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THE STRONG LIMITED p–SCHUR PROPERTY IN BANACH LATTICES

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Abstract. The concept of the strong limited p-Schur property ($1 \leqslant p \leqslant \infty$); that is, spaces on which every weakly p-compact and almost limited set is relatively compact is introduced and studied. Next, the weak DP* property of order p is defined and spaces with this property are characterized. As an application of these results, by the class of disjoint p-convergent operators, some characterizations of Banach lattices with the weak DP* property of order p are given.

1. Introduction and preliminaries

Throughout this paper X will denote a Banach space, E will denote a Banach lattice, $E^+ = \{x \in E : x \geqslant 0\}$ refers to the positive cone of E, $B_E :=$ is the closed unit ball of E and the solid hull of a subset A of E is the set $sol(A) = \{y \in E : |y| \leqslant |x|$, for some $x \in A\}$.

The concept of limited sets has been widely studied by different authors and some characterizations have been given. If A is a norm bounded subset of a Banach space X and for each weak* null sequence (x_n^*) in X^*,

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0,$$

then we say that A is *limited* and Banach spaces whose limited sets are relatively compact are called Gelfand-Phillips (GP) spaces. A Banach space X has the DP* property if each relatively weakly compact set in X is limited [8, 10, 3].

A norm bounded subset A of a Banach lattice E is said to be an *almost limited* set, if every disjoint weak* null sequence (x_n^*) in E^* converges uniformly to zero on A. A Banach lattice E has the weak DP* property if all relatively weakly compact subsets are almost limited. The weak DP* property in terms of disjoint sequences is characterized in [5].

According to the definition of almost limited sets, the stronger version of GP property is considered and the class of Banach lattices with the *strong GP property* is introduced which is shared by those Banach lattices whose almost limited subsets are

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relatively compact. The reader is referred to [2] for the definition and an extensive discussion of the strong GP property in Banach lattices.

For each $1 \leqslant p < \infty$, a sequence (x_n) of a Banach space X is called *weakly p-summable* if for each $x^* \in X^*$, $(x^*(x_n)) \in \ell_p$ and (x_n) is said to be *weakly p-convergent* to an $x \in X$ if the sequence $(x_n - x) \in \ell_p^w(X)$, where $\ell_p^w(X)$ is the space of all weakly p-summable sequences in X. The weakly ∞-convergent sequences are simply the weakly convergent sequences. A bounded set A in a Banach space X is called *relatively weakly p-compact*, $1 \leqslant p \leqslant \infty$, if each sequence in A has a weakly p-convergent subsequence. If the limit point of each weakly p-convergent subsequence is in A, then we call A a weakly p-compact set. Also a Banach space X is *weakly p-compact*, $1 \leqslant p \leqslant \infty$, if the closed unit ball B_X is a weakly p-compact set. The reader can find some useful and additional properties about these concepts in [4].

Recently in [6], the concept of the *limited p-Schur property*; that is, Banach spaces whose all limited weakly p-compact subsets are relatively compact $(1 \leqslant p \leqslant \infty)$ is introduced and some conditions under which some operator spaces have the limited p-Schur property is established. Clearly, each Banach space with the GP property has the limited p-Schur property.

In the first part of this note, we introduce a stronger form of the limited p-Schur property for Banach lattices (Definition 2.1) and characterize the class of spaces with this property. Then, using weakly p-summable sequence techniques we consider the positive Schur property of order p (i.e., the p-positive Schur property) in Banach lattices and investigate spaces in which p-positive Schur property is equivalent to the positive Schur property. A Banach lattice E has the *positive Schur property* if each positive weakly null sequence in E is norm null [17]. Finally, we introduce the weak DP* property of order p (i.e. p-weak DP* property), which is a generalization of the classical weak DP* property and by a class of disjoint p-convergent operators, research Banach lattices with the p-weak DP* property. The reader should see [5, 17] for the weak DP* and positive Schur properties in Banach lattices. Throughout the paper, p' denotes the conjugate number of p for $1 < p < \infty$; if $p = 1$, $\ell_{p'}$ plays the role of c_0. We refer the reader to references [1, 15] for the theory of operators and Banach lattices.

