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NOTE ON BOUNDS FOR EIGENVALUES USING TRACES

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Abstract. Some extensions of the results related to bounds for the eigenvalues and spreads of matrices using traces are discussed here. We also obtain bounds for the eigenvalues of a positive definite matrix in terms of its trace and trace of its inverse.

1. Introduction

In theory and its various applications it is of basic interest to locate the eigenvalues of a matrix. The inequalities involving eigenvalues provide some information about them and have been studied extensively in literature. We here focus on the bounds for the eigenvalues using traces. In this context, Wolkowicz and Styan [14, 15] have discussed the bounds for the eigenvalues of a matrix A in terms of traces of A and A^2. They showed that if the eigenvalues of a matrix A are all real and arranged as

$$\lambda_1(A) \leqslant \lambda_2(A) \leqslant \ldots \leqslant \lambda_n(A), \tag{1.1}$$

then for all $k = 1, 2, \ldots, n$,

$$\begin{aligned} \frac{\operatorname{tr}A}{n} - \sqrt{\frac{n-k}{k}}\, s_A \leqslant \frac{1}{k}\sum_{i=1}^{k}\lambda_i(A) \leqslant \lambda_k(A) \\ \leqslant \frac{1}{n-k+1}\sum_{i=k}^{n}\lambda_i(A) \leqslant \frac{\operatorname{tr}A}{n} + \sqrt{\frac{k-1}{n-k+1}}\, s_A \end{aligned} \tag{1.2}$$

where $\operatorname{tr}A$ denotes the trace of A and

$$s_A = \sqrt{\frac{\operatorname{tr}A^2}{n} - \left(\frac{\operatorname{tr}A}{n}\right)^2}. \tag{1.3}$$

Note that (1.2) is based on [[14], Theorem 2.2] which is originally formulated for the eigenvalues arranged in decreasing order.

In addition, corresponds to [[14], Theorem 2.1], we also have

$$\lambda_1(A) \leqslant \frac{\operatorname{tr}A}{n} - \frac{s_A}{\sqrt{n-1}} \quad \text{and} \quad \lambda_n(A) \geqslant \frac{\operatorname{tr}A}{n} + \frac{s_A}{\sqrt{n-1}}. \tag{1.4}$$

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As pointed out by Wolkowicz and Styan [14], the inequalities (1.2) and (1.4) are related to the statistical inequalities involving mean and variance, see [8]. The arithmetic mean and variance of n real numbers $x_1, x_2, \ldots, x_n$ are defined as, respectively,

$$a_n = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - a_n)^2. \tag{1.5}$$

Further, the inequality for the spread due to Brauer and Mewborn [5],

$$\operatorname{spd}(A) = \lambda_{\max}(A) - \lambda_{\min}(A) \geqslant 2\sqrt{\frac{\operatorname{tr}A^2}{n} - \left(\frac{\operatorname{tr}A}{n}\right)^2} \tag{1.6}$$

is related to the Popoviciu inequality [13],

$$s^2 \leqslant \frac{(b-a)^2}{4}, \tag{1.7}$$

where $a = \min_i x_i$ and $b = \max_i x_i$.

We here consider a more general case. Let $\{x_1, x_2, \ldots, x_n\} = X$ be the set of n real numbers. Let $i_1, i_2, \ldots, i_n$ be some permutation of the indices $1, 2, \ldots, n$. Then, $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ is a k-combination of the set X consisting of k elements of X with distinct indices. It is clear that the set $X(k)$ consisting of the respective means of the k-combinations from X has $\binom{n}{k}$ elements. Then, the arithmetic mean and variance of the elements of $X(k)$ are, respectively,

$$\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n} \frac{x_{i_1}+x_{i_2}+\ldots+x_{i_k}}{k} = \frac{1}{n}\sum_{i=1}^{n} x_i = a_n \tag{1.8}$$

and

$$s_k^2 = \frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n}\left(\frac{x_{i_1}+x_{i_2}+\ldots+x_{i_k}}{k} - a_n\right)^2. \tag{1.9}$$

Merikoski and Virtanen [12] have discussed the bounds for the eigenvalues using determinant and trace of the matrix A. It is costly to calculate $\operatorname{tr}A^{-1}$. It is however in interest to know if the bounds analogues to (1.2) can be derived in terms of $\operatorname{tr}A$ and $\operatorname{tr}A^{-1}$. We also consider here this problem and find the bounds for x_i's in terms of arithmetic mean a_n and harmonic mean

$$h_n = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} \tag{1.10}$$

of n positive real numbers x_i's.

We show that s_k is a constant multiple of s for fixed values of n and k and use this fact to derive some upper bounds for the variance s^2 (Theorem 1–2 and Corollary 1–2, below). The bounds for positive numbers x_i's are obtained in terms of their arithmetic mean and harmonic mean (Theorem 3–4). An extension of the inequality (1.6) is obtained and a related inequality for the ratio spread is proved (Theorem 5–6). Some easily evaluable estimates for the eigenvalues are obtained in terms of the extreme diagonal and absolute off-diagonal entries of the Hermitian matrix (Theorem 7, Corollary 3). We also discuss the bounds for the eigenvalues of a positive definite matrix in terms of the traces of the matrix and its inverse (Theorem 8–9).

2. Preliminary results

We begin with the following lemma, which we need to derive the relation between s and s_k in the next theorem.

