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STABILITY BOUNDS FOR RECONSTRUCTION FROM SAMPLING ERASURES

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Abstract. The Shannon-Whittaker Sampling Theorem states that a frequency bounded signal can be completely determined by its sampled values at a countable number of points. Thus, the theorem allows us to convert analog signals to digital signals by sampling (or evaluating) the signal at these points. In prior work, it was shown that if a signal is oversampled, and if some of the sampled values are lost when transmitting the signal, then it is still possible to reconstruct the signal. However, in certain situations, the reconstruction algorithm is very unstable. In this paper, we provide stability bounds on the reconstruction algorithm and determine when it is not feasible to perform the reconstruction.

1. Introduction

The Shannon-Whittaker Sampling Theorem allows us to reconstruct frequency bounded signals from their sampled values on a lattice. If we consider the space $\mathscr{PW}(\pi)$ (all band-limited signals with frequency band $[-\pi,\pi]$) then we can reconstruct the signal from its sampled values on $p\mathbb{Z} = \{\ldots,-2p,-p,0,p,2p,\ldots\}$ provided that $p \in (0,1]$. When $p=1$, we are sampling at the Nyquist rate, and the underlying frame is a Riesz basis. When $p \in (0,1)$, we are oversampling, and the underlying frame is a redundant tight frame. Because of this redundancy, we can still reconstruct a signal when some of the sampled values are lost or erased.

In the frame theory literature, there are several papers that deal with erasures. Frames which minimize signal error due to erasures were studied in [3] and [12] for one and two erasures, respectively. In [9] and [13], efficient algorithms for reconstructing signals from erasures were discovered. In this paper, we obtain stability bounds for the Reduced Direct Inversion algorithm from [13] applied to Shannon-Whittaker Sampling Theory. For other good references on frame erasures, and reconstruction stability, see [5], [10], [14], [15], and [16].

While the reconstruction algorithms mentioned above give perfect reconstruction, it is possible that a signal may be corrupted by noise (cf. [2]), or quantization error (cf. [1]). Moreover, the Shannon-Whittaker sampling theorem requires an infinite sequence

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of sampled values. In practice, since we cannot transmit infinitely many sampled values over a channel, we must truncate this data stream. This also contributes to the error term. If the reconstruction process is not stable, these unavoidable errors can be blown up, sometimes to the extent that it is worse to perform the reconstruction process than to perform no reconstruction from erasures at all.

In Section 3, we provide operator theoretic bounds on the various pieces that make up the *partial reconstruction operator* – the linear operator that must be inverted to reconstruct from erasures. If this operator is poorly behaved a good reconstruction is not possible. However, if the erasure set is relativley separated, we can find a good bound for the partial reconstruction operator.

Since we can only transmit finitely many sampled values, there is always truncation error involved in Shannon-Whittaker Sampling Theory. In Section 4, we bound this truncation error. We provide a bound for this error term for signals whose Fourier transforms belong to Sobolev spaces of a particular order. We show that the higher the regularity of the Fourier transform, the faster the convergence rate of the partial reconstruction. Thus, for signals with Fourier transforms of high regularity, fewer sampled values need to be transmitted.

In Section 5, we combine the results of Sections 3 and 4 to provide a total, overall bound for the error in the reconstruction. We also discuss the effects of channel noise on our reconstruction. Based on the theoretical results, qualitative guidelines are provided in Section 6 to characterize situations in which a good reconstruction is possible, and situations in which a poor reconstruction is inevitable. Numerical experiments are provided to demonstrate our theoretical results and to back up our qualitative characterizations.

2. Preliminaries

2.1. Frames and Shannon-Whittaker sampling theory

A *frame* for a Hilbert space $\mathscr{H}$ is a collection of vectors $\{f_j\}_{j\in\mathbb{J}} \subset \mathscr{H}$ for which there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{H}. \tag{1}$$

A frame is called *tight* if we can take $A = B$ in equation (1). Associated to any frame are three operators – the *analysis*, *synthesis*, and *frame operators*. The analysis operator $\Theta : \mathscr{H} \to \ell^2(\mathbb{J})$ maps a vector f to its sequence of frame coefficients:

$$\Theta f = (\langle f, f_j\rangle)_{j\in\mathbb{J}}.$$

The synthesis operator $\Theta^* : \ell^2(\mathbb{J}) \to \mathscr{H}$ is the adjoint of the analysis operator and it sums a sequence of coefficients against the frame vectors:

$$\Theta^*(c_j)_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j f_j.$$

When the frame is not understood from context, we use subscripts on Θ and Θ^* for clarity. The composition of these two operators forms the frame operator $S:\mathscr{H}\to\mathscr{H}$ defined by

$$Sf=\sum_{j\in\mathbb{J}}\langle f,f_j\rangle f_j.$$

The frame operator for a frame is an invertible operator, and the sequence $\{S^{-1}f_j\}_{j\in\mathbb{J}}$ – called the *standard dual* of $\{f_j\}_{j\in\mathbb{J}}$ – satisfies:

$$f=\sum_{j\in\mathbb{J}}\langle f,f_j\rangle S^{-1}f_j=\sum_{j\in\mathbb{J}}\langle f,S^{-1}f_j\rangle f_j,\quad \forall f\in\mathscr{H}. \tag{2}$$

In general, we say any frame $G=\{g_j\}_{j\in\mathbb{J}}$ satisfying

$$f=\sum_{j\in\mathbb{J}}\langle f,g_j\rangle f_j=\sum_{j\in\mathbb{J}}\langle f,f_j\rangle g_j$$

for all $f\in\mathscr{H}$ is called a *dual frame* to $F=\{f_j\}_{j\in\mathbb{J}}$. The pair (F,G) is called a *dual frame pair*. For more on frames, see [4], [7], and [8].