2. Strong limited p-Schur property

Recently, the notion of the p-Schur property $(1 \leqslant p < \infty)$ as a generalization of the Schur property is introduced. In fact, a Banach space X has the *p-Schur property* if every weakly p-summable sequence in X is norm null (i.e., converges to zero). Moreover, it has been shown that X has the 1-Schur property if and only if X contains no copy of c_0. As we said before, a Banach space X has the limited p-Schur property if each weakly p-compact limited set in X is relatively compact; or equivalently, every limited sequence $(x_n) \in \ell_p^w(X)$ is norm null. Every Banach space with the Schur property has the p-Schur and so the limited p-Schur prperty, but the converse is false. As an example, for $1 < p < \infty$ and $1 < q < p'$, all ℓ_q spaces have the p-Schur property, but none of them have the Schur property. Also, c_0 has the limited p-Schur property, but it does not have the p-Schur property [20, 7, 6]. Throughout this article we assume that $1 \leqslant p < \infty$, unless otherwise stated.

DEFINITION 2.1. A Banach lattice E has the *strong limited p-Schur property* if each almost limited weakly p-compact subset of E is relatively compact.

It can be easily shown that, a Banach lattice E has the strong limited p-Schur property if and only if each almost limited sequence $(x_n) \in \ell_p^w(E)$ is norm null. It is easy to see that, if $1 \leqslant p < q$ and E has the strong limited q-Schur property, then E has the strong limited p-Schur property. Notice that, the strong limited p-Schur property implies the limited p-Schur property, but the following example shows that the converse is false, in general.

EXAMPLE 2.2. $L^1[0,1]$ has the limited p-Schur property, but it does not have the strong limited p-Schur property, for $p \geqslant 2$.

Proof. Since $L^1[0,1]$ is separable, then it has the limited p-Schur property. But $L^1[0,1]$ does not have the strong limited p-Schur property, for $p \geqslant 2$. In fact, the *Rademacher sequences* (r_n) in $L^1[0,1]$ are weakly 2-summable (see [16, Proposition 3.6]) and almost limited, by the weak DP* property, but $\|r_n\| = 1$ for all n. □

Actually, $L^1[0,1]$ has the 1-Schur property, since it contains no copy of c_0.

Each Banach lattice with the strong GP property, such as the classical Banach lattices c_0, ℓ_p, Schur spaces and more generally discrete Banach lattices with order continuous norm, have the strong limited p- Schur property (see [2, Theorem 2.1]). The following example shows that the converse is not true, in general.

EXAMPLE 2.3. For each compact metric space K, $C(K)$ has the strong limited p-Schur property, but it does not have the strong GP property.

Proof. It is a well-known fact that every weakly p-summable sequence is weakly null. By [2, Theorem 2.12], every weakly null almost limited sequence in $C(K)$ is norm null, which implies that $C(K)$ has the strong limited p-Schur property. But $C(K)$ does not have the strong GP property. Indeed, there is an almost limited *Rademacher sequences* equivalent to the unit vector basis of ℓ_1 in $C(K)$ which is not relatively compact. □

Let us recall that an element $e \in E$ is a *weak unit* if $E = B_e$, where B_e is the band generated by e. For example, $C[0,1]$ has the weak unit $u(t) = t$, but $M[0,1]$ and ℓ_∞^*, do not have any weak unit.

PROPOSITION 2.4. *If E has order continuous norm or E^* has a weak unit, then the strong limited Schur (i.e., the strong limited ∞-Schur) and strong GP properties are equivalent.*

Proof. It is clear that the strong GP property implies the strong limited Schur property. For the converse, by the strong limited Schur property, each almost limited weakly null sequence in E is norm null. Then by hypothesis on E and [2, Theorem 2.15], E has the strong GP property. □

Note that, each discrete Banach lattice with order continuous norm has the strong limited p-Schur property, but the converse is not true. Indeed, discrete Banach lattice c has the strong limited p-Schur property, but it does not have order continuous norm (see [2, Theorem 2.4]). Now for the proof of the Theorem 2.7, we need the following two lemmas.

LEMMA 2.5. *Continuous linear image of an almost limited set in a Banach lattice E under a positive linear projection is almost limited.*

Proof. Assume that U is a closed sublattice of a Banach lattice E, P is a linear positive projection of E onto U and A is an almost limited set in E. To finish the proof, we have to show that $P(A)$ is an almost limited set in U; that is, for each disjoint weak* null sequence (f_n) in U^*, $\sup_{x\in A}|\langle f_n, Px\rangle| \to 0$.