LEMMA 1. *Let* $y_1, y_2, \ldots, y_n$ *denote* n *numbers. Then*

$$\sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_k}\right)^2 = \binom{n-1}{k-1}\sum_{i=1}^{n} y_i^2 + 2\binom{n-2}{k-2}\sum_{i<j}^{n} y_i y_j. \tag{2.1}$$

Proof. We prove the lemma by using the principle of mathematical induction. For $n=k$, (2.1) is obviously true. Suppose (2.1) holds for n. We then show that it also holds for $n+1$. We write,

$$\begin{aligned}
&\sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n+1} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_k}\right)^2 \\
&= \sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_k}\right)^2 \\
&\quad + \sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_{k-1}}+y_{n+1}\right)^2 \\
&= \sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_k}\right)^2 + \sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_{k-1}}\right)^2 \\
&\quad + \sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} y_{n+1}^2 + 2\sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_{k-1}}\right) y_{n+1}.
\end{aligned} \tag{2.2}$$

On applying the induction hypothesis, we find from (2.2) that

$$\begin{aligned}
&\sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n+1} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_k}\right)^2 \\
&= \left(\binom{n-1}{k-1}+\binom{n-1}{k-2}\right)\sum_{i=1}^{n} y_i^2 + \sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} y_{n+1}^2 \\
&\quad +2\sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_{k-1}}\right) y_{n+1} + 2\left(\binom{n-2}{k-2}+\binom{n-2}{k-3}\right)\sum_{i<j}^{n} y_i y_j.
\end{aligned} \tag{2.3}$$

Inserting

$$\sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} y_{n+1}^2 = \binom{n}{k-1} y_{n+1}^2,$$

$$\binom{n-1}{k-1}+\binom{n-1}{k-2} = \binom{n}{k-1}, \quad \binom{n-2}{k-2}+\binom{n-2}{k-3} = \binom{n-1}{k-2}$$

and the equation analogous to (1.8),

$$\sum_{1\leqslant i_1<i_2<\ldots<i_{k-1}}^{n} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_{k-1}}\right) = \binom{n-1}{k-2}\sum_{k=1}^{n} y_k$$

in (2.3), we get that

$$\sum_{1\leqslant i_1<i_2<\ldots<i_k}^{n+1} \left(y_{i_1}+y_{i_2}+\ldots+y_{i_k}\right)^2 = \binom{n}{k-1}\sum_{k=1}^{n+1} y_i^2 + 2\binom{n-1}{k-2}\sum_{i<j}^{n+1} y_i y_j.$$

The lemma then follows by the principle of mathematical induction. $\square$

THEOREM 1. *Let* $x_1, x_2, \ldots, x_n$ *denote* n *numbers and let* s^2 *and* s_k^2 *be defined as in* (1.5) *and* (1.9)*, respectively. Then, for all* $k = 1, 2, \ldots, n$*, we have*

$$s_k^2 = \tfrac{n-k}{(n-1)k} s^2. \tag{2.4}$$

Proof. By Lemma 1,

$$\begin{aligned} &\textstyle\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k}^{n} \left(x_{i_1} + x_{i_2} + \ldots + x_{i_k} - k a_n\right)^2 \\ &= \tbinom{n-1}{k-1} \textstyle\sum_{i=1}^{n} (x_i - a_n)^2 + 2 \tbinom{n-2}{k-2} \textstyle\sum_{i<j}^{n} (x_i - a_n)(x_j - a_n) \end{aligned} \tag{2.5}$$

and on using $\sum_{i=1}^{n} (x_i - a_n) = 0$, we have

$$2 \textstyle\sum_{i<j}^{n} (x_i - a_n)(x_j - a_n) = -\textstyle\sum_{i=1}^{n} (x_i - a_n)^2. \tag{2.6}$$

From (2.5) and (2.6), we find that

$$\textstyle\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k}^{n} \left(x_{i_1} + x_{i_2} + \ldots + x_{i_k} - k a_n\right)^2 = \tbinom{n-2}{k-1} \textstyle\sum_{i=1}^{n} (x_i - a_n)^2. \tag{2.7}$$

On inserting (2.7) in (1.9), we conclude that

$$s_k^2 = \tfrac{1}{\binom{n}{k} k^2} \tbinom{n-2}{k-1} \textstyle\sum_{i=1}^{n} (x_i - a_n)^2 = \tfrac{n-k}{(n-1)k} s^2. \qquad \square$$

On using Theorem 1, we can study various inequalities involving mean and variance of real numbers. We demonstrate some cases here.

The inequality (1.7) and its refinement

$$s^2 \leqslant (a_n - a)(b - a_n) \tag{2.8}$$

are the particular cases of the more general inequalities of interest in matrix analysis due to Bhatia and Davis [2]. We now discuss some further extensions of the inequalities (1.7) and (2.8).

THEOREM 2. *Let* n *real numbers* x_i*'s be arranged as* $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$*. Let* a_n *and* s^2 *be defined as in* (1.5)*. Then, for all* $k = 1, 2, \ldots, n-1$*, we have*

$$s^2 \leqslant \tfrac{k(n-1)}{n-k} (a_n - a_k)(b_k - a_n), \tag{2.9}$$

where $a_k = \frac{1}{k} \sum_{i=1}^{k} x_i$ *and* $b_k = \frac{1}{k} \sum_{i=n-k+1}^{n} x_i$.