Throughout this paper, we use the Fourier transform as defined by

$$\mathscr{F}\{f(x)\}=\hat{f}(\xi)=\int_{\mathbb{R}}f(x)e^{-ix\xi}\,dx.$$

A function $f\in L^2(\mathbb{R})$ is said to be *band-limited* with band Ω if $\text{spt.}(\hat{f})\subset[-\Omega,\Omega]$. We denote the space of all band-limited functions with band Ω as $\mathscr{PW}(\Omega)$. Within the space $\mathscr{PW}(\pi)$, pointwise evaluation functionals are continuous and given by inner products:

$$f(a)=\langle f(x),\text{sinc}(\pi(x-a))\rangle,\quad \forall f\in\mathscr{PW}(\pi), \tag{3}$$

where $\text{sinc}(x)=x^{-1}\sin(x)$ for $x\neq 0$ and $\text{sinc}(0)=1$. Using Parseval's equality, it is straightforward to show that $G_p=\{\text{sinc}(\pi(x-pj))\}_{j\in\mathbb{Z}}$ is a tight frame for $\mathscr{PW}(\pi)$ with frame bound $A=\frac{1}{p}$ for any $p\in(0,1]$. When $p=1$, this frame is actually an orthogonal basis because we are sampling at the Nyquist rate. For $0<p<1$, we are oversampling the signal, and thus the collection forms a redundant tight frame. The standard dual to G_p is $F_p=\{p\,\text{sinc}(\pi(x-pj))\}_{j\in\mathbb{Z}}$.

2.2. Reduced direct inversion

The Reduced Direct Inversion algorithm is a technique which can be used to reconstruct signals in a Hilbert space when part of the signal is erased. Suppose Alice wishes to send a signal $f\in\mathscr{H}$ to Bob. Given a dual frame pair (F,G), Alice can compute the coefficients $(\langle f,g_j\rangle)_{j\in\mathbb{J}}$ and then send them to Bob over some channel. Bob can then compute the recovery of the signal f:

$$f=\sum_{j\in\mathbb{J}}\langle f,g_j\rangle f_j.$$

This, of course, can be expressed in terms of the standard frame operators discussed in the previous section, i.e. $\sum_{j\in\mathbb{J}}\langle f,g_j\rangle f_j=\Theta_F^*\Theta_G f$.

We are interested in reconstructing a signal when some of the data is erased. Suppose that some of the data, indexed by an *erasure set* Λ, are lost in the channel. The *partial reconstruction operator*, $R_\Lambda : \mathscr{H} \to \mathscr{H}$, which gives the *partial reconstruction* f_R, is defined as

$$f_R = R_\Lambda f = \sum_{j \in \Lambda^c} \langle f, g_j \rangle f_j.$$

The *error operator*, $E_\Lambda : \mathscr{H} \to \mathscr{H}$, is defined as

$$E_\Lambda f = (I - R_\Lambda) f = \sum_{j \in \Lambda} \langle f, g_j \rangle f_j = \Theta^*_{F_\Lambda} \Theta_{G_\Lambda} f.$$

Notice that $E_\Lambda f$ is the erased portion of the signal. Pay close attention to the subscripts, which tell us which frames and operators we are considering. For example, Θ_{G_Λ} denotes the analysis operator for the reduced collection $\{g_j\}_{j \in \Lambda}$, and $\Theta^*_{F_\Lambda}$ is the synthesis operator for $\{f_j\}_{j \in \Lambda}$.

Since the partial reconstruction operator gives rise to the partial reconstruction in the sense that $f_R = R_\Lambda f$, recovering the signal f would amount to inverting R_Λ. It can be shown that R_Λ is invertible if and only if $I - \Theta_{G_\Lambda} \Theta^*_{F_\Lambda}$ is invertible, with inverse

$$R_\Lambda^{-1} = I + \Theta^*_{F_\Lambda} (I - \Theta_{G_\Lambda} \Theta^*_{F_\Lambda})^{-1} \Theta_{G_\Lambda}.$$

We would then be able to recover f by the formula

$$f = R_\Lambda^{-1} f_R = [I + \Theta^*_{F_\Lambda} (I - \Theta_{G_\Lambda} \Theta^*_{F_\Lambda})^{-1} \Theta_{G_\Lambda}] f_R. \tag{4}$$

Given a vector $\mathbf{x} = (x_k)_{k=1}^L$ and an erasure set $\Lambda = \{n_k\}_{k=1}^L$, it can be shown that

$$\Theta_{G_\Lambda} \Theta^*_{F_\Lambda} \mathbf{x} = p \left[\sum_{k=1}^{L} x_k \operatorname{sinc}(p\pi(n_j - n_k)) \right]_{j=1}^{L} = M_\Lambda \mathbf{x},$$

where

$$M_\Lambda = p \begin{bmatrix} 1 & \operatorname{sinc}(p\pi(n_1 - n_2)) & \cdots & \operatorname{sinc}(p\pi(n_1 - n_L)) \\ \operatorname{sinc}(p\pi(n_2 - n_1)) & 1 & & \operatorname{sinc}(p\pi(n_2 - n_L)) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{sinc}(p\pi(n_L - n_1)) & \operatorname{sinc}(p\pi(n_L - n_2)) & \cdots & 1 \end{bmatrix}. \tag{5}$$

Substituting this matrix representation into (4) allows us to rewrite the reconstruction as the following formula:

$$f = R_\Lambda^{-1} f_R = [I + \Theta^*_{F_\Lambda} (I - M_\Lambda)^{-1} \Theta_{G_\Lambda}] f_R. \tag{6}$$

Directly inverting R_Λ, which is an infinite dimensional operator, is not possible. However, inverting the $L \times L$ matrix $I - M_\Lambda$ is possible in most cases.

2.3. Sobolev spaces

Sobolev spaces consist of integrable functions whose weak derivatives are also integrable up to some order. More specifically, for $s \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, the Sobolev space $W_p^s(\Omega)$ is defined as the space of all functions $u \in L^p(\Omega)$ whose α-th weak derivatives, $u^{(\alpha)}$, exist and also belong to $L^p(\Omega)$ for all $\alpha \leqslant s$. That is,

$$W_p^s(\Omega) = \{u \in L^p(\Omega) : u^{(\alpha)} \in L^p(\Omega) \text{ for all } \alpha \leqslant s\}.$$

For $p = 2$, we often write $H^s(\Omega) = W_2^s(\Omega)$, for the well known fact that these spaces are Hilbert spaces.

Given a function $f \in L^2(\Omega)$, we define its Fourier coefficients by

$$c_k = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} f(x) e^{-\frac{ik\pi x}{\Omega}}\, dx.$$

For a function $f \in H^s(\Omega)$, it can be shown using Parseval's equation for f and $f^{(s)}$ that

$$\sum_{\ell \in \mathbb{Z}} (|c_\ell|(1 + |\ell|^s))^2 < \infty.$$

The following simple observation relates decay of Fourier coefficients to the order of weak-differentiability a function obtains.