By [15, 1.4. E4], P^* is a lattice homomorphism and so $(P^* f_n)$ is a disjoint weak* null sequence in E^*. Since A is an almost limited set in E, then $\sup_{x\in A}|\langle f_n, Px\rangle| = \sup_{x\in A}|\langle P^* f_n, x\rangle| \to 0$, as $n\to\infty$ and so $P(A)$ is an almost limited set. □

LEMMA 2.6. *Let Y be a complemented subspace of a Banach space X and $A\subseteq Y$ is a limited set in X. Then A is a limited set in Y.*

Proof. Since Y is a complemented subspace of X, then there is an onto positive projection $P: X\to Y$. If (x_n^*) is a weak* null sequence in Y^*, then we can also extend each x_n^* to the whole of X by putting $u_n^* = x_n^*\ o\ P$. It is clear that (u_n^*) is a weak* null sequence in X^*. Since A is a limited set in X, then

$$\sup_{a\in A}|\langle x_n^*, a\rangle| = \sup_{a\in A}|\langle x_n^*\ o\ P, a\rangle| = \sup_{a\in A}|\langle u_n^*, a\rangle| \to 0. \quad \square$$

It is a well-known result that a Banach space X has the Schur (resp. p-Schur) property if and only if each closed separable linear subspace of X has the Schur (resp. p-Schur) property.

THEOREM 2.7. *For each $1\leqslant p\leqslant\infty$, the following are equivalent:*

(a) E has the strong limited p-Schur property,

(b) every closed separable sublattice of E is contained in a complemented sublattice Z of E with the strong limited p-Schur property,

(c) E is the direct sum of two spaces with the strong limited p-Schur property.

Proof. $(a)\Rightarrow(b)$. It is obvious. In fact, the strong limited p-Schur property is inherited by closed sublattices.

$(b)\Rightarrow(a)$. Suppose that A is a weakly p-compact almost limited subset of E and that $(x_n)\subseteq A$. Then there is a subsequence (x_{n_k}) of (x_n) that is almost limited weakly p-convergent to some $x\in A$. Consider the closed linear span of (x_n). By hypothesis, it is contained in a complemented sublattice Z of E with the strong limited p-Schur property. By Lemma 2.6, (x_n) is an almost limited sequence in Z. Since Z has the strong limited p-Schur property, then (x_{n_k}) is norm convergent to x.

$(a) \Rightarrow (c)$. Consider $E = E \oplus \{0\}$.

$(c) \Rightarrow (a)$. Let $E = Y \oplus Z$ such that Y and Z have the strong limited p-Schur property. Consider the positive linear projections $P_Y : E \to Y$ and $P_Z : E \to Z$ and assume that A is a weakly p-compact almost limited subset of E. So by Lemma 2.5, two sets $P_Y(A)$ and $P_Z(A)$ are weakly p-compact almost limited. Then for each sequence $(x_n) \subseteq A$ there is $y_n \in P_Y(A)$ and $z_n \in P_Z(A)$ such that $x_n = y_n + z_n$. Since $P_Y(A)$ and $P_Z(A)$ have the strong limited p-Schur property, so the sequences (y_n) and (z_n) have convergent subsequences (y_{n_k}) and (z_{n_k}); that is, there are $y \in P_Y(A)$ and $z \in P_Z(A)$ such that $y_{n_k} \to y$ and $z_{n_k} \to z$. Since A is a weakly p-compact set, then $x_{n_k} \to y + z \in A$. Hence A is compact and so E has the strong limited p-Schur property. □

Note that the complemented ness of the lemma 2.6 cannot be removed. In fact, the unit vector basis (e_n) of c_0, as a closed separable sublattice of ℓ_∞, is limited in ℓ_∞, but it is not limited (almost limited) in c_0.

It can be easily shown that, a Banach space X has the p-Schur property if and only if X has the limited p-Schur and DP* properties. So one can conclude that, ℓ_∞ does not have the limited p-Schur property. In fact, ℓ_∞ has the DP* property, but it does not have the p-Schur property. So, we can say more:

COROLLARY 2.8. *If a Banach space X contains a sublattice isomorphic to ℓ_∞, then X does not have the limited p-Schur property.*

For each σ–Dedekind complete Banach lattice, we have the following theorem.

THEOREM 2.9. *If E is a σ–Dedekind complete Banach lattice, then the following are equivalent:*

(a) E has the strong GP property,

(b) E is discrete with the strong limited p-Schur property,

(c) E is discrete with the limited p-Schur property.

Proof. $(a) \Rightarrow (b)$. It is clear that E has the strong limited p-Schur property. Also from [2, Theorem 2.3], E is discrete.

$(b) \Rightarrow (c)$. It is clear.

$(c) \Rightarrow (a)$. Since E has the limited p-Schur property, then by Corollary 2.8, E contains no sublattice isomorphic to ℓ_∞ and so E has order continuous norm [15, Corollary 2.4.3]. Also E is discrete and then by [2, Theorem 2.1], E has the strong GP property. □

Each weakly p-compact set is weakly compact, but the converse is not true. In fact, non-discrete Banach lattice $L^1[0,1]$ has the 1-Schur property and so there is an order interval in $L^1[0,1]$ which is not weakly 1-compact (see [1, Proposition 3.6]). But $L^1[0,1]$ has order continuous norm and so each order interval in $L^1[0,1]$ is weakly compact.