Proof. Note that a_n and s_k^2 are respectively the arithmetic mean and variance of the $\binom{n}{k}$ numbers $\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k}^{n} \frac{x_{i_1} + x_{i_2} + \ldots + x_{i_k}}{k}$, $1 \leqslant i_1 < \ldots < i_k \leqslant n$ in the set $X(k)$. For $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, the smallest and largest numbers in $X(k)$ are a_k and b_k, respectively. Then, on applying the inequality (2.8) to the $\binom{n}{k}$ numbers in $X(k)$, we get that

$$s_k^2 \leqslant (a_n - a_k)(b_k - a_n). \tag{2.10}$$

On combining (2.4) and (2.10), we immediately get (2.9). $\square$

COROLLARY 1. *With notations and conditions as in Theorem 2, we have, for all* $k = 1, 2, \ldots, n-1$,

$$s^2 \leqslant \frac{k(n-1)}{(n-k)} \frac{(b_k - a_k)^2}{4}. \tag{2.11}$$

Proof. It is clear that $a_k \leqslant a_n \leqslant b_k$. So, on using the arithmetic mean-geometric mean inequality for two positive numbers, we get

$$(a_n - a_k)(b_k - a_n) \leqslant \tfrac{1}{4}(b_k - a_k)^2. \tag{2.12}$$

The inequality (2.11) then follows from (2.9) and (2.12). □

It may be noted that for $k = 1$, the inequalities (2.8) and (2.9) are identical and same is true for (1.7) and (2.11).

In the following corollary, we extend the Brunk inequalities [6],

$$s^2 \leqslant (n-1)(a_n - a)^2 \quad \text{and} \quad s^2 \leqslant (n-1)(b - a_n)^2.$$

COROLLARY 2. *With notations and conditions as in Theorem 2, we have, for* $k \leqslant \frac{n}{2}$,

$$s^2 \leqslant (n-1)(a_n - a_k)^2 \tag{2.13}$$

and

$$s^2 \leqslant (n-1)(b_k - a_n)^2. \tag{2.14}$$

Proof. For $k \leqslant \frac{n}{2}$, we have

$$na_n = \textstyle\sum_{i=1}^{k} x_i + \ldots + \sum_{i=n-k+1}^{n} x_i \geqslant (n-k)a_k + kb_k$$

and therefore

$$b_k - a_n \leqslant \tfrac{n-k}{k}(a_n - a_k). \tag{2.15}$$

On combining (2.9) and (2.15), we immediately get (2.13).

Likewise, (2.14) follows from (2.9) and the fact that for $k \leqslant \frac{n}{2}$,

$$a_n - a_k \leqslant \tfrac{n-k}{k}(b_k - a_n). \qquad \square$$

Since $a_k \leqslant a_n \leqslant b_k$, by (2.13) and (2.14), we respectively have

$$a_k \leqslant a_n - \frac{s}{\sqrt{n-1}} \quad \text{and} \quad b_k \geqslant a_n + \frac{s}{\sqrt{n-1}}, \tag{2.16}$$

for $k \leqslant \frac{n}{2}$.

The inequalities (2.16) also yield the inequalities

$$b_{n-k} \geqslant a_n + \frac{k}{(n-k)\sqrt{n-1}} s \quad \text{and} \quad a_{n-k} \leqslant a_n - \frac{k}{(n-k)\sqrt{n-1}} s, \tag{2.17}$$

on using the fact that $ka_k + (n-k)b_{n-k} = (n-k)a_{n-k} + kb_k = na_n$. The inequalities (2.17) correspond to the inequalities [[14], Theorem 2.3]. Wolkowicz and Styan [14] have remarked that the inequalities in (2.16) and (2.17) also follow from the Corollary 6.1 of Mallows and Ritcher [8]. We here get the alternative and simple arithmetic proofs of these inequalities.

Let $x_1, x_2, \ldots, x_n$ denote n real numbers. The weighted arithmetic mean and variance of these n numbers with corresponding weights $p_1, p_2, \ldots, p_n$ are, respectively,

$$\mu_1' = \sum_{i=1}^n p_i x_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^n p_i \left(x_i - \mu_1'\right)^2, \tag{2.18}$$

where $p_i \geqslant 0$ and $\sum_{i=1}^n p_i = 1$.

Let r numbers chosen from the n real numbers x_i's be denoted as $y_1, y_2, \ldots, y_r$ and let $q_1, q_2, \ldots, q_r$ be their respective weights. Let

$$\gamma_r = \frac{1}{\sum_{j=1}^r q_j} \sum_{j=1}^r q_j y_j \quad \text{and} \quad \sigma_r^2 = \frac{1}{\sum_{j=1}^r q_j} \sum_{j=1}^r q_j (y_j - \gamma_r)^2. \tag{2.19}$$

Denote the weighted arithmetic mean and variance of the remaining $n-r$ numbers y_i's by γ_{n-r} and σ_{n-r}^2, respectively.