PROPOSITION 2.1. *If $f \in H^s(\Omega)$, then $|c_k| = \mathscr{O}(|k|^{-s})$.*

Proof. If $f \in H^s(\Omega)$, then

$$\sum_{\ell \in \mathbb{Z}} (|c_\ell|(1 + |\ell|^s))^2 < \infty.$$

For each k, $|k|^{2s}|c_k|^2 \leqslant \sum_{\ell \in \mathbb{Z}} |c_\ell|^2 |\ell|^{2s} \leqslant \sum_{\ell \in \mathbb{Z}} |c_\ell|^2 (1 + |\ell|^{2s})$. It follows that

$$\begin{aligned} |k|^s |c_k| &\leqslant \left(\sum_{\ell \in \mathbb{Z}} |c_\ell|^2 (1 + |\ell|^{2s}) \right)^{1/2} \\ &\leqslant \left(\sum_{\ell \in \mathbb{Z}} |c_\ell|^2 (1 + |\ell|^s)^2 \right)^{1/2} \\ &= \left(\sum_{\ell \in \mathbb{Z}} (|c_\ell|(1 + |\ell|^s))^2 \right)^{1/2} < \infty. \end{aligned}$$

Therefore, $|k|^s |c_k| \leqslant C = \left(\sum_{\ell \in \mathbb{Z}} (|c_\ell|(1 + |\ell|^s))^2 \right)^{1/2}$, so $|c_k| = \mathscr{O}(|k|^{-s})$. $\square$

3. Stability bounds for the partial reconstruction operator

The first step in bounding the reconstruction error is to bound the operator norm of the partial reconstruction operator, R_Λ. The smaller the norm is, the more stable our reconstruction will be. To do this, we will individually bound all of the operators in equation (6). The most challenging piece to bound is $\|M_\Lambda\|$, so this is where we will start.

In preliminary experiments, we noticed a very clear trend. The more separated the erasure set is, the better the reconstruction. By contrast, when a sequence of consecutive sampled values is erased, then a very poor reconstruction seemed inevitable. Thus, the *separation constant* will play a key role in our analysis going forward. For an erasure set $\Lambda = \{n_j\}_{j=1}^L \subset \mathbb{Z}$ the separation constant is defined as

$$\delta = \inf\{|n_j - n_k| : j \neq k \text{ and } 1 \leqslant j,k \leqslant L\}.$$

The bound on $\|M_\Lambda\|$ in the next theorem is a function of the separation constant and the erasure set size.

THEOREM 3.1. *Let δ be the separation constant for the erasure set $\Lambda = \{n_j\}_{j=1}^L$. If M_Λ is defined as in equation (5), then*

$$\|M_\Lambda\| \leqslant p + \frac{2}{\pi\delta}\left(1 + \log(L-1)\right). \tag{7}$$

Proof. Since M_Λ is a self-adjoint operator, we have the following:

$$\begin{aligned}\|M_\Lambda\| = \sup_{\|\mathbf{x}\|=1} |\langle M_\Lambda \mathbf{x}, \mathbf{x}\rangle| &= \sup_{\|\mathbf{x}\|=1} \left| \left\langle p \left[\sum_{k=1}^L x_k \operatorname{sinc}(p\pi(n_j - n_k)) \right]_{j=1}^L, (x_j)_{j=1}^L \right\rangle \right| \\ &\leqslant \sup_{\|\mathbf{x}\|=1} p \sum_{j=1}^L \sum_{k=1}^L |x_k \overline{x}_j \operatorname{sinc}(p\pi(n_j - n_k))|.\end{aligned}$$

By splitting the sum between $j = k$ and $j \neq k$ we obtain

$$\begin{aligned}\|M_\Lambda\| &\leqslant \sup_{\|\mathbf{x}\|=1} p \left(\sum_{j=1}^L |x_j|^2 \operatorname{sinc}(0) + \sum_{j\neq k} |x_k \overline{x}_j| |\operatorname{sinc}(p\pi(n_j - n_k))| \right) \\ &\leqslant \sup_{\|\mathbf{x}\|=1} p \left(1 + \sum_{j\neq k} |x_k \overline{x}_j| \frac{1}{p\pi|n_j - n_k|)} \right) \\ &\leqslant \sup_{\|\mathbf{x}\|=1} \left(p + \frac{1}{2\pi} \sum_{j\neq k} \frac{|x_k|^2 + |x_j|^2}{|n_j - n_k|} \right) \\ &= p + \frac{1}{2\pi} \sup_{\|\mathbf{x}\|=1} \sum_{j\neq k} \frac{|x_k|^2 + |x_j|^2}{|n_j - n_k|}.\end{aligned}$$

(Note the use of the inequality $2ab \leqslant a^2+b^2$ in the second to the last inequality above.)

Without loss of generality, we can assume that $\Lambda = \{n_j\}_{j=1}^{L}$ is an ordered set. Then, by arranging these points in ascending order on a number line, we can see that $|n_{j+1}-n_j| \geqslant \delta$. More generally, by the triangle inequality, we see that $|n_j - n_k| \geqslant \delta|j-k|$. So,

$$\frac{1}{\delta|j-k|} \geqslant \frac{1}{|n_j-n_k|}.$$

Thus, from above, we have

$$\|M_\Lambda\| \leqslant p + \frac{1}{2\pi} \sup_{\|\mathbf{x}\|=1} \sum_{j\neq k} \frac{|x_k|^2+|x_j|^2}{|n_j-n_k|} \leqslant p + \frac{1}{2\pi\delta} \sup_{\|\mathbf{x}\|=1} \sum_{j\neq k} \frac{|x_k|^2+|x_j|^2}{|j-k|}. \tag{8}$$