LEMMA 2.10. *Every order bounded disjoint sequence in a Banach lattice E is weakly p-summable, for all $1 \leqslant p \leqslant \infty$.*

Proof. It is useful to know that, every order bounded disjoint sequence in a Banach lattice E converges weakly to zero (see [1, p. 192]). Actually, it has been shown that, this sequence is weakly 1-summable. Since $\ell_p^w(X) \subset \ell_q^w(X)$ for all $1 \leqslant p \leqslant q$, then every weakly 1-summable sequence is weakly p-summable and this finishes the proof. □

It is proved in [18] that, a σ–Dedekind complete Banach lattice has order continuous norm if and only if it has the GP property. In the following theorem, we obtain the same result with the limited p-Schur property that is weaker than the GP property. Recall that a Banach lattice E has the *property* (d) if the sequence $(|f_n|)$ is weak* null for every disjoint weak* null sequence (f_n) in E^*. Cleraly, each σ–Dedekind complete Banach lattice has the property (d), but the converse is false, in general. In fact, ℓ_∞/c_0 has the property (d), but it is not σ–Dedekind complete [14].

THEOREM 2.11. *Let E be a Banach lattice. Then for the following assertions:*

(a) E has the property (d) and the strong limited p-Schur property,

(b) E has order continuous norm,

(c) E is σ–Dedekind complete with the limited p-Schur property,

the implications $(a) \Rightarrow (b) \Leftrightarrow (c)$ are valid.

Proof. $(a) \Rightarrow (b)$. Let (x_n) be an order bounded disjoint sequence in E. Then (x_n) is weakly p-summable and almost limited by the property (d) (see [14, Proposition 2.1]). Also E has the strong limited p-Schur property and then (x_n) is norm null. Hence by [1, Theorem 4.14], E has order continuous norm.

$(b) \Rightarrow (c)$. Each Banach lattice with order continuous norm has the GP (and so the limited p-Schur) property. It is clear that E is σ–Dedekind complete.

$(c) \Rightarrow (b)$. Let (x_n) be an order bounded disjoint sequence in E. Then (x_n) is weakly p-summable and limited in a σ–Dedekind complete Banach lattice E (see [12, Lemma 3.7]). Also E has the limited p-Schur property and then (x_n) is norm null. Hence by [1, Theorem 4.14], the norm of E is order continuous. □

It can be concluded that, a σ–Dedekind complete Banach lattice E has the limited p-Schur property if and only if E has the GP property. If E is discrete, then all the statements of the Theorem 2.11 are equivalent. From Corollary 2.8, we have the following two corollaries:

COROLLARY 2.12. *If a Banach lattice E contains a sublattice isomorphic to ℓ_1, then its dual E^* does not have the limited p-Schur property.*

Proof. If E contains a sublattice isomorphic to ℓ_1, then by [15, Proposition 2.3.12], E^* contains a sublattice isomorphic to ℓ_∞ and so E^* does not have the limited p-Schur property. $\square$

COROLLARY 2.13. *If a Banach space X contains a complemented copy of ℓ_1, then X^* does not have the limited p-Schur property.*

Proof. If X contains a complemented copy of ℓ_1, then by [8, Theorem 10], X^* contains an isomorphic copy of ℓ_∞ and so X^* does not have the limited p-Schur property. $\square$

Note that the converse of Corollary 2.13 is true for Banach lattices. In fact, By [8, Theorem 10], if a Banach lattice E contains no complemented copy of ℓ_1, then E^* contains no copy of c_0. By [15, Theorem 2.5.6], E^* is a KB-space and so it has order contiunous norm. Hence E^* has the limited p-Schur property.

PROPOSITION 2.14. *If a Banach space X contains no copy of ℓ_1, then X^* has the 1-Schur property.*

Proof. If X contains no copy of ℓ_1, then by [8, Theorem 10], X^* contains no copy of c_0 and so by [20, Proposition 3.2.3], X^* has the 1-Schur property. $\square$

PROPOSITION 2.15. *For each Banach lattice E, the following assertions hold:*

(a) If E has the p-Schur property, then it has the GP property.

(b) If E is discrete with the p-Schur property, then it has the strong GP property.

Proof. To prove that (a), it suffice to note that if E has the p-Schur property, then E contains no copy of c_0; that is, E is a KB-space. Hence E has order continuous norm and so it has the GP property.