LEMMA 2. *Let μ_1' and σ^2 be the weighted arithmetic mean and variance of n real numbers x_i's as defined in (2.18). Let γ_r and σ_r^2 (as defined in (2.19)) be the arithmetic mean and variance of any r numbers $(r < n)$ chosen from the numbers x_i's, respectively. Then,*

$$\sigma^2 = \sigma_r^2 \sum_{i=1}^r q_i + \sigma_{n-r}^2 \left(1 - \sum_{i=1}^r q_i\right) + \frac{\sum_{i=1}^r q_i}{1 - \sum_{i=1}^r q_i} \left(\mu_1' - \gamma_r\right)^2. \tag{2.20}$$

Proof. We note that

$$\sum_{i=1}^n q_i = \sum_{i=1}^n p_i = 1, \quad \sum_{i=1}^n q_i y_i = \mu_1', \quad \mu_1' - \gamma_r = \left(1 - \sum_{i=1}^r q_i\right)(\gamma_{n-r} - \gamma_r). \tag{2.21}$$

Further, a little computation shows that

$$\sigma^2 = \sum_{i=1}^n q_i \left(y_i - \mu_1'\right)^2 = \sum_{i=1}^r q_i \sigma_r^2 + \left(1 - \sum_{i=1}^r q_i\right)(y_i - \gamma_r)^2 - \left(\mu_1' - \gamma_r\right)^2 \tag{2.22}$$

and

$$\left(1 - \sum_{i=1}^r q_i\right)(y_i - \gamma_r)^2 = \left(1 - \sum_{i=1}^r q_i\right)\left(\sigma_{n-r}^2 + (\gamma_{n-r} - \gamma_r)^2\right). \tag{2.23}$$

On inserting (2.23) into (2.22), we find that

$$\sigma^2 = \sum_{i=1}^r q_i \sigma_r^2 + \left(1 - \sum_{i=1}^r q_i\right)\sigma_{n-r}^2 + \left(1 - \sum_{i=1}^r q_i\right)(\gamma_{n-r} - \gamma_r)^2 - \left(\mu_1' - \gamma_r\right)^2. \tag{2.24}$$

On combining (2.21) and (2.24) and simplifying the resulting expression, we immediately get (2.20). □

THEOREM 3. *Let n positive real numbers x_i's be arranged as $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. Let a_n and h_n be defined as in (1.5) and (1.10), respectively. Then*

$$\frac{1}{2k}\left(\alpha(k)-\sqrt{\beta(k)}\right) \leqslant \frac{k}{\sum_{i=1}^{k}\frac{1}{x_i}} \leqslant x_k \leqslant \frac{n-k+1}{\sum_{i=k}^{n}\frac{1}{x_i}} \leqslant \frac{1}{2(n-k+1)}\left(\alpha(n-k+1)+\sqrt{\beta(n-k+1)}\right) \tag{2.25}$$

where

$$\alpha(r) = na_n - (n-2r)h_n, \quad \beta(r) = \alpha^2(r) - 4r^2 a_n h_n \tag{2.26}$$

for all $k, r = 1, 2, \ldots, n$.

Proof. Consider n positive real numbers x_i's with corresponding weights p_i's. Let x_i's be arranged as $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ and let $\gamma_r = \frac{1}{\sum_{i=1}^{r} p_i}\sum_{i=1}^{r} p_i x_i$, $\mu_1^{'} = \sum_{i=1}^{n} p_i x_i$ and $\sigma^2 = \sum_{i=1}^{n} p_i (x_i - \mu_1^{'})^2$. For $r < n$, on using (2.20), it is easily seen that

$$\sigma^2 \geqslant \frac{\sum_{i=1}^{r} p_i}{1-\sum_{i=1}^{r} p_i}\left(\mu_1^{'} - \gamma_r\right)^2. \tag{2.27}$$

Then, for $p_i = \frac{1}{x_i}\frac{h_n}{n} > 0$, we have $\sum_{i=1}^{r} p_i = \frac{r}{n}\frac{h_n}{h_r}$, $\gamma_r = h_r$, $\mu_1^{'} = h_n$ and $\sigma^2 = h_n(a_n - h_n)$. Thus, by (2.27), we get

$$a_n - h_n \geqslant \frac{r}{nh_r - rh_n}(h_n - h_r)^2. \tag{2.28}$$

Since $nh_r - rh_n > 0$, we find from (2.28) that

$$rh_r^2 - (na_n - (n-2r)h_n)h_r + ra_n h_n \leqslant 0. \tag{2.29}$$

With notations as in (2.26), the roots of the quadratic equation in (2.29) can be written as

$$g_1(r) = \frac{1}{2r}\left(\alpha(r) - \sqrt{\beta(r)}\right) \quad \text{and} \quad g_2(r) = \frac{1}{2r}\left(\alpha(r) + \sqrt{\beta(r)}\right). \tag{2.30}$$

So, (2.29) holds if and only if

$$g_1(r) \leqslant h_r \leqslant g_2(r). \tag{2.31}$$

From (2.31), for $r = k$, $g_1(k) \leqslant h_k$. Therefore,

$$g_1(k) \leqslant \frac{k}{\sum_{i=1}^{k}\frac{1}{x_i}} \leqslant x_k. \tag{2.32}$$

Likewise, for $r = n-k+1$, $h_{n-k+1} \leqslant g_2(n-k+1)$ and

$$x_k \leqslant \frac{n-k+1}{\sum_{i=k}^{n}\frac{1}{x_i}} \leqslant g_2(n-k+1). \tag{2.33}$$

On combining (2.32) and (2.33) and using (2.30), we immediately get (2.25) for $k < n$.

For $k = n$, the first three inequalities (2.25) are immediate. Further, for $r = 1$, $\gamma_1 = x_n$ and $q_1 = p_n$, we find from (2.27) that $\sigma^2 \geqslant \frac{p_n}{1-p_n}(\mu_1^{'} - x_n)^2$. Then, the arguments similar to the above yield the fourth inequality (2.25) for $k = n$. $\square$

THEOREM 4. *With notations and conditions as in Theorem 3, we have*

$$x_1 \leqslant h_n - \frac{2(a_n-h_n)h_n}{n(a_n-h_n)+\sqrt{\beta(1)}} \tag{2.34}$$

and

$$x_n \geqslant h_n + \frac{2(a_n-h_n)h_n}{\sqrt{\beta(1)}-n(a_n-h_n)} \tag{2.35}$$

where $\beta(1) = (na_n-(n-2)h_n)^2 - 4a_nh_n$.

