0 & B + \lambda A^* \end{bmatrix} U_1.$$

Thus

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} (U_1^*)^{-1} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A+\lambda B^* & C-\lambda C^* \\ 0 & B+\lambda A^* \end{bmatrix} = \begin{bmatrix} A+\lambda B^* & 0 \\ 0 & B+\lambda A^* \end{bmatrix} U_1,$$

which, together with the equality (4.3), implies that

$$U_2 \begin{bmatrix} A+\lambda B^* & C-\lambda C^* \\ 0 & B+\lambda A^* \end{bmatrix} = \begin{bmatrix} A+\lambda B^* & 0 \\ 0 & B+\lambda A^* \end{bmatrix} U_1.$$

Applying Theorem 3.2, we then know that the system

$$AX_1 - X_2 B = C, \tag{4.5}$$

$$-B^* X_1 + X_2 A^* = C^*. \tag{4.6}$$

has a solution $(X_1, X_2)$. From (4.6), it is clear that $AX_2^* - X_1^* B = C$. Combining with (4.5), we conclude that

$$A(X_1 + X_2^*) - (X_2 + X_1^*)B = 2C,$$

Therefore $X = \frac{1}{2}(X_1 + X_2^*)$ is a solution of the equation $AX - X^*B = C$. $\square$

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