The second part can be deduced from [2, Theorem 2.3]. In fact, in this case E is a discrete KB-space. $\square$

3. p-positive Schur and p-weak DP* properties

A Banach lattice E has the positive Schur property if each positive weakly null sequence in E is norm null [17, 18]. It has been shown, for discrete Banach lattices and also for Banach lattices with the strong GP property the positive Schur and Schur properties are equivalent [18, 2].

Using weakly p-summable sequences, the notion of the p-positive Schur property has been introduced. A Banach lattice E has the *p-positive Schur property* if every sequence $(x_n) \in \ell_p^w(E)_+$ is norm null, alternatively, E has the p-positive Schur property if and only if every disjoint sequence $(x_n) \in \ell_p^w(E)_+$ is norm null.

Clearly, the positive Schur property implies the p-positive Schur property, but the converse is not true. In fact, ℓ_p spaces $(1 < p < \infty)$, weakly sequentially complete (wsc) Banach lattices and reflexive spaces do not have the positive Schur property, but all of them have the 1-Schur property [20].

In the following theorem, we show that the positive Schur and p-positive Schur properties coincide in each Banach lattice with the finite type. In Chapter 16 of [9] one finds a discussion of type and cotype in Banach lattices. Moreover, we introduce and study the concept of the p-weak DP* property and then by disjoint p-convergent operators, characterize some Banach lattices with the p-weak DP* property.

THEOREM 3.1. *Let E be a Banach lattice with the type q (with $1 < q \leqslant 2$) and let $p \geqslant q'$. Then the following are equivalent:*

(a) E has the p-positive Schur property,

(b) E has the positive Schur property.

Proof. $(a) \Rightarrow (b)$. From [15, Corollary 3.6.8], it is enough to show that every relatively weakly compact set A in E is an L-weakly compact set (i.e., $\|x_n\| \to 0$ for every disjoint sequence (x_n) in the $sol(A)$). For this, let (x_n) be a disjoint sequence in $sol(A)$. Then by [20, Lemma 4.2.1, Proposition 3.1.5], two sequences (x_n) and $(|x_n|)$ are weakly p-summable. Hence by the p-positive Schur property of E, the sequence $(|x_n|)$ (and hence (x_n) itself) is norm null. Therefore A is an L-weakly compact set, as desired.

$(b) \Rightarrow (a)$. It is evident. □

A Banach lattice E is *weak p-consistent* if it follows from $(x_n) \in \ell_p^w(E)$ that $(|x_n|) \in \ell_p^w(E)$. If E is a weak p-consistent Banach lattice (for instance an AM-space with unit), then E has the p-positive Schur property if and only if E has the p-Schur property [20]. In the following theorem, we show that the same result holds for some Banach lattices with the strong limited p-Schur property, too.

THEOREM 3.2. *Let E be a Banach lattice with the type q (with $1 < q \leqslant 2$) and let $p \geqslant q'$. Then the following are equivalent:*

(a) E has the p-Schur property,

(b) E has the strong limited p-Schur and p-positive Schur properties.

Proof. $(a) \Rightarrow (b)$. It is obvious.

$(b) \Rightarrow (a)$. Let A be a weakly p-compact set in E. Clearly A is a relatively weakly compact set. Since E has the p-positive Schur property, so by Theorem 3.1 A is L-weakly compact. Then from [5, Theorem 2.6], A is an almost limited set in E. On the other hand, E has the strong limited p-Schur property and then A is relatively compact. Hence E has the p-Schur property. □

PROPOSITION 3.3. *Every Banach lattice E with the p-positive Schur property has the 1-Schur property.*

Proof. Note that c_0 does not have the p-positive Schur property. So if a Banach lattice E has the p-positive Schur property, then E contains no copy of c_0 and by [20, Proposition 3.2.3], E has the 1-Schur property. □

Similar to the DP* property, the so-called *weak DP* property* of a Banach lattice is introduced in [5]. In fact, a Banach lattice E has the weak DP* property if all relatively weakly compact subsets are almost limited. In other words, E has the weak DP* property if and only if for every weakly null sequence (x_n) in E and every disjoint weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$.

THEOREM 3.4. *If E^* has the limited p-Schur property and a weak unit, Y is a Banach space and $T : E \to Y$ is an operator, then we have:*

(a) If Y is wsc, then T is a weakly compact operator.

(b) If Y has the Schur property, then T is a compact operator.

(c) If Y has the DP* property, then T is a limited operator.

(d) If Y has the weak DP* property, then T is an almost limited operator.