Proof. For $x_1 \leqslant x_i \leqslant x_n$, we have $(x_i-x_1)(x_i-x_n) \leqslant 0$, therefore for all $i = 1,2,\ldots,n$,

$$x_i \leqslant (x_1+x_n) - \frac{x_1x_n}{x_i}. \tag{2.36}$$

On adding the n inequalities (2.36), and on using (1.5) and (1.10), we find that

$$a_n \leqslant (x_1+x_n) - \frac{x_1x_n}{h_n}$$

and hence

$$h_n(a_n-h_n) \leqslant (x_n-h_n)(h_n-x_1). \tag{2.37}$$

On the other hand, for $k=n$, the fourth inequality (2.25) gives

$$x_n \leqslant \frac{1}{2}\left(na_n-(n-2)h_n+\sqrt{\beta(1)}\right), \tag{2.38}$$

where $\beta(1) = \alpha^2(1) - 4a_nh_n$ and $\alpha(1) = na_n-(n-2)h_n$.

On combining (2.37) and (2.38) and simplifying the resulting expression, we immediately get (2.34). Likewise, (2.35) follows on combining (2.37) and the inequality

$$x_1 \geqslant \frac{1}{2}\left(na_n-(n-2)h_n-\sqrt{\beta(1)}\right). \qquad \square$$

3. Main results

We mainly consider here Hermitian matrices and discuss some new inequalities involving eigenvalues and the entries of the Hermitian matrix. These results are based on (1.6), [[3], Theorem 3.7] and the reformulation of [[14], Theorem 2.2] in (1.2). Further, some estimates obtained here might be interesting when applied to the approximate evaluation of certain error bounds for Hermitian matrices such as those using a vector-valued generalised spread, presented recently in Massey et. al. [10].

Let $A=(a_{ij})$ be an $n\times n$ Hermitian matrix. Denote the kth smallest diagonal entry of A by d_k and let its eigenvalues be arranged as in (1.1). Then, the Schur majorization inequalities say that, see [1],

$$\sum_{i=1}^{k}\lambda_i(A) \leqslant \sum_{i=1}^{k} d_i \quad \text{and} \quad \sum_{i=n-k+1}^{n}\lambda_i(A) \geqslant \sum_{i=n-k+1}^{n} d_i. \tag{3.1}$$

It is evident from (3.1) that

$$\sum_{i=n-k+1}^{n}\lambda_i(A) - \sum_{i=1}^{k}\lambda_i(A) \geqslant \sum_{i=n-k+1}^{n} d_i - \sum_{i=1}^{k} d_i. \tag{3.2}$$

The left hand side expression (3.2) can be regarded as the generalised form of spread $(k=1)$. The inequalities (3.2) provide the extension of the well known theorem in linear algebra that says that $\operatorname{spd}(A)=\lambda_n(A)-\lambda_1(A)\geqslant \max_i a_{ii}-\min_i a_{ii}$, see [4]. We now prove a similar extension of (1.6).

THEOREM 5. *Let A be a complex $n\times n$ matrix with real eigenvalues $\lambda_1(A),\lambda_2(A),\ldots,\lambda_n(A)$, arranged as in (1.1). Then, for all $k=1,2,\ldots,n$, we have*

$$\sum_{i=n-k+1}^{n}\lambda_i(A)-\sum_{i=1}^{k}\lambda_i(A)\geqslant 2\sqrt{\frac{k(n-k)}{(n-1)n}trB^2}, \tag{3.3}$$

where $B=A-\frac{trA}{n}I$.

Proof. Note that the arithmetic mean of the eigenvalues of A is $\frac{\operatorname{tr}A}{n}=a_n$ and

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i^2(A)-\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i(A)\right)^2=\frac{\operatorname{tr}A^2}{n}-\left(\frac{\operatorname{tr}A}{n}\right)^2=\frac{\operatorname{tr}B^2}{n}=s_A^2$$

is the variance of the eigenvalues of A. Also, if the eigenvalues of A are arranged as in (1.1), then $a_k=\frac{1}{k}\sum_{i=1}^{k}\lambda_i(A)$ and $b_k=\frac{1}{k}\sum_{i=n-k+1}^{n}\lambda_i(A)$. On applying the Corollary 1 to the n eigenvalues of A, we get

$$b_k-a_k\geqslant 2\sqrt{\frac{n-k}{(n-1)k}\frac{\operatorname{tr}B^2}{n}},$$

and this yields (3.3). □

It may be noted here that $\frac{\operatorname{tr}B^2}{n}$ also appears in [[14], (1.8)] for describing the variance of the eigenvalues of A.

The condition number of a positive definite Hermitian matrix is defined as $c(A)=\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ and is also known as the ratio spread of A. The Schur inequalities (3.1) implies that for positive definite Hermitian matrix A,

$$\frac{\sum_{i=n-k+1}^{n}\lambda_i(A)}{\sum_{i=1}^{k}\lambda_i(A)}\geqslant\frac{\sum_{i=n-k+1}^{n}d_i}{\sum_{i=1}^{k}d_i}. \tag{3.4}$$

A particular case of (3.4) gives $c(A)\geqslant\frac{\max_i a_{ii}}{\min_i a_{ii}}$, see [16]. Also, the inequality

$$\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}\geqslant\left(c_1+\sqrt{1+c_1^2}\right)^2 \tag{3.5}$$

for $c_1=\frac{\sqrt{n\operatorname{tr}B^2}}{\operatorname{tr}A}$ is related to the inequality

$$\frac{b}{a}\geqslant\left(\frac{s}{a_n}+\sqrt{1+\left(\frac{s}{a_n}\right)^2}\right)^2 \tag{3.6}$$

which is a special case of a more general result, see Bhatia and Sharma [Theorem 3.7, [3]]. We extend this result in the next theorem.