The right most sum in (8) can be seen as the sum of the off-diagonal terms of the following matrix:

$$\begin{bmatrix}
1 & \frac{|x_1|^2+|x_2|^2}{1} & \frac{|x_1|^2+|x_3|^2}{2} & \frac{|x_1|^2+|x_4|^2}{3} & \cdots & \frac{|x_1|^2+|x_L|^2}{(L-1)} \\
\frac{|x_2|^2+|x_1|^2}{1} & 1 & \frac{|x_2|^2+|x_3|^2}{1} & \frac{|x_2|^2+|x_4|^2}{2} & \cdots & \frac{|x_2|^2+|x_L|^2}{(L-2)} \\
\frac{|x_3|^2+|x_1|^2}{2} & \frac{|x_3|^2+|x_2|^2}{1} & 1 & \frac{|x_3|^2+|x_4|^2}{1} & \cdots & \frac{|x_3|^2+|x_L|^2}{(L-3)} \\
\frac{|x_4|^2+|x_1|^2}{3} & \frac{|x_4|^2+|x_2|^2}{2} & \frac{|x_4|^2+|x_3|^2}{1} & 1 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 1 & \frac{|x_{L-1}|^2+|x_L|^2}{1} \\
\frac{|x_L|^2+|x_1|^2}{(L-1)} & \frac{|x_L|^2+|x_2|^3}{(L-2)} & \frac{|x_L|^2+|x_3|^3}{(L-3)} & \cdots & \frac{|x_L|^2+|x_{L-1}|^2}{1} & 1
\end{bmatrix}$$

Since the matrix is symmetric about the main diagonal, we can just double the sum over the upper triangle (excluding the diagonal) of the matrix to get

$$\sum_{j\neq k} \frac{|x_k|^2+|x_j|^2}{|j-k|} = 2\sum_{k<j} \frac{|x_k|^2+|x_j|^2}{|j-k|}.$$

Since $|x_j|^2$ appears at most twice along the m^{th} diagonal, we can convert to a sum over the m^{th} diagonal:

$$2\sum_{k<j} \frac{|x_k|^2+|x_j|^2}{|j-k|} \leqslant 2\sum_{m=1}^{L-1} \frac{2\|\mathbf{x}\|^2}{m} = 4\|\mathbf{x}\|^2 \sum_{m=1}^{L-1} \frac{1}{m}. \tag{9}$$

Therefore, replacing the sum $\sum_{j\neq k} \frac{|x_k|^2+|x_j|^2}{|j-k|}$ in (8) with (9), we obtain

$$\|M_\Lambda\| \leqslant p + \frac{2}{\pi\delta} \sup_{\|\mathbf{x}\|=1} \|\mathbf{x}\|^2 \sum_{m=1}^{L-1} \frac{1}{m} = p + \frac{2}{\pi\delta} \sum_{m=1}^{L-1} \frac{1}{m}.$$

We complete the proof by using the integral test to bound the sum:

$$\|M_\Lambda\| \leqslant p + \frac{2}{\pi\delta}\left(1 + \int_1^{L-1} \frac{1}{x}\,dx\right) = p + \frac{2}{\pi\delta}\left(1 + \log(L-1)\right). \quad \square$$

In analyzing the bound from Theorem 3.1 we see that the norm of M_Λ is small provided $\delta = \mathscr{O}(\log L)$. If the separation constant δ is large relative to $\log L$, the operator M_Λ is well behaved.

Similar to geometric series, if $\|T\| < 1$ for a linear operator $T : \mathscr{H} \to \mathscr{H}$, then

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k,$$

with convergence in the sense of the operator norm. Such a series is called a *Neumann series*. We will use Neumann series to extend the bound in Theorem 3.1 to a bound on the term $(I-M_\Lambda)^{-1}$ in equation (6).

COROLLARY 3.2. *Let δ be the separation constant for the erasure set $\Lambda = \{n_j\}_{j=1}^{L}$. If $L < 1+\exp(\frac{\pi\delta(1-p)}{2}-1)$, then $\sum_{k=0}^{\infty} \|M_\Lambda\|^k$, converges and*

$$\|(I-M_\Lambda)^{-1}\| \leqslant \frac{\pi\delta}{\pi\delta(1-p)+2(\log(L-1)+1)}.$$

Proof. Note that $\sum_{k=0}^{\infty} \|M_\Lambda\|^k$ converges whenever $\|M_\Lambda\| < 1$. It is straightforward to show this is true provided

$$L < 1+\exp\left(\frac{\pi\delta(1-p)}{2}-1\right).$$

Applying a Neumann series gives

$$\|(I-M_\Lambda)^{-1}\| = \left\|\sum_{k=0}^{\infty} M_\Lambda^k\right\| \leqslant \sum_{k=0}^{\infty} \|M_\Lambda\|^k.$$

Using our expression for $\|M_\Lambda\|$ from Theorem 3.1 we have

$$\sum_{k=0}^{\infty} \|M_\Lambda\|^k \leqslant \frac{1}{1-\left(\frac{p\pi\delta+2\log(L-1)+2}{\pi\delta}\right)} = \frac{\pi\delta}{\pi\delta(1-p)+2(\log(L-1)+1)}. \qquad \square$$

In the next lemma, we will compute the bounds on the remaining terms in equation (6) on our way to a total bound for $\|R_\Lambda^{-1}\|$.

LEMMA 3.3. *Let $\Lambda = \{n_j\}_{j=1}^{L}$. Then,*

$$\|\Theta_{G_\Lambda}\| \leqslant \sqrt{\frac{1}{p}}, \quad \|\Theta_{F_\Lambda}^*\| \leqslant \sqrt{p}, \text{ and } \quad \|\Theta_{F_{\Lambda^c}}^*\| \leqslant \sqrt{p}.$$

Proof. Since G_p is a tight frame for $\mathscr{PW}(\pi)$ with frame bound $\frac{1}{p}$, we have

$$\begin{aligned}\|\Theta_{G_\Lambda}\|^2 &= \sup_{\|f\|=1} \|\Theta_{G_\Lambda} f\|^2 = \sup_{\|f\|=1} \|(\langle f, g_j\rangle)_{j\in\Lambda}\|^2_{\ell^2(\Lambda)} \\ &= \sup_{\|f\|=1} \sum_{j\in\Lambda} |\langle f, g_j\rangle|^2 \leqslant \sup_{\|f\|=1} \sum_{j\in\mathbb{Z}} |\langle f, g_j\rangle|^2 = \sup_{\|f\|=1} \frac{1}{p}\|f\|^2 = \frac{1}{p}.\end{aligned}$$

Thus, $\|\Theta_{G_\Lambda}\| \leqslant \sqrt{\frac{1}{p}}$.