Proof. Since E^* has the limited p-Schur property, then E^* does not contain any sublattice isomorphic to ℓ_∞ and by [15, Theorem 2.4.14] E^* is a KB-space. On the other hand, E^* has a weak unit and so by [15, 2.5.E1] the closed unit ball B_E is weakly conditionally compact. Hence each of the statements of the theorem can be concluded easily. □

From [11], a Banach space X is said to have the *DP* property of order p* or briefly the *p-DP* property*, whenever every weakly p-compact set in X is limited. So, it is natural to study the Banach lattices E satisfying the p-weak DP* property. Thereby, we have a scale of properties, in the sense that all Banach lattices have the 1-weak DP* property and if $p < q$ and E has the q-weak DP* property then it has the p-weak DP* property. The strongest property, the ∞-weak DP* property coincides with the weak DP* property. We characterize some Banach lattices with the p-weak DP* property, discuss some examples and then consider some applications of disjoint p-convergent operators on Banach lattices.

DEFINITION 3.5. A Banach lattice E has the *p-weak DP* property* if all weakly p-compact subsets are almost limited.

It can be easily shown that, each Banach lattice with the positive Schur property is a KB-space with the p-weak DP* property. Also, the p-weak DP* property is equivalent to, for every sequence $(x_n) \in \ell_p^w(E)$ and every disjoint weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$. The weak DP* property implies the p-weak DP* property, but the converse is false.

THEOREM 3.6. *Every discrete Banach lattice E with the p-Schur and without the Schur property has the p-weak DP* property, but it does not have the weak DP* property.*

Proof. If E has the p-Schur property, then it is clear that E has the p-weak DP* property. Moreover, E contains no copy of c_0; that is, E is a KB-space. Also E is discrete and so E has the strong GP property. But E does not have the weak DP* property, since E does not have the Schur property [2]. □

For examples, for each $1 < p < \infty$ and $1 < q < p'$, all the spaces ℓ_q have the p-Schur (and so the p-weak DP*) and strong GP properties. But, none of them have the weak DP* property (see [2, Corollary 2.7]). Of course, none of the spaces ℓ_q have the Schur property.

The following result is easily verified:

PROPOSITION 3.7. *For a Banach lattice E, the following assertinos are equivalent:*

(a) E has the p-DP* and limited p-Schur properties,

(b) E has the p-weak DP* and strong limited p-Schur properties,

(c) E has the p-Schur property.

THEOREM 3.8. *Let E be a σ–Dedekind complete Banach lattice with the type q (with $1 < q \leqslant 2$) and let $p \geqslant q'$. Then the following are equivalent:*

(a) E has the p-weak DP* property,

(b) for every disjoint sequence $(x_n) \in \ell_p^w(E)$ and every disjoint weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$,

(c) for every disjoint sequence $(x_n) \in \ell_p^w(E)_+$ and every disjoint weak* null sequence (x_n^*) in E_+^*, $x_n^*(x_n) \to 0$,

(d) the solid hull of a weakly p-compact set in E is almost limited.

Proof. We only prove $(c) \Rightarrow (d)$. Let W be a weakly p-compact set in E and let $B := sol(W)$. Then by [20, Lemma 4.2.1] each disjoint sequence in B is weakly p-summable. So by hypothesis, for every disjoint sequence (x_n) in $B \cap E^+$ and every disjoint weak* null sequence (x_n^*) in E_+^*, $x_n^*(x_n) \to 0$. Hence by [5, Theorem 2.5], B is almost limited. □

Now, we define the p-bi-sequence property and then we characterise Banach lattices with this property. Also, we provide an example of a Banach lattice on which the p-weak DP* and p-bi-sequence properties are equivalent.

DEFINITION 3.9. A Banach lattice E has the *p-bi-sequence property* if for every disjoint sequence $(x_n) \in \ell_p^w(E)_+$ and every weak* null sequence (x_n^*) in E_+^*, $x_n^*(x_n) \to 0$.

If a Banach lattice has the p-weak DP* property, then it has the p-bi-sequence property, but the converse is false in general. In fact, Banach lattice c of all convergent sequence of scalars has the bi-sequence property, but it does not have the p-weak DP* property. Note that, c is a Banach lattice with the strong limited p-Schur proeprty and without the p-Schur property and so by Theorem 3.7, it does not have the p-weak DP* property.

THEOREM 3.10. *Let E be a Banach lattice with the type q (with $1 < q \leqslant 2$) and let $p \geqslant q'$. Then these are equivalent:*

(a) for each disjoint sequence $(x_n) \in \ell_p^w(E)$ and every disjoint weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$,

(b) E has the p-bi-sequence property,

(c) for every sequence $(x_n) \in \ell_p^w(E)_+$ and every disjoint weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$,

(d) for every sequence $(x_n) \in \ell_p^w(E)_+$ and every weak* null sequence (x_n^*) in E_+^*, $x_n^*(x_n) \to 0$.