THEOREM 6. *Let the eigenvalues of a complex $n \times n$ matrix A be all positive and arranged as in* (1.1). *Then,*

$$\frac{\sum_{i=n-k+1}^{n} \lambda_i(A)}{\sum_{i=1}^{k} \lambda_i(A)} \geqslant \left(c_k + \sqrt{1+c_k^2}\right)^2, \tag{3.7}$$

where $c_k = \sqrt{\frac{n(n-k)}{(n-1)k} \frac{trB^2}{(trA)^2}}$.

Proof. On using the arguments similar to those used in the proof of the Theorem 2, namely, by regarding a_n and s_k^2 as mean and variance and on using (3.6), we find that

$$\frac{b_k}{a_k} \geqslant \left(\frac{s_k}{a_n} + \sqrt{1+\left(\frac{s_k}{a_n}\right)^2}\right)^2$$

and hence by (2.4),

$$\frac{b_k}{a_k} \geqslant \left(\sqrt{\frac{n-k}{(n-1)k}} \frac{s}{a_n} + \sqrt{1+\frac{n-k}{(n-1)k} \frac{s^2}{a_n^2}}\right)^2. \tag{3.8}$$

As in Theorem 5, $a_k = \frac{1}{k}\sum_{i=1}^{k} \lambda_i(A)$, $b_k = \frac{1}{k}\sum_{i=n-k+1}^{n} \lambda_i(A)$, $a_n = \frac{\text{tr}A}{n}$ and $s = \sqrt{\frac{\text{tr}B^2}{n}}$. On inserting these values in (3.8), we immediately get (3.7). $\square$

It is always interesting to note easily evaluable estimates for the eigenvalues in terms of the expressions involving one or two entries of the matrix. One such result due to Hirsch [7] says that for a complex $n \times n$ matrix $A = (a_{ij})$, we have

$$|\lambda_k(A)| \leqslant n \max_{i,j} |a_{ij}| \tag{3.9}$$

for all $k = 1, 2, \ldots, n$. Also, see Marcus and Minc [9].

We here consider Hermitian matrices only and discuss the refinements of the inequality (3.9). The Cauchy interlacing principle of Hermitian matrices says that if P is any $r \times r$ principal submatrix of A and eigenvalues of A and P are arranged as in (1.1), then for $1 \leqslant i \leqslant r$, see [1],

$$\lambda_i(A) \leqslant \lambda_i(P) \leqslant \lambda_{n-r+i}(A). \tag{3.10}$$

It is immediate from (3.9) and (3.10) that for $r = 1, 2, \ldots, n$,

$$\lambda_r(A) \leqslant \lambda_r(P) \leqslant |\lambda_r(P)| \leqslant rC_r \tag{3.11}$$

where C_r is the maximum of absolute values of entries of P. Then, (3.11) provides an improvement in (3.9) for positive semidefinite matrices. Furthermore, a reformulation of [[14], Theorem 2.2] for $\frac{1}{k}\sum_{i=1}^{k} \lambda_i(A)$ and $\frac{1}{n-k+1}\sum_{i=k}^{n} \lambda_i(A)$ in (1.2) approximately locates the kth smallest eigenvalue of A by using traces of A and A^2. Applying this to the principal submatrices (for Hermitian A) leads to simpler bounds depending on fewer matrix entries. We demonstrate this in next theorem.

THEOREM 7. *Let A be an $n\times n$ Hermitian matrix and that $P=(p_{ij})$ be any $r\times r$ principal submatrix of A. Let $\lambda_k(A)$ be the kth smallest eigenvalue of A. Then, for $1\leqslant k\leqslant r\leqslant n$, we have*

$$\lambda_k(A)\leqslant \tfrac{1}{r-k+1}\textstyle\sum_{i=k}^{r}\lambda_i(A)\leqslant \max_i p_{ii}+\sqrt{\tfrac{(k-1)(r-1)}{r-k+1}}\max_{i\neq j}|p_{ij}| \tag{3.12}$$

and

$$\lambda_{n-k+1}(A)\geqslant \tfrac{1}{r-k+1}\textstyle\sum_{i=n-r+1}^{n-k+1}\lambda_i(A)\geqslant \min_i p_{ii}-\sqrt{\tfrac{(k-1)(r-1)}{r-k+1}}\max_{i\neq j}|p_{ij}|. \tag{3.13}$$

Proof. We first show that

$$\lambda_k(A)\leqslant \tfrac{1}{n-k+1}\textstyle\sum_{i=k}^{n}\lambda_i(A)\leqslant \max_i a_{ii}+\sqrt{\tfrac{(k-1)(n-1)}{n-k+1}}\max_{i\neq j}|a_{ij}| \tag{3.14}$$

for all $k=1,2,\ldots,n$.