Since $f_j = pg_j$, we have

$$\sqrt{\frac{1}{p}} \geqslant \|\Theta_{G_\Lambda}\| = \|\Theta^*_{G_\Lambda}\| = \left\|\frac{1}{p}\Theta^*_{F_\Lambda}\right\| = \frac{1}{p}\|\Theta^*_{F_\Lambda}\|. \tag{10}$$

Hence, $\|\Theta^*_{F_\Lambda}\| \leqslant p\sqrt{\frac{1}{p}} = \sqrt{p}$.

The same technique can be used to show that $\|\Theta^*_{F_{\Lambda^c}}\| \leqslant \sqrt{p}$. (Just repeat the computation with Λ^c in place of Λ.) □

In Theorem 3.4, we knit together our individual bounds from Corollary 3.2 and Lemma 3.3 to obtain an overall bound for R_Λ^{-1}.

THEOREM 3.4. *Let δ be the separation constant for the erasure set $\Lambda = \{n_j\}_{j=1}^L$. Define γ by*

$$\gamma := \pi(1-p) + \frac{2}{\delta}(\log(L-1)+1). \tag{11}$$

If $p \in (0,1)$ and $L < 1 + \exp(\frac{\pi\delta(1-p)}{2} - 1)$, then

$$\|R_\Lambda^{-1}\| \leqslant 1 + \frac{\pi}{\gamma}.$$

Proof. From the reduced direct inversion algorithm, $R_\Lambda^{-1} = I + \Theta^*_{F_\Lambda}(I - M_\Lambda)^{-1}\Theta_{G_\Lambda}$. The triangle inequality, Corollary 3.2, and Lemma 3.3 then yield

$$\begin{aligned}\|R_\Lambda^{-1}\| &\leqslant 1 + \|\Theta^*_{F_\Lambda}\|\|(I-M_\Lambda)^{-1}\|\|\Theta_{G_\Lambda}\| \\ &\leqslant 1 + \sqrt{p}\left(\frac{\pi\delta}{\pi\delta(1-p) + 2(\log(L-1)+1)}\right)\sqrt{\frac{1}{p}} \\ &= 1 + \frac{\pi\delta}{\pi\delta(1-p) + 2(\log(L-1)+1)} \\ &= 1 + \frac{\pi}{\gamma}. \quad \square\end{aligned}$$

4. Error in the N-term approximation

We now consider the error associated with the partial reconstruction, f_R, which may arise from approximations to an infinite sum. Any errors in f_R can be blown up when computing $f = R_\Lambda^{-1} f_R$, so it is important that these errors are small. The primary source of error is truncation error, since the infinite sum $f_R(t) = p\sum_{j\in\Lambda^c} f(pj)\mathrm{sinc}(\pi(t-pj))$ can only be approximated with a finite sum:

$$f_R^{(N)}(t) = p \sum_{\substack{j=-N \\ j\in\Lambda^c}}^{N} f(pj)\mathrm{sinc}(\pi(t-pj)). \tag{12}$$

In reality, since we are only considering coefficients indexed from $j=-N,\dots,N$, we use the convention $\Lambda \subset \{-N,\dots,N\}$. We define the j $^{\text{th}}$ *Fourier frame coefficient* to be

$$\widehat{c}_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\xi)e^{ipj\xi}\, d\xi. \tag{13}$$

We now provide a bound on the error in the partial reconstruction resulting from taking an N-term approximation.

LEMMA 4.1. *Let $f \in \mathscr{PW}(\pi)$ and f_R denote the partial reconstruction of the signal f. Then the error in the partial reconstruction is given by*

$$\|f_R - f_R^{(N)}\|_{L^2(\mathbb{R})} \leqslant p \sum_{|j|>N} |\widehat{c}_j|.$$

Proof. For any $t\in\mathbb{R}$ we have the following:

$$\begin{aligned}(f_R - f_R^{(N)})(t) &= p\sum_{j\in\Lambda^c} f(pj)\mathrm{sinc}(\pi(t-pj)) - p\sum_{\substack{|j|\leqslant N\\ j\in\Lambda^c}} f(pj)\mathrm{sinc}(\pi(t-pj)) \\ &= p\sum_{|j|>N} f(pj)\mathrm{sinc}(\pi(t-pj)),\end{aligned}$$

so taking L^2-norms results in

$$\begin{aligned}\|f_R - f_R^{(N)}\|_{L^2(\mathbb{R})} &\leqslant p\sum_{|j|>N} |f(pj)| \|\mathrm{sinc}(\pi(t-pj))\|_{L^2(\mathbb{R})} \\ &= p\sum_{|j|>N} \left|\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)e^{ipj\xi}\, d\xi\right| \|\mathrm{sinc}(\pi(t-pj))\|_{L^2(\mathbb{R})}.\end{aligned}$$

By equation (3),

$$\begin{aligned}\|\mathrm{sinc}(\pi(t-pj))\|_{L^2(\mathbb{R})} &= \sqrt{\langle \mathrm{sinc}(\pi(t-pj)), \mathrm{sinc}(\pi(t-pj))\rangle} \\ &= \sqrt{\mathrm{sinc}(0)} = 1.\end{aligned}$$

So,

$$\begin{aligned}\|f_R - f_R^{(N)}\|_{L^2(\mathbb{R})} &\leqslant p \sum_{|j|>N} \left| \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{ipj\xi}\, d\xi \right| \\ &= p \sum_{|j|>N} |\widehat{c}_j|. \quad \square\end{aligned}$$

Lemma 4.1 shows that the rate at which $f_R^{(N)}$ converges to f_R is proportional to the decay rate of the Fourier coefficients of $\widehat{f}$. From basic Fourier analysis, we know that this decay rate is dependent on the smoothness, or regularity of $\widehat{f}$. This relationship is formalized in Lemma 4.2 below.

LEMMA 4.2. *Suppose* $\widehat{f} \in H^s(\mathbb{R})$ *and* $f \in \mathscr{PW}(\pi)$, *then* $|\widehat{c}_j| = \mathscr{O}(|j|^{-s})$.

Proof. Since $p \in (0,1)$, spt.$(\widehat{f}) \subset [-\pi,\pi] \subset \left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$. The Fourier basis for $L^2\left[-\frac{\pi}{p}, \frac{\pi}{p}\right]$ is $\left\{\sqrt{\frac{p}{2\pi}} e^{ipk\xi}\right\}_{k\in\mathbb{Z}}$. Since $\widehat{f} \in H^s(\frac{\pi}{p})$, its Fourier coefficients

$$\begin{aligned}\widehat{c}_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f}(\xi) e^{ipk\xi}\, d\xi \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{p}}^{\frac{\pi}{p}} \widehat{f}(\xi) e^{ipk\xi}\, d\xi\end{aligned}$$

satisfy $\widehat{c}_k \in \mathscr{O}(|k|^{-s})$ by Proposition 2.1. $\square$

In Theorem 4.3, we combine Lemmas 4.1 and 4.2 to determine the rate at which the N-term approximation, $f_R^{(N)}$ converges to the true partial reconstruction, f_R.