Proof. $(a) \Rightarrow (b)$. It follows from [19, Proposition 2.4].

$(b) \Rightarrow (d)$. Let $(x_n) \in \ell_p^w(E)$ be a positive sequence and let (x_n^*) be a positive weak* null sequence of E^*. Then the set $A = \{x_n : n \in \mathbb{N}\} \cup \{0\}$ is weakly p-compact and so it is relatively weakly compact. Hence by [20, Lemma 4.2.1], every disjoint sequence (y_n) in $sol(A)$ is weakly p-summable. Since E has the p-bi-sequence property, then $\sup_{k \in A} |\langle x_k^*, y_n \rangle| \to 0$ and so for an operator $T : E \to c_0$ defined by

$$Tx = (\langle x, x_n^* \rangle), \qquad x \in E,$$

we have $\|Ty_n\| \to 0$. Now by the same method in [19, Proposition 2.4], we conclude that $x_n^*(x_n) \to 0$.

$(d) \Rightarrow (c) \Rightarrow (a)$ Obvious. □

In the following theorem we show that, for some Banach lattices with the property (d), the p-weak DP* and p-bi-sequence properties are equivalent. We state the following result without proof and refer the reader to [14, Theorem 3.1]. Also it describes another characterisation for the p-weak DP* property.

THEOREM 3.11. *Let E be a Banach lattice with the property (d) and the type q (with $1 < q \leqslant 2$) and let $p \geqslant q'$. Then the following are equivalent:*

(a) E has the p-weak DP* property,

(b) for every disjoint sequence $(x_n) \in \ell_p^w(E)$ *and every disjoint weak** *null sequence* (x_n^*) *in* E^*, $x_n^*(x_n) \to 0$,

(c) for every disjoint sequence $(x_n) \in \ell_p^w(E)_+$ *and every disjoint weak** *null sequence* (x_n^*) *in* E_+^*, $x_n^*(x_n) \to 0$,

(d) the solid hull of a weakly p*-compact set is almost limited,*

(e) for every sequence $(x_n) \in \ell_p^w(E)_+$ *and every weak** *null sequence* (x_n^*) *in* E_+^*, $x_n^*(x_n) \to 0$,

(f) E *has the* p*-bi-sequence property.*

If a Banach lattice has the p-Schur or the p-DP* property, then it also has the p-weak DP* property, but the converse is false. In fact, Banach lattice $\ell_\infty \oplus L^1[0,1]$ has the p-weak DP* property, but it has neither the p-positive Schur nor the p-DP* property. Since ℓ_∞ has the DP* property, but it does not have the p-positive Schur property, also $L^1[0,1]$ has the positive Schur and weak DP* properties, but it does not have the p-DP* property. Note that, each separale Banach space with the p-DP* property has the p-Schur property.

Castillo mentioned that p*-convergent* operators in [4]. In fact, p-convergent operators are precisely those operators which transformed weakly p-compact subsets into realtively compact subsets. Equivalently, an operator $T: X \to Y$ between two Banach spaces is called p-convergent, if $\|Tx_n\| \to 0$, for every sequence $(x_n) \in \ell_p^w(X)$. Also, an operator $T: E \to Y$ is called *disjoint* p*-convergent* if it takes disjoint sequences $(x_n) \in \ell_p^w(E)$ to norm null ones [20]. It is clear that every p-convergent operator is disjoint p-convergent. For the converse, we prove the following practical lemma.

LEMMA 3.12. *Let* E *be a Banach lattice with the type* q *(with* $1 < q \leqslant 2$*), let* $p \geqslant q'$ *and* F *is discrete with order continuous norm. Then every positive operator* $T: E \to F$ *is disjoint* p*-convergent if and only if it is* p*-convergent.*

Proof. Let W be a weakly p-compact subset of E and let $T: E \to F$ be a positive disjoint p-convergent operator. From [4], it is enough to show that $T(W)$ is relatively compact. let A be the solid hull of W, then by [20, Lemma 4.2.1] every disjoint sequence (x_n) in A is weakly p-summable and so by hypothesis, $\|Tx_n\| \to 0$. By a consequence of [1] theorems 13.3 and 13.5, $T(A)$ is an almost order bounded set in F. Since F is discrete with order continuous norm, then $T(A)$ is relatively compact. Hence, T is a p-convergent operator. □

An operator T on a Banach lattice E is said to be an *almost DP operator* if the sequence $\|Tx_n\| \to 0$ for every disjoint weakly null sequence (x_n) in E. By [5, Theorem 3.5], E has the weak DP* property if and only if each operator $T: E \to c_0$ is an almost DP operator. With the same techniques of [5, Theorem 3.5], we can characterize some σ–Dedekind complete Banach lattices with the p-weak DP* property.