Let a_{ij}'s be the entries of A and let $C=(c_{ij})$ be an $n\times n$ matrix such that $c_{jj}=\max_i a_{ii}$ and $c_{ij}=a_{ij}$ for $i\neq j$, $i,j=1,2,\ldots,n$. It is clear that

$$\tfrac{\mathrm{tr}C}{n}=\tfrac{1}{n}\textstyle\sum_{j=1}^{n}c_{jj}=\max_i a_{ii}. \tag{3.15}$$

Further, for any $n\times n$ Hermitian matrix $A=(a_{ij})$, the s_A^2 (as defined in (1.3)) can be written as

$$s_A^2=\tfrac{1}{n}\textstyle\sum_{i=1}^{n}a_{ii}^2-\left(\tfrac{1}{n}\sum_{i=1}^{n}a_{ii}\right)^2+\tfrac{1}{n}\sum_{i\neq j}|a_{ij}|^2. \tag{3.16}$$

For the matrix C, we have

$$\tfrac{1}{n}\textstyle\sum_{i=1}^{n}c_{ii}^2-\left(\tfrac{1}{n}\sum_{i=1}^{n}c_{ii}\right)^2=0,\quad \text{and}\quad \sqrt{\textstyle\sum_{i\neq j}|c_{ij}|^2}\leqslant\sqrt{n(n-1)}\max_{i\neq j}|a_{ij}|.$$

So, from (3.16) ,

$$s_C=\sqrt{\tfrac{1}{n}\textstyle\sum_{i\neq j}|c_{ij}|^2}\leqslant\sqrt{n-1}\,\max_{i\neq j}|a_{ij}|. \tag{3.17}$$

On using (3.15) and (3.17), the fourth inequality from (1.2) applied to C yields

$$\tfrac{1}{n-k+1}\textstyle\sum_{i=k}^{n}\lambda_i(C)\leqslant \max_i a_{ii}+\sqrt{\tfrac{(k-1)(n-1)}{n-k+1}}\max_{i\neq j}|a_{ij}|. \tag{3.18}$$

It is clear that $C-A$ is positive semidefinite and therefore by Weyl's inequality $\lambda_i(A)\leqslant\lambda_i(C)$ for all $i=1,2,\ldots,n$. So, (3.18) implies the second inequality (3.14). The first inequality (3.14) is evident. On applying (3.18) to the principal submatrix P and using the Cauchy interlacing inequalities (3.10), we immediately get (3.12).

Likewise, the first inequality (3.13) is evident and second is based on the inequality analogous to that in (3.18) and follows on using similar arguments. □

COROLLARY 3. *Under the conditions of the Theorem 7, for $k\leqslant\frac{n}{2}$, we have*

$$\tfrac{1}{k}\textstyle\sum_{i=1}^{k}\lambda_i(A)\leqslant\max_i a_{ii}-\min_{i\neq j}|a_{ij}|,$$

$$\frac{1}{n-k}\sum_{i=1}^{n-k}\lambda_i(A) \leqslant \max_i a_{ii} - \frac{k}{n-k}\min_{i\neq j}|a_{ij}|,$$
$$\frac{1}{k}\sum_{i=n-k+1}^{n}\lambda_i(A) \geqslant \min_i a_{ii} + \min_{i\neq j}|a_{ij}|,$$

and

$$\frac{1}{n-k}\sum_{i=k+1}^{n}\lambda_i(A) \geqslant \min_i a_{ii} + \frac{k}{n-k}\min_{i\neq j}|a_{ij}|.$$

Proof. The assertions of the corollary follow from the inequalities (2.16) and (2.17) on using the arguments similar to those used in the proof of the Theorem 7. Furthermore, the derivation of the inequalities in this corollary requires a lower bound for s_C which is analogous to (3.17). □

We finally consider the case when the eigenvalues of A are all positive as in case of positive definite matrices. The trace of A^{-1} in this case provides an additional useful information in estimating the eigenvalues of A. It is worth noting here that the approaches for an inverse-free approximation of $\mathrm{tr}A^{-1}$ are also available in literature, for example, see Meurant [11]. We now derive bounds for the eigenvalues of A in terms of $\mathrm{tr}A$ and $\mathrm{tr}A^{-1}$.

THEOREM 8. *Let the eigenvalues of a complex $n\times n$ matrix A be all positive and arranged as in (1.1). Then*

$$\frac{1}{2k}\left(\alpha_1(k)-\sqrt{\beta_1(k)}\right) \leqslant \lambda_k(A) \leqslant \frac{1}{2(n-k+1)}\left(\alpha_1(n-k+1)+\sqrt{\beta_1(n-k+1)}\right) \tag{3.19}$$

where

$$\alpha_1(r) = trA - \frac{n(n-2r)}{trA^{-1}} \quad \textit{and} \quad \beta_1(r) = \alpha_1^2(r) - 4r^2\frac{trA}{trA^{-1}} \tag{3.20}$$

for all $k, r = 1, 2, \ldots, n$.