THEOREM 4.3. *Suppose* $\widehat{f} \in H^s(\mathbb{R})$ *for* $s > 1$ *and* $f \in \mathscr{PW}(\pi)$. *Then*

$$\|f_R - f_R^{(N)}\|_{L^2(\mathbb{R})} = \mathscr{O}(|N|^{1-s}).$$

Proof. By Lemma 4.2, $|\widehat{c}_j| = \mathscr{O}(|j|^{-s})$. So we can find a constant $K > 0$ such that $|\widehat{c}_j| < \frac{K}{|j|^s}$. Using Lemma 4.1, we see that

$$\begin{aligned}\|f_R - f_R^{(N)}\| &\leqslant p \sum_{|j|>N} |\widehat{c}_j| \leqslant pK \sum_{|j|>N} \frac{1}{|j|^s} \\ &= 2pK \sum_{j=N+1}^{\infty} \frac{1}{j^s} \leqslant 2pK \int_N^{\infty} \frac{1}{x^s}\, dx \\ &= \frac{2pK}{s-1} N^{1-s}.\end{aligned}$$

Therefore, $\|f_R - f_R^{(N)}\| = \mathscr{O}(N^{1-s})$. $\square$

5. Total error in the reconstruction

In this section, we tie our error bounds together to establish overall bounds on the error resulting from the reconstruction algorithm. In doing so, we can determine when the reconstruction is feasible. We also consider estimates resulting from error introduced into the coefficients.

The first result gives an upper bound on the error in the reconstruction resulting from truncation. We recall that $f = R_\Lambda^{-1} f_R$, and define the *N-term approximation in the reconstruction* by $\widetilde{f} = R_\Lambda^{-1} f_R^{(N)}$. Also, recall that γ, as a function of p, δ, and L, is given in (11).

THEOREM 5.1. *Let $f \in \mathscr{PW}(\pi)$ and $p \in (0,1)$. If $L < 1 + \exp(\frac{\pi\delta(1-p)}{2} - 1)$ and $\widehat{f} \in H^s(\mathbb{R})$ for $s > 1$, then there exists $K > 0$ such that*

$$\|f - \widetilde{f}\| \leqslant K\left(1 + \frac{\pi}{\gamma}\right) N^{1-s}.$$

Proof. The true reconstruction can be obtained as $f = R_\Lambda^{-1} f_R$. Consider

$$\begin{aligned}\|f - \widetilde{f}\| &= \|R_\Lambda^{-1}(f_R - f_R^{(N)})\| \\ &\leqslant \|R_\Lambda^{-1}\| \|f_R - f_R^{(N)}\|.\end{aligned}$$

By Theorem 3.4,

$$\|R_\Lambda^{-1}\| \leqslant 1 + \frac{\pi}{\gamma}$$

and, by Theorem 4.3, there is some $K > 0$ such that $\|f_R - f_R^{(N)}\| \leqslant K N^{1-s}$. Combining these results, we find

$$\begin{aligned}\|f - \widetilde{f}\| &\leqslant \|R_\Lambda^{-1}\| \|f_R - f_R^{(N)}\| \\ &\leqslant K\left(1 + \frac{\pi}{\gamma}\right) N^{1-s}. \quad \square\end{aligned}$$

The previous result is an overall error estimate on the possible reconstruction $\widetilde{f}$ with the theoretically defined reconstruction f. However, it is also possible that there could be error introduced into the coefficients of the signal. In this case, we need a way to look at these more specific error terms. Recall that

$$\begin{aligned}f_R(t) &= p \sum_{j \in \Lambda^c} f(pj) \operatorname{sinc}(\pi(t - pj)) \\ &= \Theta^*_{F_{\Lambda^c}} (f(pj))_{j \in \Lambda^c}.\end{aligned}$$

Suppose the signal is subject to an error $\boldsymbol{\varepsilon} = (\varepsilon_j)_{j \in \Lambda^c}$ in the coefficients, which may be a result of quantization error or channel noise [1, 6]. Then Alice would compute $f(pj)$, and Bob would receive $f(pj) + \varepsilon_j$. So, during the reconstruction phase, $\Theta^*_{F_{\Lambda^c}} (f(pj))_{j \in \Lambda^c}$ would become $\Theta^*_{F^c_\Lambda} (f(pj) + \varepsilon_j)_{j \in \Lambda^c}$.

Now, let us return to the reconstruction, considering error in the coefficients. Define $\Delta = R_\Lambda^{-1}\Theta^*_{F_{\Lambda^c}}$ so that our reconstruction would be $\Delta(f(pj)+\varepsilon_j)_{j\in\Lambda^c}$. Since Δ is a linear operator,

$$\begin{aligned}\Delta(f(pj)+\varepsilon_j)_{j\in\Lambda^c} &= \Delta(f(pj))_{j\in\Lambda^c} + \Delta(\varepsilon_j)_{j\in\Lambda^c}\\ &= \Delta(f(pj))_{j\in\Lambda^c} + \Delta\boldsymbol{\varepsilon}\\ &= f + \Delta\boldsymbol{\varepsilon}.\end{aligned}$$

Thus, the error in the reconstruction is $\Delta\boldsymbol{\varepsilon}$. The next result gives a bound on the coefficient error in the reconstruction, given by $\|\Delta\boldsymbol{\varepsilon}\|$.