THEOREM 3.13. *Let* E *be a* σ*–Dedekind complete Banach lattice with the type* q *(with* $1 < q \leqslant 2$*) and let* $p \geqslant q'$*. Then the following are equivalent:*

(a) E has the p-weak DP* property,

(b) every continuous operator $T: E \to c_0$ is disjoint p-convergent,

(c) every positive operator $T: E \to c_0$ is disjoint p-convergent,

(d) every positive operator $T: E \to c_0$ is p-convergent.

Proof. $(b) \Rightarrow (c)$. It is clear.

$(c) \Leftrightarrow (d)$. Because c_0 is discrete with order continuous norm, it follows from Lemma 3.12.

$(c) \Rightarrow (a)$. By Theorem 3.8, we have to show that for each disjoint sequence $(x_n) \in \ell_p^w(E)_+$ and each disjoint weak* null sequence (x_n^*) in E_+^*, $x_n^*(x_n) \to 0$. Consider the positive opeartor $T: E \to c_0$ defined by $Tx = (\langle x, x_n^* \rangle)$, for all $x \in E$. According to (c), T is a disjoint p-convergent operator. Therefore, $\|Tx_n\| \to 0$, and hence $x_n^*(x_n) \to 0$, as desired.

$(a) \Rightarrow (b)$. Consider the operator $T: E \to c_0$. We have to show that $\|Tx_n\| \to 0$, for each disjoint squence $(x_n) \in \ell_p^w(E)$. Assume by way of contradiction that $\|Tx_n\| \nrightarrow 0$ for some disjoint squence $(x_n) \in \ell_p^w(E)$. Then, we can suppose that there would exist some $\varepsilon > 0$ satisfying $\|Tx_n\| > \varepsilon$ for all $n \in \mathbb{N}$. Since (x_n) is weakly p-summable, then it is weakly null and so by [1, Ex. 22, p. 73] and by the similar idea in Theorem 3.5 of [5], one can find a disjoint weak* null sequence (x_n^*) in E^*. Since E has the p-weak DP* property, then by Theorem 3.8, $x_n^*(x_n) \to 0$. This is a contradiction. Hence, $\|Tx_n\| \to 0$, for each disjoint squence $(x_n) \in \ell_p^w(E)$ and so T is a disjoint p-convergent operator. □

We can uniquely determine every bounded linear operator $T: E \to c_0$ by a weak* null sequence $(x_n^*) \subset E^*$ such that $Tx = (\langle x, x_n^* \rangle)$, for all $x \in E$. But, if $(x_n^*) \subset E^*$ is a disjoint weak* null sequence, then T is called a disjoint operator.

The final result deals the relationship between the weak DP* (resp. the p-weak DP*) property and disjoint completely continous (resp. disjoint p-convergent) operators. Recall that, an operator on a Banach lattice E is called *completely continous*, if it maps weakly null sequences in E to norm null ones.

THEOREM 3.14. *If E is a Banach lattice with the property (d), then for the following assertions:*

(a) every disjoint operator $T: E \to c_0$ is completely continous,

(b) E has the weak DP* property,

(c) E has the p-weak DP* property,

(d) every disjoint operator $T: E \to c_0$ is p-convergent.

the implications $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$ are valid.

Proof. $(a) \Rightarrow (b)$. Assume by way of contradiction that there is a weakly null sequence (x_n) in E and a disjoint weak* null sequence (x_n^*) in E^* such that $|\langle x_n^*, x_n\rangle| > \varepsilon$, for all n and some $\varepsilon > 0$. Consider the operator $T : E \to c_0$ defined by

$$Tx = (\langle x, x_n^*\rangle), \qquad x \in E.$$

It is clear that T is a disjoint operator, but T is not completely continous. In fact, the sequence (x_n) is weakly null, but $\|Tx_n\| \geqslant x_n^*(x_n) \geqslant \varepsilon$, for all n and some $\varepsilon > 0$.

$(b) \Rightarrow (a)$. Let $T : E \to c_0$ be a disjoint operator and let (x_n) be a weakly null sequence in E. Since E has the weak DP* proeprty, then (x_n) is almost limited. By [13, Theorem 2.7], the operator T is order bounded. Moreover E and c_0 have the property (d) and c_0 has the strong GP property. Then by [13, Proposition 3.9] and [2, Definition 3.1], $\|Tx_n\| \to 0$ and so T is completely continous.

$(b) \Rightarrow (c)$. It is clear.

$(c) \Leftrightarrow (d)$. It is enough to repeat the techniques of Theorem 3.13. □

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