Proof. The eigenvalues of A are all positive real numbers. Then, the arithmetic mean and harmonic mean of the eigenvalues of A can be written as, respectively,

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i(A) = \frac{\mathrm{tr}A}{n} = a_n \quad \text{and} \quad \frac{n}{\sum_{i=1}^{n}\frac{1}{\lambda_i(A)}} = \frac{n}{\mathrm{tr}A^{-1}} = h_n. \tag{3.21}$$

The assertions of the theorem now follow on using Theorem 3. □

THEOREM 9. *With notations and conditions as in Theorem 8, we have*

$$\lambda_1(A) \leqslant \frac{n}{trA^{-1}} - \frac{2\left(\frac{trA}{n}-\frac{n}{trA^{-1}}\right)\frac{n}{trA^{-1}}}{trA-\frac{n^2}{trA^{-1}}+\sqrt{\beta_1(1)}} \tag{3.22}$$

and

$$\lambda_n(A) \geqslant \frac{n}{trA^{-1}} + \frac{2\left(\frac{trA}{n}-\frac{n}{trA^{-1}}\right)\frac{n}{trA^{-1}}}{\sqrt{\beta_1(1)}-trA+\frac{n^2}{trA^{-1}}} \tag{3.23}$$

where $\beta_1(1) = \left(trA - \frac{n(n-2)}{trA^{-1}}\right)^2 - 4\frac{trA}{trA^{-1}}$.

Proof. The inequalities (3.22) and (3.23) follow respectively from the inequalities (2.34) and (2.35) on using (3.21) and arguments similar to those used in the proof of Theorem 8. □

4. Examples

We consider some simple examples and compare our results favourably with those in literature.

EXAMPLE 1. Let

$$E_1 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & 3 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Denote $\sum_{i=3}^{4} \lambda_i(E_j) - \sum_{i=1}^{2} \lambda_i(E_j)$ by $\alpha(E_j)$. Then, from (3.2) and (3.3), $\alpha(E_1) \geqslant 0, \alpha(E_2) \geqslant 8$ and $\alpha(E_1) \geqslant 7.3030, \alpha(E_2) \geqslant 6.2183$, respectively. The inequalities (3.2) and (3.3) are therefore independent. The eigenvalues of E_1 and E_2 are approximately $-2.2806, -0.8373, 1.0326, 6.0853$ and $0.3344, 1.8180, 4.4403, 7.4072$, respectively.

EXAMPLE 2. Let

$$E_3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The matrices E_3 and E_4 are positive definite. Denote $\frac{\lambda_2(E_j)+\lambda_3(E_j)}{\lambda_1(E_j)+\lambda_2(E_j)}$ by $\beta(E_j)$. Then, from (3.4) and (3.7), we respectively have $\beta(E_3) \geqslant 1$, $\beta(E_4) \geqslant 5.5$ and $\beta(E_3) \geqslant 2$, $\beta(E_4) \geqslant 2.8585$.

EXAMPLE 3. Let

$$E_5 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

From (1.2) and (3.19), the eigenvalues of E_5 lie in the interval $[0.9028, 4.4305]$ and $[0.9872, 5.1181]$. Also, from (1.4) and the Theorem 9, we have $\lambda_1(E_5) \leqslant 1.7847$, $\lambda_3(E_5) \geqslant 3.5486$ and $\lambda_1(E_5) \leqslant 1.4410$ and $\lambda_3(E_5) \geqslant 3.5064$. Also, note that if the eigenvalues of A are all positive and $(\mathrm{tr}A)^2 \leqslant (n-k)\mathrm{tr}A^2$, then the first inequality (1.2) is not useful as it gives negative lower bound for $\lambda_k(A)$ but (3.19) will always give positive lower bound. For example, for the diagonal matrix $A = \mathrm{diag}(1,1,2,10)$, we respectively have from (1.2) and (3.19), $-0.2749 \leqslant \lambda_2(A) \leqslant 5.6794$ and $0.8797 \leqslant \lambda_2(A) \leqslant 4.4942$.

EXAMPLE 4. (Example 4 and 5, [14]). Let

$$E_6 = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix} \quad \text{and} \quad E_7 = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix}.$$

The Theorem 7 yield the better estimates $3 \leqslant \lambda_2(E_6) \leqslant 5$ and $\lambda_3(E_6) \geqslant 6$, respectively, on chosing P to be

$$\begin{bmatrix} 5 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}.$$

Likewise, we have $\lambda_1(E_7) \leqslant 4$, $\lambda_2(E_7) \leqslant 6$, $4 \leqslant \lambda_3(E_7) \leqslant 8$, $\lambda_4(E_7) \geqslant 6$. The remaining estimates for the eigenvalues of E_6 and E_7 are better in [Example 4 and 5, [14]]. We conclude that the estimates in our Theorem 7 are also independent of the corresponding estimates in [[14], Theorem 2.2.]. In addition, by Theorem 8, $\lambda_1(E_6) \geqslant 1.3565$ and $\lambda_1(E_7) \geqslant 2.1061$.

EXAMPLE 5. Let $A = (a_{ij})$ be a 10×10 real symmetric matrix with $a_{ij} = \min(i, j)$. Then, A is positive definite, $\mathrm{tr}A = 55$, $\mathrm{tr}A^2 = 2035$ and $\mathrm{tr}A^{-1} = 19$. The inequality (1.2) gives a negative lower bound -33.987 for the smallest eigenvalue of A while (3.19) yields a better estimate, $\lambda_1(A) \geqslant 0.057$. From (1.4) and (3.22), we respectively have $\lambda_1(A) \leqslant 1.1125$ and $\lambda_1(A) \leqslant 0.47418$. Further, from (1.2) and (3.19), we have $\lambda_{10}(A) \leqslant 44.987$ and $\lambda_{10}(A) \leqslant 50.732$, respectively. The inequalities (1.4) and (3.22) respectively give $\lambda_{10}(A) \geqslant 9.8875$ and $\lambda_{10}(A) \geqslant 6.1048$.

We conclude from the above examples that our bounds for the eigenvalues derived here are all independent of the corresponding bounds in literature.

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