COROLLARY 5.2. *Let $f \in \mathscr{PW}(\pi)$ and $p \in (0,1)$. If $L < 1+\exp(\frac{\pi\delta(1-p)}{2}-1)$ and $\widehat{f} \in H^s(\mathbb{R})$ for $s > 1$, then*

$$\|\Delta\boldsymbol{\varepsilon}\| \leqslant \left(1+\frac{\pi}{\gamma}\right)\sqrt{p}\,\|\boldsymbol{\varepsilon}\|.$$

Proof. From Lemma 3.3, $\|\Theta^*_{F_{\Lambda^c}}\| \leqslant \sqrt{p}$. Therefore, by Theorem 3.4,

$$\begin{aligned}\|\Delta\boldsymbol{\varepsilon}\| &\leqslant \|R_\Lambda^{-1}\|\|\Theta^*_{F_{\Lambda^c}}\|\|\boldsymbol{\varepsilon}\|\\ &\leqslant \left(1+\frac{\pi}{\gamma}\right)\sqrt{p}\,\|\boldsymbol{\varepsilon}\|. \quad \square\end{aligned}$$

Though theoretically defined, in order to actually compute the reconstruction for applications, we again need an N-term approximation. Let $\Omega = \Lambda^c \cap \{-N,\dots,N\}$ denote the N-term approximation of Λ^c. Define $\Delta_\Omega = R_\Lambda^{-1}\Theta^*_{F_\Omega}$ and $\boldsymbol{\varepsilon}_\Omega = (\varepsilon_j)_{j\in\Omega}$. Then the N-term approximation to the reconstruction, with coefficient error, is given by

$$\widetilde{\widetilde{f}} = \Delta_\Omega(f(pj)+\varepsilon_j)_{j\in\Omega}.$$

The following theorem presents a final bound for the norm of the error associated with this reconstruction.

THEOREM 5.3. *Let $f \in \mathscr{PW}(\pi)$ and $p \in (0,1)$. Suppose $L < 1+\exp(\frac{\pi\delta(1-p)}{2}-1)$ and $\widehat{f} \in H^s(\mathbb{R})$ for $s > 1$. Then there exists $K > 0$ such that*

$$\|f-\widetilde{\widetilde{f}}\| \leqslant \left(1+\frac{\pi}{\gamma}\right)(KN^{1-s}+\sqrt{p}\|\boldsymbol{\varepsilon}_\Omega\|).$$

Proof. Since $\Delta_\Omega = R_\Lambda^{-1}\Theta^*_{F_\Omega}$, we may write $\widetilde{f} = R_\Lambda^{-1}f_R^{(N)} = \Delta_\Omega(f(pj))_{j\in\Omega}$. Therefore,

$$\begin{aligned}\|f-\widetilde{\widetilde{f}}\| &\leqslant \|f-\widetilde{f}\| + \|\widetilde{f}-\widetilde{\widetilde{f}}\|\\ &= \|f-\widetilde{f}\| + \|\Delta_\Omega(f(pj))_{j\in\Omega} - \Delta_\Omega(f(pj)+\varepsilon_j)_{j\in\Omega}\|\\ &= \|f-\widetilde{f}\| + \|\Delta_\Omega\boldsymbol{\varepsilon}_\Omega\|.\end{aligned}$$

Since $\Omega \subseteq \Lambda^c$, we can use Corollary 5.2 to conclude that

$$\|\Delta_\Omega \boldsymbol{\varepsilon}_\Omega\| \leqslant \left(1+\frac{\pi}{\gamma}\right)\sqrt{p}\,\|\boldsymbol{\varepsilon}_\Omega\|.$$

Also, by Theorem 5.1,

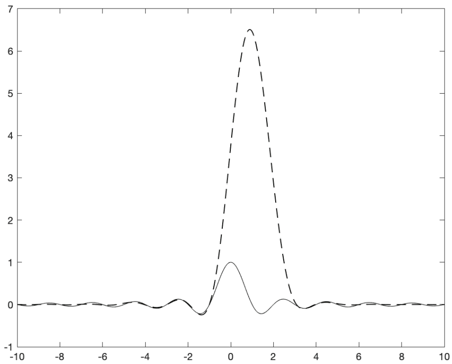
$$\|f-\widetilde{f}\| \leqslant K\left(1+\frac{\pi}{\gamma}\right)N^{1-s}.$$

Putting this together, we obtain the desired result:

$$\|f-\widetilde{\widetilde{f}}\| \leqslant \left(1+\frac{\pi}{\gamma}\right)(KN^{1-s}+\sqrt{p}\|\boldsymbol{\varepsilon}_\Omega\|). \quad \square$$

6. Numerical experiments and concluding remarks

The theoretical results obtained in this work allow us to make some important observations relating to feasibility of reconstructions. We see that a large separation in erasures (δ) tells us that smaller partial sum numbers (N) will suffice in order to obtain a good reconstruction. However, small separations in erasures requires the partial sum number to be large in order to obtain a good reconstruction. These observations are made clear in the next three figures.



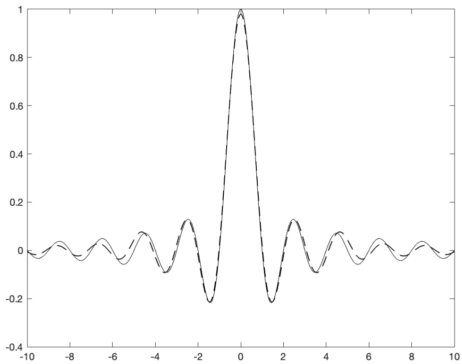


Figure 1: *Reduced direct inversion algorithm for* $f(x) = sinc(\pi x)$ *(pictured in black solid line) with* $p = \frac{1}{2}$ *and* $N = 10$*. For the left figure (consecutive erasures),* $\delta = 1$ *and* $\Lambda = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\}$*. For the right figure (erase every other sampled value),* $\delta = 2$ *and* $\Lambda = \{0,1,2,3,4,5,6,7\}$*. The dashed line graph represents the reconstruction from erasures. We see that the larger* δ *gives a better reconstruction.*

In the next experiment, we will run the same tests on a function that has a Fourier transform with a higher degree of regularity. It is not too difficult to show that the Fourier transform of

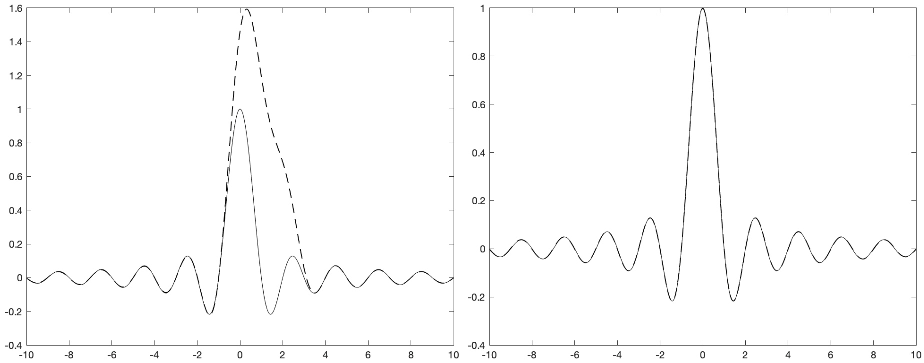


Figure 2: *Reduced direct inversion algorithm for* $f(x) = sinc(\pi x)$ *with* $p = \frac{1}{2}$ *and* $N = 100$*. For the left figure (consecutive erasures),* $\delta = 1$ *and* $\Lambda = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\}$*. For the right figure (erase every other sampled value),* $\delta = 2$ *and* $\Lambda = \{0,1,2,3,4,5,6,7\}$*. For increased* N*, a better reconstruction is obtained.*

$$\tau(x) = \begin{cases} \pi - |x| & -\pi \leqslant x \leqslant \pi \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

is $\hat{\tau}(\xi) = -\pi^2 \mathrm{sinc}^2\left(\frac{\pi\xi}{2}\right)$. Using this and the identity $\hat{\hat{f}}(x) = 2\pi f(-x)$, we see that the Fourier transform of $h(x) = \mathrm{sinc}^2\left(\frac{\pi x}{2}\right)$ is $-\frac{2}{\pi}\tau(\xi)$. That is, $h(x) \in \mathscr{PW}(\pi)$, and $\hat{h}(\xi)$ is of a higher regularity than the function from the previous example since it is continuous.

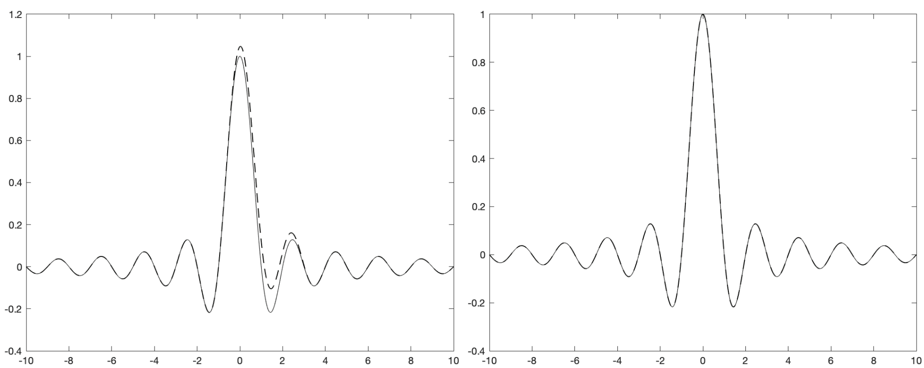


Figure 3: *Reduced direct inversion algorithm for* $f(x) = sinc(\pi x)$ *with* $p = \frac{1}{2}$ *and* $N =$ *1,000. For the left figure (consecutive erasures),* $\delta = 1$ *and* $\Lambda = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\}$*. For the right figure (erase every other sampled value),* $\delta = 2$ *and* $\Lambda = \{0,1,2,3,4,5,6,7\}$.

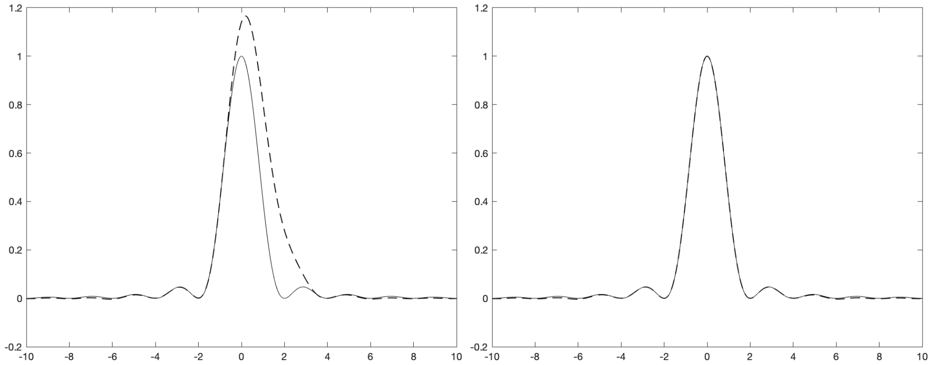


Figure 4: *This is a repeat of the experiment from Figure 1 with* $h(x) = sinc^2\left(\frac{\pi x}{2}\right)$*. Again, a larger* δ *gives a better reconstruction. Moreover, we see that a smaller* N *value is less detrimental when dealing with a function of a higher regularity.*

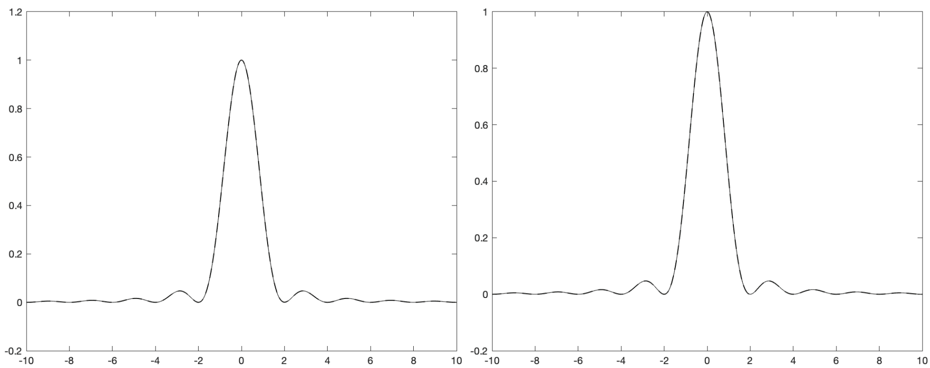


Figure 5: *This is a repeat of the experiment from Figure 2 with* $h(x) = sinc^2\left(\frac{\pi x}{2}\right)$*. Again we see that a more regular function,* $h(x)$ *requires a smaller* N *value to acheive a good reconstruction. In both figures, we see very little difference in dashed line (reconstructed signal) and black solid line (true signal) graphs meaning that the reconstruction process is highly stable. This should be contrasted with the left image in Figure 3 where even* $N = 1,000$ *is insufficient to obtain a good reconstruction when the separation constant* δ *is small.*

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